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Hybrid Fuzzy Fractional Differential Equations by Hybrid Functions Method

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Abstract In this paper, we study a new operational numerical method for hybrid fuzzy fractional differential equations by using of the hybrid functions under generalized Caputo- type fuzzy fractional derivative. Solving two examples of hybrid fuzzy fractional differential equations illustrate the method.

Keywords Fuzzy Caputo derivative · Hybrid fuzzy fractional differential equations · Hybrid functions method.

Mathematics Subject Classification (2010) 34A08 · 34A07

1 Introduction

The fractional calculus is a field of science that involves both integrals and derivatives of any arbitrary order. Dynamical systems can be efficiently characterized by fractional differential equations. In the recent years, fractional differential equations have attracted a considerable interest due to their numerous appearance in various fields and their more accurate models of systems by considering fractional derivative. Thus it is very momentous to find efficient methods for solving fractional differential equations.
The hybrid fuzzy differential equations (HFDEs) is a natural way to model dynamic systems with embedded uncertainty. Therefore, they have a wide range of applications in science and engineering.
This work is devoted to studying the hybrid fuzzy fractional differential equations. So far, some researcher have used various numerical for solve the hybrid

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fuzzy fractional differential equations [10,12]. However, our method is a operational method based on hybrid of block-pulse functions and the second kind Chebyshev polynomials will be presented.
The paper is organized as follows. Behind preliminaries, Section is devoted to definitions of fuzzy Riemann-Liouville integral and fuzzy Caputo derivative. After this section, we will study the hybrid fuzzy fractional differential equations using the concept of fuzzy generalized Caputo differentiability. The next section, we briefly describe the hybrid functions method. In the last section we present two numerical examples to illustrate the method.

2 Preliminaries

Definition 1 A fuzzy number u is a fuzzy subset of the real line with a normal, convex and upper semicontinuous membership function of bounded support. The family of fuzzy numbers will be denoted by $\mathbb{R}_{\mathcal{F}}$. An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leqslant r \leqslant 1$ that, satisfies the following requirements:

- $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$, with respect to any r.
- $\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0, 1]$, with respect to any r.
- $\underline{u}(r) \leqslant \overline{u}(r)$, $0 \leqslant r \leqslant 1$.

then, the r-level set $[v]^r = \{s|v(s) \geq 0\}$ is a closed bounded interval, denoted $[v]^r = [\underline{v}^r, \overline{v}^r]$.

Definition 2 A triangular fuzzy number is a fuzzy set u in $\mathbb{R}_{\mathcal{F}}$ that is characterized by an ordered triple $(u_l, u_c, u_r) \in \mathbb{R}^3$ with $u_l \leqslant u_c \leqslant u_r$ such that $[u]^0 = [u_l; u_r]$ and $[u]^1 = \{u_c\}$.

The r-level set of a triangular fuzzy number u is given by

$$[u]^r = [u_c - (1-r)(u_c - u_l), u_c + (1-r)(u_r - u_c)$$

for any $r \in I = [0, 1]$

Definition 3 Let A, B two nonempty bounded subset of $\mathbb{R}$. The Hausdorff distance between A and B is

$$d_H(A, B) = \max[\sup_{a\in A} \inf_{b\in B} |a-b|, \sup_{b\in B} \inf_{a\in A} |a-b|].$$

The supremum metric D on $\mathbb{R}_{\mathcal{F}}$ is as follows:

$$D(u, v) = \sup\{d_H([u]^r, [v]^r), r \in I\}.$$

With the supremum metric, the space $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

Definition 4 Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v + w$, then w is called the H-difference of u and v, and it is denoted by $u \ominus v$.

Definition 5 The generalized Hukuhara difference of two fuzzy number $u, v \in \mathbb{R}_{\mathcal{F}}$ (gH- difference for short) is $w \in \mathbb{R}_{\mathcal{F}}$, that defined as follows

$$u \ominus_{gH} v = w \leftrightarrow \begin{cases} (i)\ u = v + w, \\ (ii)\ v = u + (-1)w. \end{cases}$$

Definition 6 The generalized Hukuhara derivative of a fuzzy-valued function $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ at x_0 is defined as

$$(f)'_{gH}(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) \ominus_{gH} f(x_0)}{h}$$

If $(f)'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$, we say that f is generalized Hukuhara differentiable at x_0, Also, we say that f is $[(i) - gH]-$ differentiable at x_0 if

$$(i)\ (f)'_{gH}(x_0; r) = [(\underline{f})'(x_0; r), (\overline{f})'(x_0; r)], \quad 0 \le r \le 1$$

and that f is $[(ii) - gH]-$differentiable at x_0 if

$$(ii)\ (f)'_{gH}(x_0; r) = [(\overline{f})'(x_0; r), (\underline{f})'(x_0; r)], \quad 0 \le r \le 1$$

Definition 7 Consider $f : [a, b] \to \mathbb{R}$, fractional derivative of $f(t)$ in the Caputo sense is defined as

$$(D_*^\alpha f)(x) = \frac{q}{\Gamma(m - \alpha)} \int_a^x (x-t)^{(m-\alpha-1)} f^{(m)}(t) dt \quad m-1 < \alpha \le m, m \in \mathbb{N},\ x > a \tag{1}$$

3 Fuzzy Riemann-Liouville integral and fuzzy Caputo derivative

Definition 8 Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$; the fuzzy Riemann-Liouville integral of fuzzy-valued function f is denoted as follows:

$$(J_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x - t)^{1-\alpha}} dt$$

for $a \le x$, $0 \le \alpha \le 1$. For $\alpha = 1$, we set $J_a^1 = I$, the identity operator.

Let us denote $C^{\mathcal{F}}[a, b]$ as the space of all continuous fuzzy valued functions on interval $[a, b]$. Also, we denote the space of all Lebesque integrable fuzzy-valued functions on interval $[a, b] \subset \mathbb{R}$ by $L^{\mathcal{F}}[a, b]$.

Definition 9 Let $f_{gH}^{(m)} \in C^{\mathcal{F}}[a, b] \cap L^{\mathcal{F}}[a, b]$. The fuzzy gH-fractional Caputo differentiability of fuzzy valued function f (${}^{CF}[gH]-$differentiability for short) is defined as following:

$$({}_{gH}D_*^\alpha f)(x) = J_a^{m-\alpha}(f_{gH}^{(m)}(x)) = \frac{1}{m - \alpha} \int_a^t (x - t)^{m-\alpha-1} (f_{gH}^{(m)})(t) dt, \tag{2}$$

where $m - 1 < \alpha \le m$, $m \in \mathbb{N}$, $x > a$.

Theorem 1 *Let ${f'}_{gH} \in C^{\mathcal{F}}[a,b] \cap L^{\mathcal{F}}[a,b]$, and $f(x;r) = [\underline{f}(x;r), \overline{f}(x;r)]$ for $0 \le r \le 1$, $x \in [a,b]$. Let $\underline{f}(x;r)$ and $\overline{f}(x;r)$ are Caputo differentiable functions then the function f is ${}^{CF}[gH]-$differentiable. Furthermore*

$$({}_{gH}D_*^{\alpha} f)(x;r) = [\min\{(D_*^{\alpha}\underline{f})(x;r), (D_*^{\alpha}\overline{f})(x;r)\}, \max\{(D_*^{\alpha}\underline{f})(x;r), (D_*^{\alpha}\overline{f})(x;r)\}] \tag{3}$$

where $(D_*^{\alpha}\underline{f})(x;r)$ and $(D_*^{\alpha}\overline{f})(x;r)$ defined in Definition 7.

Definition 10 Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be ${}^{CF}[gH]-$differentiable at $x_0 \in (a,b)$. We say that f is ${}^{CF}[(i) - gH]-$differentiable at x_0 if

$$({}_{gH}D_*^{\alpha} f)(x_0;r) = [(D_*^{\alpha}\underline{f})(x_0;r), (D_*^{\alpha}\overline{f})(x_0;r)], \quad 0 \le r \le 1 \tag{4}$$

and that f is ${}^{CF}[(ii) - gH]-$differentiable at x_0 if

$$({}_{gH}D_*^{\alpha} f)(x_0;r) = [(D_*^{\alpha}\overline{f})(x_0,r), (D_*^{\alpha}\underline{f})(x_0;r)], \quad 0 \le r \le 1 \tag{5}$$

where

$$(D_*^{\alpha}\underline{f})(x_0;r) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{(\underline{f}')(t;r)}{(x-t)^{\alpha}} dt, \tag{6}$$

$$(D_*^{\alpha}\overline{f})(x_0;r) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{(\overline{f}')(t;r)}{(x-t)^{\alpha}} dt. \tag{7}$$

Theorem 2 *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function on $[a,b]$.*

(a) If f is [(i)-gH]-differentiable at $x_0 \in [a,b]$ then f is ${}^{CF}[(i) - gH]-$differentiable at x_0.
(b) If f is is [(ii)-gH]-differentiable at $x_0 \in [a,b]$ then f is ${}^{CF}[(ii) - gH]-$differentiable at x_0.

4 Hybrid fuzzy fractional differential equation

Consider the hybrid fuzzy differential equation:

$$\begin{cases} {}_{gH}D_*^{\alpha} x(t) = f(t, x(t), \lambda_k(x(t_k))), & t \in [t_k, t_{k+1}] \\ x(t_k) = x_k, \end{cases} \tag{8}$$

where $0 \le t_0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$, $f \in C[\mathbb{R}^+ \times \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}}]$, $\lambda_k \in C[\mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}}]$. be as follows:

$$ {}_{gH}D_*^{\alpha} x(t) = \begin{cases} {}_{gH}D_*^{\alpha} x_0(t) = f(t, x_0(t), \lambda_0(x_0)), & x_0(t_0) = x_0, & t \in [t_0, t_1] \\ {}_{gH}D_*^{\alpha} x_1(t) = f(t, x_1(t), \lambda_1(x_1)), & x_1(t_1) = x_1, & t \in [t_1, t_2] \\ {}_{gH}D_*^{\alpha} x_2(t) = f(t, x_2(t), \lambda_2(x_2)), & x_2(t_2) = x_2, & t \in [t_2, t_3] \\ \vdots \\ {}_{gH}D_*^{\alpha} x_k(t) = f(t, x_k(t), \lambda_k(x_k)), & x_k(t_k) = x_k, & t \in [t_k, t_{k+1}] \\ \vdots \end{cases} \tag{9}$$

With respect to the solution of (8), we determine the following function:

$$x(t,t_0,x_0) = \begin{cases} x_0(t), & t \in [t_0,t_1] \\ x_1(t), & t \in [t_1,t_2] \\ x_2(t), & t \in [t_2,t_3] \\ \vdots & \\ x_k(t), & t \in [t_k,t_{k+1}] \\ \vdots & \end{cases} \tag{10}$$

We note that the solutions of (8) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in \mathbb{R}_{\mathcal{F}}$ and $k = 0, 1, 2, \ldots$. We may replace (8) by the following ODEs:

$$\begin{cases} D_*^\alpha x^-(t) = f^-(t, x, \lambda_k(x_k)) = f_k^-(t, x^-, x^1, x^+, \lambda_k(x_k)), \\ x^-(t_k) = x_k^-, \\ D_*^\alpha x^1(t) = f^1(t, x, \lambda_k(x_k)) = f_k^1(t, x^-, x^1, x^+, \lambda_k(x_k)), \\ x^1(t_k) = x_k^1, \\ D_*^\alpha x^+(t) = f^+(t, x, \lambda_k(x_k)) = f_k^+(t, x^-, x^1, x^+, \lambda_k(x_k)), \\ x^+(t_k) = x_k^+. \end{cases} \tag{11}$$

That is, for each t, the pair $[x^+(t,r), x^-(t,r)]$ is a fuzzy number, where $x^-(t,r), x^+(t,r)$ are, respectively, the solutions of the parametric form given by:

$$\begin{cases} D_*^\alpha x^-(t;r) = F_k^-(t, x^-(t;r), x^+(t;r), \lambda_k(x_k)), \\ x^-(t_k;r) = x_k^-(r), \\ D_*^\alpha x^+(t;r) = F_k^+(t, x^-(t;r), x^+(t;r), \lambda_k(x_k)), \\ x^+(t_k;r) = x_k^+(r). \end{cases} \tag{12}$$

for each $0 \le r \le 1$. For a fixed r, to integrate the system (12) in

$$[t_0,t_1], [t_1,t_2], \cdots, [t_k,t_{k+1}], \cdots.$$

5 Hybrid functions

Hybrid functions which consist of block-pulse functions and different polynomials, such as Legendre polynomials, Chebyshev polynomials, bernoulli polynomials and Bernstein polynomial, have a special place in differential equations and integral equations. Here we apply hybrid functions based upon block-pulse functions and the second kind Chebyshev polynomials.

5.1 hybrid functions based upon block-pulse functions and the second kind Chebyshev polynomials

Definition 11 The second kind Chebyshev Polynomials are defined on interval $[-1, 1]$ by

$$U_m(t) = \frac{\sin(m+1)\theta}{\sin\theta}, \quad t = \cos\theta, \quad m = 0, 1, 2, \ldots.$$

These polynomial functions are orthogonal with respect to the weighted function $w(t) = \sqrt{1-t^2}$, on the interval $[-1, 1]$ and satisfy the following recursive formulas

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, ...$$

Definition 12 The hybrid functions $h_{nm},\ n = 1, 2, ..., N,\ m = 0, 1, ..., M-1$, on the interval $[0, t_f)$ are defined as, h_{nm} are defined as,

$$h_{nm}(t) = \begin{cases} U_m(2Nt - 2n + 1), & t \in [\frac{n-1}{N}, \frac{n}{N}), \\ 0, & o.w., \end{cases}$$

where n and m are the order of the block-pulse functions and the second kind Chebyshev polynomials, respectively. Since $h_{nm}(t)$ consists of the block-pulse functions and the chebyshev polynomials, which are both complete and orthogonal, so a set of the hybrid functions based on them is complete orthogonal set.

5.2 Approximations and operations

In this section, we apply the Hybrid functions to approximate an arbitrary function. A function $f(t) \in L^2_w[0, t_f]$, may be expanded as the following:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} h_{nm}(t) \cong \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} h_{nm}(t) = C^T H_\mu(t). \qquad (13)$$

where $\mu(t) = NM$,

$$C = [c_{10}, ..., c_{1(M-1)}, c_{20}, ..., c_{2(M-1)}, ..., c_{N0}, ..., c_{N(M-1)}]^T,$$

and

$$H_\mu(t) = [h_{10}, ..., h_{1(M-1)}, h_{20}, ..., h_{2(M-1)}, ..., h_{N0}, ..., h_{N(M-1)}]^T.$$

Let $\mu = NM$, The integration of the vector $H_\mu(t)$, is given by

$$\int H_{\mu(t)} \approx P_{\mu\times\mu} H_\mu(t),$$

where $P_{\mu\times\mu}$ is the $\mu\times\mu$ operational matrix for integration, and can be obtained as follows $P_{\mu\times\mu} \approx \frac{1}{2\mu} \Psi_{\mu\times\mu} F^\alpha \Psi^{-1}_{\mu\times\mu}$. In above relation, square matrix $\Psi_{\mu\times\mu}$ by using collocation points $t_i = \frac{2i-1}{2\mu}, i = 1, 2, ..., \mu$, and F^α matrix are defined with

$$\Psi_{\mu\times\mu} = [H_\mu(\frac{1}{\mu})\ H_\mu(\frac{3}{2\mu})\ \cdots\ H(\frac{2\mu-1}{2\mu})],$$

$$F^{\alpha} = \frac{1}{\mu^{\alpha}} \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} 1 & \epsilon_1 & \epsilon_2 & \cdots & \epsilon_{\mu-1} \\ 0 & 1 & \epsilon_1 & \cdots & \epsilon_{\mu-2} \\ 0 & 0 & 1 & \cdots & \epsilon_{\mu-3} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\epsilon_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$, for $k = 1, 2, ..., \mu - 1$.

Theorem 3 *Assume that $f(t) \in C^{\mathcal{F}}M[0,1]$ is a fuzzy-valued function with bounded M^{th} derivative, i.e. $D(f^{(M)}), 0) \leq L$. If the function f can be expanded as a fuzzy finite sum of hybrid functions*

$$\{h_{10}, \cdots, h_{1(M-1)}, h_{20}, \cdots, h_{2(M-1)}, \cdots, h_{N0}, \cdots, h_{N(M-1)}\},$$

then by using hybrid basis functions, the mean error bound is presented as follow: $D_w(f(t), C^T H_\mu(t)) \leq \dfrac{2L}{N^M M!}$. where $D_w(u,v) = (\int_0^1 D(u(t), v(t))w(t)dt)$ and $w = \sqrt{1-t^2}$

Proof. The function f is a fuzzy function thus it can be written as $f^r = [\underline{f}^r, \overline{f}^r]$, for all $r \in [0,1]$, for which $\underline{f}^r$ and $\overline{f}^r$ are real functions. In addition, the assumptions follow that $|(\underline{f}^r)^M| \leq L$ and $|(\overline{f}^r)^M| \leq L$

$$D_w(f(t), C^T H_\mu(t)) = \int_0^1 (f(t), C^T H_\mu(t))w(t)dt \tag{14}$$

$$\left(\int_0^1 (\sup_{r\in[0,1)} \max\{|\underline{f}^r - \underline{c}^{rT} H_\mu(t)|, |\overline{f}^r - \overline{c}^{rT}|H_\mu(t)\})w(t)dt \right) \tag{15}$$

$$\sup_{r\in[0,1)} \max \left\{ \left(\int_0^1 |\underline{f}^r - \underline{c}^{rT} H_\mu(t)|w(t)dt \right), \left(\int_0^1 |\overline{f}^r - \overline{c}^{rT} H_\mu(t)|w(t)dt \right) \right\} \tag{16}$$

$$\leq \frac{2L}{N^M M!} \tag{17}$$

the last inequality is according to Theorem 1 in [8]. □

In Theorem 3 we get a upper bounded that shows for M, N are sufficiently large, we have $C^T H_\mu(t) \to f(t)$.

6 Numerical examples

To give a clear overview of our study and to illustrate the above discussed technique, we consider the following examples.

Example 1 Consider the following hybrid fuzzy IVP,

$$\begin{cases} {}_{gH}D_*^{\alpha}x(t) = x(t) + m(t)\lambda_k(x(t_k)), \quad t \in [t_k, t_k+1], \\ \quad t_k = k, \quad k = 0, 1, 2, \ldots \\ x(0) = [0.75, 1, 1.125], \end{cases} \tag{18}$$

$$m(t) = \begin{cases} 2(t(mod1)), & \text{if} \quad t(mod1) \leq 0.5 \\ 2(1 - t(mod1)), & \text{if} \quad t(mod1) > 0.5, \end{cases}$$

$$\lambda_k(\mu) = \begin{cases} \hat{0}, & \text{if} \quad k = 0 \\ \mu, & \quad k \in \{1, 2, \ldots\}. \end{cases} \tag{19}$$

For the which $\hat{0} \in \mathbb{R}_{\mathcal{F}}$ define as $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x \neq 0$.
The hybrid fuzzy initial value problem (18) is equivalent to the following system of fuzzy initial value problems (for (i)-differentiability):

$$\begin{cases} D_*^{\alpha}(x_0^-) = x_0^-(t), \\ D_*^{\alpha}(x_0^1) = x_0^1(t), \\ D_*^{\alpha}(x_0^+) = x_0^+(t), \quad t \in [0, 1], \\ x^-(0) = 0.75, \ x^1(0) = 1, \ x^+(0) = 1.125, \\ D_*^{\alpha}(x_i^-)(t) = x_i^-(t) + m(t)x_i^-(t_i), \\ D_*^{\alpha}(x_i^1)(t) = x_i^1(t) + m(t)x_i^1(t_i), \\ D_*^{\alpha}(x_i^+)(t) = x_i^+(t) + m(t)x_i^+(t_i), \quad t \in [t_i, t_{i+1}], \\ x_i(t_i) = x_{i-1}(t_i), \quad i = 1, 2, 3, ... \end{cases}$$

In (18), $x(t) + m(t)\lambda_k(x_k(t))$ is a continuous function of t, x and $\lambda_k(x(t_k))$ and the fuzzy IVP

$$\begin{cases} {}_{gH}D_*^{\alpha}x(t) = x(t) + m(t)\lambda_k(x(t_k)), \quad t \in [t_k, t_{k+1}], t_k = k, \\ x(t_k) = x_{t_k}, \end{cases} \tag{20}$$

has the exact solution as ${}_{CF}[(i) - gH]$ [1]:
For $[0, 1]$, the exact solution of (18) satisfies

$$x(t) = [0.75E_{\alpha}(t^{\alpha}), E_{\alpha}(t^{\alpha}), 1.125E_{\alpha}(t^{\alpha})].$$

For $[1, 1.5]$, the exact solution of (18) satisfies

$$x(t) = x(1)E_{\alpha}[(t-1)^{\alpha}] + \int_a^t (t-x)^{\alpha-1}E_{\alpha,\alpha}[\lambda(t-x)^{\alpha}]2(x-1)dx.$$

For $[1.5, 2]$, the exact solution of (18) satisfies

$$x(t) = x(1)E_{\alpha}[(t-1)^{\alpha}] + \int_a^t (t-x)^{\alpha-1}E_{\alpha,\alpha}[\lambda(t-x)^{\alpha}]2(2-x)dx.$$

But solve integrals arise in solution is not easy, So we don't have exact explicit solution, Hence approximate solutions will be very useful. To numerically solve the hybrid fuzzy IVP (18) we will apply the hybrid functions method with

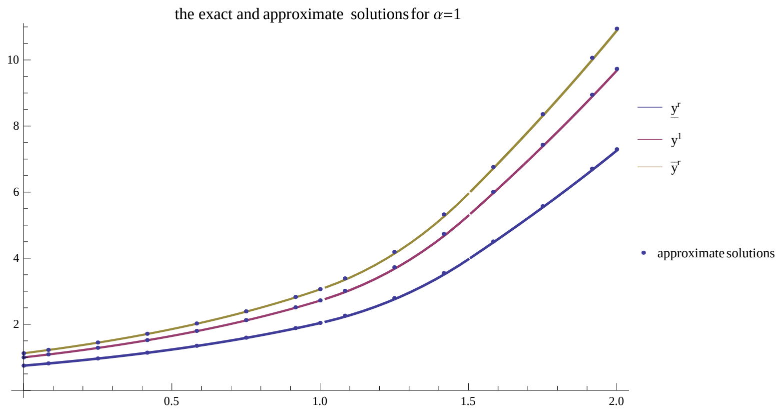

Fig. 1 $\underline{y}^r, \overline{y}^r$ and y^1 with $\alpha = 1$ for example 1 on [0,2]

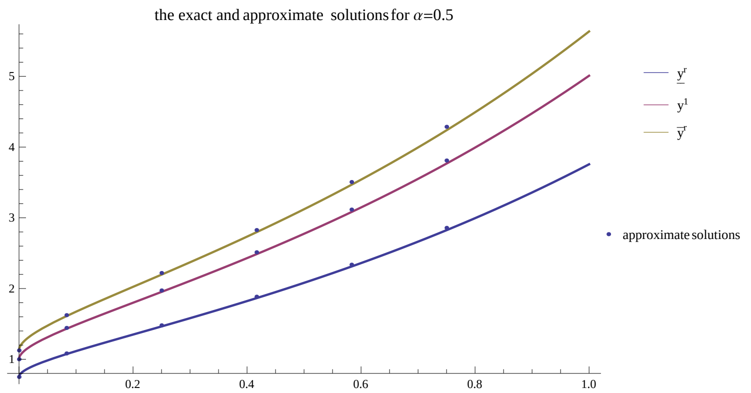

Fig. 2 $\underline{y}^r, \overline{y}^r$ and y^1 with $\alpha = 0.5$ for example 1 on [0,1]

$M = 3$ and $N = 2$ for the following hybrid fuzzy differential equations systems. The comparison between the exact and numerical solutions on $[0, 2]$ or $[0, 1]$ is shown in Figures 1,2,3,4.

Example 2 Consider the following hybrid fuzzy IVP,

$$\begin{cases} {}_{gH}D_*^{\alpha} x(t) = -x(t) + m(t)\lambda_k(x_k)), \ t \in [t_k, t_{k+1}], \\ \qquad t_k = k, k = 0, 1, 2, \ldots, \\ x(0) = [0.75, 1, 1.125], \end{cases} \tag{21}$$

where

$$m(t) = sin(\pi t), \quad k = 0, 1, 2, \ldots, \tag{22}$$

$$\lambda_k(\mu) = \begin{cases} \hat{0}, & if \ k = 0 \\ \mu, & if \ k \in \{0, 1, 2, \ldots\} \end{cases} \tag{23}$$

now, by (ii)-differential The hybrid fuzzy initial value problem (21) is equivalent to the following system:

$$\begin{cases} D_*^{\alpha} x_0^-(t) = -x_0^-(t), \\ D_*^{\alpha} x_0^1(t) = -x_0^1(t), \\ D_*^{\alpha} x_0^+(t) = -x_0^+(t), \quad t \in [0,1], \\ x(0) = [0.75, 1, 1.125], \\ D_*^{\alpha} x_i^-(t) = -x_i^-(t) + m(t)x_i^-(t_i), \\ D_*^{\alpha} x_i^1(t) = -x_i^1(t) + m(t)x_i^1(t_i), \\ D_*^{\alpha} x_i^+(t) = -x_i^+(t) + m(t)x_i^+(t_i), \quad t \in [t_i, t_{i+1}], \\ x_i(t_i) = x_{i-1}(t_i), \qquad \text{if } i \text{ is odd} \\ D_*^{\alpha} x_i^{-\prime}(t) = -x_i^-(t) + m(t)x_i^+(t_i), \\ D_*^{\alpha} x_i^{1(}t) = -x_i^1(t) + m(t)x_i^1(t_i), \\ D_*^{\alpha} x_i^{+(}t) = -x_i^+(t) + m(t)x_i^-(t_i), \quad t \in [t_i, t_{i+1}], \\ x_i(t_i) = x_{i-1}(t_i), \qquad \text{if } i \text{ is even} \end{cases} \tag{24}$$

For [0,1], the exact solution of Eq. (21) satisfies

$$x(t) = [0.75E_{\alpha}(-t^{\alpha}), E_{\alpha}(-t^{\alpha}), 1.125E_{\alpha}(-t^{\alpha})]$$

For [1,2], the exact solution of Eq. (21) satisfies,

$$x(t) = x(1)E_{\alpha}[-t^{\alpha}] + \int_0^t (t-x)^{\alpha-1} E_{\alpha,\alpha}[-(t-x)^{\alpha}] sin(\pi x) dx$$

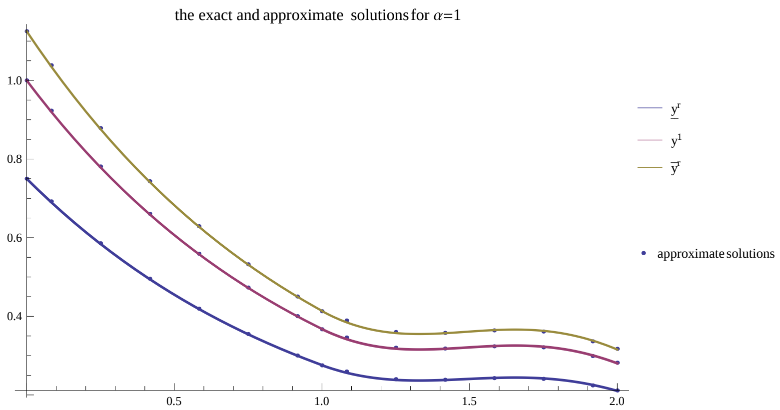

Fig. 3 $\underline{y}^r, \overline{y}^r$ and y^1 with $\alpha = 1$ for example 2 on [0,2]

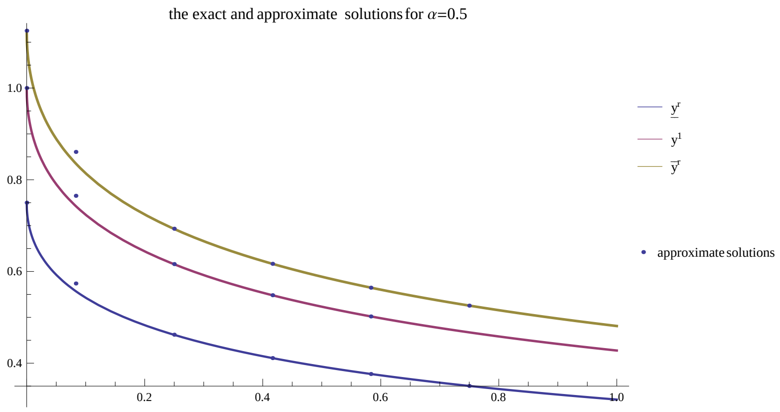

Fig. 4 $\underline{y}^r$, $\overline{y}^r$ and y^1 with $\alpha = 0.5$ for example 2 on [0,1]

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