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Analytical and Verified Numerical Results Concerning Interval Continuous-time Algebraic Riccati Equations

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Abstract This paper focuses on studying the interval continuous-time algebraic Riccati equation $\mathbf{A}^*X + X\mathbf{A} + \mathbf{Q} - X\mathbf{G}X = 0$, both from the theoretical aspects and the computational ones. In theoretical parts, we show that Shary's results for interval linear systems can only be partially generalized to this interval Riccati matrix equation. We then derive an efficient technique for enclosing the united stable solution set based on a modified variant of the Krawczyk method which enables us to reduce the computational complexity, significantly. Various numerical experiments are also given to show the efficiency of proposed scheme.

Keywords Interval continuous-time algebraic Riccati equation · AE-solution set · United stable solution set · Krawczyk's method · Verified computation · Interval analysis

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1 Introduction

The continuous-time algebraic Riccati equation ($CARE$)

$$A^*X + XA + Q - XGX = 0; \quad A, G, Q \in \mathbb{C}^{n\times n}, G^* = G, Q^* = Q, \qquad (1.1)$$

which is a famous matrix equation with linear and quadratic terms, has attracted several authors. Recent examples include books by D. A. Bini, B. Iannazzo, B. Meini, [7], and by H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, [2]. An old source is a book by P. Lancaster and L. Rodman, [16]. For a newer, see V. Simoncini, D. B. Szyld, M. Monsalve, [30]. (1.1) arises in, for example, classical problems of systems theory, differential equations, and filter design, as well as in differential games. References to these motivating problems are strewn throughout the literature and particularly in several earlier works. References [1, 7, 8, 16, 18] deal in varying detail with equations of type (1.1) in which A^* denotes the conjugate transpose of the matrix A and $n \times n$ matrix solutions X are to be found.

However, engineers lack precise knowledge regarding the process and its input data. This lack of knowledge and the inherent inexactness in measurement make a verification method cycle tasks as design of a formal model and definition of relevant parameters. Nevertheless, less attention has been paid to the form of uncertainties that may occur in the matrix coefficients in (1.1).

One of the most well-known methods of representing uncertainty and/or ambiguity in mathematics is *interval analysis* [4]. In interval analysis, uncertain parameters are described by a lower and upper bound then, (nearly) sharp bounds on the solution(s) are computed. So, the following *interval continuous-time algebraic Riccati equation* (**ICARE**) should be solved

$$\mathbf{A}^*X + X\mathbf{A} + \mathbf{Q} - X\mathbf{G}X = 0, \qquad (1.2)$$

where $\mathbf{A}, \mathbf{G}$ and $\mathbf{Q}$ are known complex interval matrices and $\mathbf{G}$ and $\mathbf{Q}$ are interval Hermitian.

Up to our knowledge, a few works concerning to the interval form of CARE (1.1) have already been done; see, e.g., [14, 17, 28, 29]. In [17], the authors apply Brouwer's fixed point theorem to calculate verified solutions of the ARE

$$A^TX + XA + Q = XBR^{-1}B^TX; \quad Q, R \in \mathbb{R}^{n\times n}, \qquad (1.3)$$

with symmetric matrices Q and R, Q positive semi definite and R positive definite while here and everywhere in this note, A^T represents the transpose of A. They find an interval matrix including a positive definite solution of (1.3). Then, they claim that this verification procedure can extend to ARE problem for interval matrices, as well. In [28, 29], Shashikhin uses an interval linear system to find an interval enclosure for the united solution set of the interval Sylvester matrix equation. Hashemi and Dehghan [14] suggest a modification of Krawczyk operator with a reduction in computational complexity of obtaining an outer estimation of the united solution set to $\mathbf{A}X + X\mathbf{A}^T = \mathbf{Q}$, provided that the matrix mid($\mathbf{A}$) is diagonalizable. The same paper also includes the generalized concept of AE-solution sets to the interval Lyapunov matrix equation together with some analytical characterizations for these solution sets. In addition to an iterative method, the recent paper [9] recommends some efficient approaches for enclosing the united solution sets of preconditioned interval generalized Sylvester matrix equations which are again based on a modified variant of Krawczyk operator and require only $\mathcal{O}(n^3)$ operations under some assumptions on some spectral decompositions. Necessary conditions for characterizing the solution set and a sufficient condition for boundedness of the united solution set are also suggested.

In this paper, we will present a *tight verified* enclosure for the so-called united stable solution set of the interval continuous-time algebraic Riccati equation which is not mentioned in the literature in this general form, as we know. We develop and characterize the concept of AE-solution sets of ICARE (1.2) and then address the problem of computing verified outer estimations for the united stable solution set of ICARE (1.2), that is, determining an interval matrix which is guaranteed to contain the united stable solution set of (1.2). Indeed, our approach is based on the Krawczyk method, which we modify in such a manner that the computational complexity for the ICARE is reduced to n^3.

Moreover, to verify the stabilizing property of all solutions in the computed interval matrix, we have used Algorithm 7 from [13] based on the method described in [19].

In the following, we introduce some symbols and notation in Section 2. Then, we define and characterize the generalized AE-solution sets to the interval continuous-time algebraic Riccati equation (1.2) in Section 3. In Section 4, we focus on the united stable solution set to the ICARE and develop an approach for outer estimation of the united stable solution set. We test the performance of our algorithm on some standard examples in Section 5. Section 6 ends this paper with a short summary.

2 Preliminaries

We use the following abbreviations: $\mathbb{K}$ – either of the fields of real, $\mathbb{R}$, or complex numbers, $\mathbb{C}$; $\mathbb{K}^n$ – the space of n-dimensional vectors over $\mathbb{K}$; $\mathbb{K}^{n\times n}$ – the space of $n \times n$ matrices over $\mathbb{K}$; $\mathbb{IK}^n$ – the set of all n-dimensional interval vectors over $\mathbb{K}$; $\mathbb{IK}^{n\times n}$– the set of all $n \times n$ interval matrices over $\mathbb{K}$; I_n – the unit $n \times n$ matrix; $\overline{A} \in \mathbb{C}^{m\times n}$, $A^T \in \mathbb{C}^{n\times m}$ and $A^H = \overline{A}^T \in \mathbb{C}^{n\times m}$ – the complex conjugate, transpose and complex conjugate transpose of $A \in \mathbb{C}^{m\times n}$; $\mathrm{vec}\,(Z) \in \mathbb{C}^{mn}$ – the vector column-wise representation of the matrix $Z \in \mathbb{C}^{m\times n}$; $A \otimes B \in \mathbb{C}^{mp\times nq}$ – the Kronecker product of an $m \times n$ matrix $A = (A_{ij})$ and a $p \times q$ matrix B; $A./B$ – the element-wise division of a matrix $A = (A_{ij}) \in \mathbb{C}^{m\times n}$ by a matrix $B = (B_{ij}) \in \mathbb{C}^{m\times n}$ provided that $B_{ij} \neq 0$, for each $1 \leq i \leq m$ and $1 \leq j \leq n$; $\mathrm{Diag}(d) \in \mathbb{C}^{n\times n}$ – the diagonal matrix whose (i, i) entry is d_i where $d = (d_1, d_2, \ldots, d_n)^T \in \mathbb{C}^n$; $\mathrm{diag}(D)$ – the vector whose elements are the diagonal entries of D where D is a diagonal matrix; `realmin` – the smallest positive normalized floating point number. Most of these operations and notion are analogously defined for interval quantities.

Also, all interval quantities will be typeset in boldface whereas lower case will imply scalar quantities or vectors and upper case will denote matrices. Underscores and over scores will show lower bounds and upper bounds of interval quantities, correspondingly.

Complex intervals can be defined either as rectangles or as discs. We use the definition as discs for the circular complex interval or simply the complex interval $\mathbf{x}$, i.e., $\mathbf{x} := \{z \in \mathbb{C} \colon |z - \mathrm{mid}\,(\mathbf{x})| \le \mathrm{rad}\,(\mathbf{x})\} = \langle \mathrm{mid}\,(\mathbf{x}), \mathrm{rad}\,(\mathbf{x})\rangle$ when the radius, $\mathrm{rad}\,(\mathbf{x})$ belongs to $\mathbb{R}$ with $\mathrm{rad}\,(\mathbf{x}) \ge 0$ and the center, $\mathrm{mid}\,(\mathbf{x})$ is in $\mathbb{C}$.

The operations on the circular complex intervals, $\mathbb{IC}$, are introduced as generalizations of operations on complex numbers [4]. We emphasize that the definitions of addition, subtraction and inversion substantially coincide with their set theoretic definitions but the set $\{xy \colon x \in \mathbf{x}, y \in \mathbf{y}\}$, in general, is not a disc. Fortunately, one of the basic properties of interval arithmetic, which makes its use well-founded, is that respects inclusion: for *all* the four basic arithmetic operations $\circ \in \{+, -, \cdot, /\}$ one has

$$\{x \circ y : x \in \mathbf{x}, y \in \mathbf{y}\} \subseteq \mathbf{x} \circ \mathbf{y},$$

in which $\mathbf{x}$ and $\mathbf{y}$ are two real or circular complex intervals. In the case of division, we need to assume that $0 \notin \mathbf{y}$ for the operation to be well-defined.

Other abbreviations are as follows: $\square(\mathbf{x}, \mathbf{y})$ – the interval hull of two intervals $\mathbf{x}$ and $\mathbf{y}$: the smallest interval containing $\mathbf{x}$ and $\mathbf{y}$; $\mathrm{mig}(\mathbf{x}) := \min\{|x| : x \in \mathbf{x}\}$ – the mignitude of $\mathbf{x} \in \mathbb{IC}$; $\mathbf{A} = \langle\, \mathrm{mid}(\mathbf{A}), \mathrm{rad}(\mathbf{A})\rangle \in \mathbb{IC}^{m\times n}$ – the $m \times n$ interval matrix $\mathbf{A}$ whose (i,j) element is the complex interval $\langle\, \mathrm{mid}(\mathbf{A}_{ij}), \mathrm{rad}(\mathbf{A}_{ij})\rangle$, with $\mathrm{rad}(\mathbf{A}_{ij}) \ge 0$, $1 \le i \le m, 1 \le j \le n$. For interval vectors and matrices, $\mathrm{mid}, \mathrm{rad}, \mathrm{mig}$ and $\square$ will be applied component-wise.

Now, we shall recall some basic facts about Kronecker product, vec operator, $\mathrm{mid}, \mathrm{rad}$ and $\square$. For Lemmas 21 and 22 see for instance [10] and [15] and for Lemmas 23 and 24 see for instance [21] and [5].

Lemma 21 *Assume that $A = (A_{ij})$, $B = (B_{ij})$, $C = (C_{ij})$ and $D = (D_{ij})$ be complex matrices with compatible sizes. Then,*

1. $(A \otimes B)(C \otimes D) = AC \otimes BD$,
2. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$,
3. $(A \otimes B)^* = A^* \otimes B^*$,
4. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, *if A and B are invertible,*
5. $\mathrm{vec}(ABC) = (C^T \otimes A)\,\mathrm{vec}\,(B)$,
6. $\big(\mathrm{Diag}(\mathrm{vec}\,(A))\big)^{-1}\,\mathrm{vec}\,(B) = \mathrm{vec}\,(B./A)$, *if $A_{ij} \ne 0$ for each (i,j).*

Notice that if $\mathbf{D} = \mathrm{Diag}(\mathbf{d})$ is a diagonal interval matrix, with $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_N)^T$ and $0 \notin \mathbf{d}_i$ for each $i = 1, 2, \ldots, N$, then we may define $\mathbf{D}^{-1} := \mathrm{Diag}\,\big((\mathbf{d}_1^{-1}, \mathbf{d}_2^{-1}, \ldots, \mathbf{d}_N^{-1})^T\big)$.

Lemma 22 *Let $\mathbf{A} = (\mathbf{A}_{ij})$, $\mathbf{B} = (\mathbf{B}_{ij})$ and $\mathbf{C} = (\mathbf{C}_{ij})$ be complex interval matrices of compatible sizes. Then,*

1. $\Big\{(C^T \otimes A)\,\mathrm{vec}\,(B) \colon A \in \mathbf{A},\ B \in \mathbf{B},\ C \in \mathbf{C}\Big\} \subseteq \mathrm{vec}\Big(\mathbf{A}(\mathbf{B}\mathbf{C})\Big)$,
2. $\Big\{(C^T \otimes A)\,\mathrm{vec}\,(B) \colon A \in \mathbf{A},\ B \in \mathbf{B},\ C \in \mathbf{C}\Big\} \subseteq \mathrm{vec}\Big((\mathbf{A}\mathbf{B})\mathbf{C}\Big)$,
3. $\Big(\mathrm{Diag}(\mathrm{vec}\,(\mathbf{A}))\Big)^{-1}\mathrm{vec}\,(\mathbf{B}) = \mathrm{vec}(\mathbf{B}./\mathbf{A})$, *if* $0 \notin \mathbf{A}_{ij}$ *for all* (i,j).

Lemma 23 *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{IC}^{n\times n}$ *and* X *be a matrix with complex elements and compatible size. Then,*

1. $\mathbf{A} \subseteq \mathbf{B} \Leftrightarrow |\,\mathrm{mid}(\mathbf{B}) - \mathrm{mid}(\mathbf{A})| \le \mathrm{rad}(\mathbf{B}) - \mathrm{rad}(\mathbf{A})$,
2. $\mathrm{mid}(\mathbf{A} \pm \mathbf{B}) = \mathrm{mid}(\mathbf{A}) \pm \mathrm{mid}(\mathbf{B})$,
3. $\mathrm{rad}(\mathbf{A} \pm \mathbf{B}) = \mathrm{rad}(\mathbf{A}) + \mathrm{rad}(\mathbf{B})$,
4. $\mathrm{mid}(\mathbf{A}X) = \mathrm{mid}(\mathbf{A})X$,
5. $\mathrm{mid}(X\mathbf{A}) = X\,\mathrm{mid}(\mathbf{A})$,
6. $\mathrm{rad}(\mathbf{A}X) = \mathrm{rad}(\mathbf{A})|X|$,
7. $\mathrm{rad}(X\mathbf{A}) = |X|\,\mathrm{rad}(\mathbf{A})$.

Lemma 24 *Let* $\mathbf{A} \in \mathbb{IC}^{n\times n}$ *and suppose that* Φ *and* Ω *be two bounded sets of complex point matrices, all of the same size. Then,*

1. $\Phi \subseteq \Omega \Rightarrow \Box\Phi \subseteq \Box\Omega$,
2. $\Phi \subseteq \mathbf{A} \Rightarrow \Box\Phi \subseteq \mathbf{A}$.

3 Generalization and characterization of AE-solution sets to ICAREs

The AE-solution sets are studied frequently, e.g., in Shary [25, 27], Goldsztejn [11] and Goldsztejn and Chabert [12]. Shary introduced the concept of generalized solution sets and AE(AllExist)-solution sets to a linear interval system of equations [26,27]. In [27], quantifiers are used to describe and recognize various kinds of interval uncertainty in the course of modeling. The concepts of generalized solution sets and AE-solution sets to other interval (system of) equations have been utilized by other authors, see for example [14] and [23]. By a similar convention, we consider the different possible styles of describing the uncertainty type distributions with respect to the interval parameters of ICARE (1.2). Then, we study AE-solution sets, which are defined by universally and existentially quantified parameters where the former precede the latter.

In order to define generalized AE-solution sets, we need a way to describe the uncertainty type distribution for the interval equation (1.2). It seems to be adequate if we fix *disjoint decompositions* of all the interval matrices $\mathbf{A}, \mathbf{G}$ and $\mathbf{Q}$. First, we set interval matrices $\mathbf{A}^{\forall} := (\mathbf{A}^{\forall}_{ij})$ and $\mathbf{A}^{\exists} := (\mathbf{A}^{\exists}_{ij})$ of the same size as $\mathbf{A}$ as follows:

$$\mathbf{A}^{\forall}_{ij} := \begin{cases} \mathbf{A}_{ij}, & \text{if } \alpha_{ij} = \forall, \\ 0, & \text{o.w.}, \end{cases} \quad \text{and} \quad \mathbf{A}^{\exists}_{ij} := \begin{cases} \mathbf{A}_{ij}, & \text{if } \alpha_{ij} = \exists, \\ 0, & \text{o.w.}, \end{cases}$$

where α_{ij} is the (i,j) element of matrix $\alpha := (\alpha_{ij})$ defined as

$$\alpha_{ij} := \begin{cases} \forall & \text{if } (i,j) \in \Pi', \\ \exists & \text{if } (i,j) \in \Pi''. \end{cases}$$

Indeed, one can partition the entire set of the indices (i,j) of the elements $\mathbf{A}_{ij}$ of the matrix $\mathbf{A}$ by means of the sets $\Pi' = \{\pi'_1, \pi'_2, \ldots, \pi'_p\}$ and $\Pi'' = \{\pi''_{p+1}, \pi''_{p+2}, \ldots, \pi''_{n^2}\}$ such that $\Pi' \cap \Pi'' = \emptyset$. For $(i,j) \in \Pi'$, $\mathbf{A}_{ij}$ is one of the interval *A-uncertainty* [27], i.e., a specific property (here, satisfying in a point matrix equation) holds for all members of $\mathbf{A}_{ij}$. $\mathbf{A}_{ij}$ is of the interval *E-uncertainty* [27] when $(i,j) \in \Pi''$ that means only some, not all, members of $\mathbf{A}_{ij}$ have a desired property. Similarly, one can introduce non-interesting sets of integer pairs (i,j) of entry indices of the matrices $\mathbf{G}$ and $\mathbf{Q}$, respectively as $\Theta' = \{\theta'_1, \theta'_2, \ldots, \theta'_r\}, \Theta'' = \{\theta''_{r+1}, \theta''_{r+2}, \ldots, \theta''_{n^2}\}$ and $\Psi' = \{\psi'_1, \psi'_2, \ldots, \psi'_t\}$ and $\Psi'' = \{\psi''_{t+1}, \psi''_{t+2}, \ldots, \psi''_{n^2}\}$ such that all elements whose indices belong to Θ' and Ψ' are of interval A-uncertainty while all entries whose indices are chosen from Θ'' and Ψ'' have the interval E-uncertainty form. There is also this normal possibility that one of the sets in each pair $(\Pi', \Pi'), (\Theta', \Theta'')$ or (Ψ', Ψ''), but not both of them, to be empty.

We do similar work for explaining $\mathbf{G}^{\forall} := (\mathbf{G}^{\forall}_{ij}), \mathbf{G}^{\exists} := (\mathbf{G}^{\exists}_{ij}), \mathbf{Q}^{\forall} := (\mathbf{Q}^{\forall}_{ij})$ and $\mathbf{Q}^{\exists} := (\mathbf{Q}^{\exists}_{ij})$

$$\mathbf{G}^{\forall}_{ij} := \begin{cases} \mathbf{G}_{ij}, & \text{if } \beta_{ij} = \forall, \\ 0, & \text{o.w.,} \end{cases} \quad \text{and} \quad \mathbf{G}^{\exists}_{ij} := \begin{cases} \mathbf{G}_{ij}, & \text{if } \beta_{ij} = \exists, \\ 0, & \text{o.w.,} \end{cases}$$

and

$$\mathbf{Q}^{\forall}_{ij} := \begin{cases} \mathbf{Q}_{ij}, & \text{if } \gamma_{ij} = \forall, \\ 0, & \text{o.w.,} \end{cases} \quad \text{and} \quad \mathbf{Q}^{\exists}_{ij} := \begin{cases} \mathbf{Q}_{ij}, & \text{if } \gamma_{ij} = \exists, \\ 0, & \text{o.w.,} \end{cases}$$

as well as the quantifier matrices β and γ

$$\beta_{ij} := \begin{cases} \forall & \text{if } (i,j) \in \Theta', \\ \exists & \text{if } (i,j) \in \Theta'', \end{cases} \quad \gamma_{ij} := \begin{cases} \forall & \text{if } (i,j) \in \Psi', \\ \exists & \text{if } (i,j) \in \Psi''. \end{cases}$$

Therefore, for all $(i,j) \in \{1, 2, \ldots, n^2\}$ and the matrix $\mathbf{M} \in \{\mathbf{A}, \mathbf{G}, \mathbf{Q}\}$ we have $\mathbf{M}^{\forall}_{ij}\mathbf{M}^{\exists}_{ij} = 0$ and so $\mathbf{M} = \mathbf{M}^{\forall} + \mathbf{M}^{\exists}$.

Now, we can write down the formal extended definition of AE-solution sets to ICARE (1.2).

Definition 31 *We shall call the set*

$$\begin{aligned}
&\{X \in \mathbb{K}^{n\times n} : ((\forall A_{\pi'_1} \in \mathbf{A}_{\pi'_1}) \ldots (\forall A_{\pi'_p} \in \mathbf{A}_{\pi'_p}) \\
&(\forall G_{\theta'_1} \in \mathbf{G}_{\theta'_1}) \ldots (\forall G_{\theta'_r} \in \mathbf{G}_{\theta'_r})(\forall Q_{\psi'_1} \in \mathbf{Q}_{\psi'_1}) \ldots (\forall Q_{\psi'_t} \in \mathbf{Q}_{\psi'_t}) \\
&(\exists A_{\pi''_{p+1}} \in \mathbf{A}_{\pi''_{p+1}}) \ldots (\exists A_{\pi''_{n^2}} \in \mathbf{A}_{\pi''_{n^2}})(\exists G_{\theta''_{r+1}} \in \mathbf{G}_{\theta''_{r+1}}) \ldots (\exists G_{\theta''_{n^2}} \in \mathbf{G}_{\theta''_{n^2}}) \\
&(\exists Q_{\psi''_{t+1}} \in \mathbf{Q}_{\psi''_{t+1}}) \ldots (\exists Q_{\psi''_{n^2}} \in \mathbf{Q}_{\psi''_{n^2}})(A^*X + XA + Q = XGX))\},
\end{aligned}$$

the *AE-solution set of type* $\alpha\beta\gamma$ *to ICARE* (1.2) or briefly $\alpha\beta\gamma$ *solution set to ICARE* (1.2) and denote it by $\Sigma_{\alpha\beta\gamma}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$.

Definition 31 is a general definition while only particular cases are the ones on interest in almost all current research in advanced interval analysis. The basic types of AE-solution sets are:

- The *united solution set* to ICARE (1.2):

  $$\Sigma_{\exists\exists\exists}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) = \Sigma_{unt}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) := \{\, X \in \mathbb{K}^{n\times n} : ((\exists A \in \mathbf{A})(\exists G \in \mathbf{G})(\exists Q \in \mathbf{Q})(A^*X + XA + Q = XGX))\,\},$$

  which is the most interesting case among various possible choices occurs when we pick out the existential quantifier for all the components of quantifier matrices α, β and γ, i.e., where $\Pi' = \Theta' = \Psi' = \emptyset$. It is formed by all possible solutions of all point CAREs, $A^*X + XA + Q = XGX$ with $A \in \mathbf{A}, G \in \mathbf{G}$ and $Q \in \mathbf{Q}$.
- The *tolerable solution set* to ICARE (1.2):

  $$\Sigma_{\forall\forall\exists}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) = \Sigma_{tlr}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) := \{\, X \in \mathbb{K}^{n\times n} : ((\forall A \in \mathbf{A})(\forall G \in \mathbf{G})(\exists Q \in \mathbf{Q})(A^*X + XA + Q = XGX))\,\}.$$

  Hence, the tolerable solution set is that set such that if $X \in \mathbb{K}^{n\times n}$ belongs to it, then for each $A \in \mathbf{A}$ and each $G \in \mathbf{G}$, there exists at least one $Q \in \mathbf{Q}$ with $A^*X + XA - XGX = -Q$ or $A^*X + XA - XGX \in -\mathbf{Q}$.
- The *controllable solution set* to ICARE (1.2):

  $$\Sigma_{\exists\exists\forall}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) = \Sigma_{cnt}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) := \{\, X \in \mathbb{K}^{n\times n} : ((\forall Q \in \mathbf{Q})(\exists A \in \mathbf{A})(\exists G \in \mathbf{G})(A^*X + XA + Q = XGX))\,\},$$

  formed by all matrices as $X \in \mathbb{K}^{n\times n}$ such that for any $Q \in \mathbf{Q}$, one could determine some $A \in \mathbf{A}$ and some $G \in \mathbf{G}$ satisfying $A^*X+XA+Q = XGX$.

Remark 32 *The united solution set to ICARE is the widest solution set of all possible AE-solution sets to ICARE, i.e.,*

$$\Sigma_{\alpha\beta\gamma}(\textit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) \subseteq \Sigma_{unt}(\textit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}).$$

Theorem 33

$$\Sigma_{\alpha\beta\gamma}(\textit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) = \bigcap_{A'\in\mathbf{A}^\forall} \bigcap_{G'\in\mathbf{G}^\forall} \bigcap_{Q'\in\mathbf{Q}^\forall} \bigcup_{A''\in\mathbf{A}^\exists} \bigcup_{G''\in\mathbf{G}^\exists} \bigcup_{Q''\in\mathbf{Q}^\exists} \{\, X \in \mathbb{K}^{n\times n} : ((A' + A'')^*X + X(A' + A'') + (Q' + Q'') = X(G' + G'')X)\,\}. \tag{3.1}$$

Specifically,

$$\Sigma_{unt}(\textit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) = \cup_{A\in\mathbf{A}} \cup_{G\in\mathbf{G}} \cup_{Q\in\mathbf{Q}} \{X \in \mathbb{K}^{n\times n} : A^*X + XA + Q = XGX\},$$
$$\Sigma_{tlr}(\textit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) = \cap_{A\in\mathbf{A}} \cap_{G\in\mathbf{G}} \cup_{Q\in\mathbf{Q}} \{X \in \mathbb{K}^{n\times n} : A^*X + XA + Q = XGX\},$$
$$\Sigma_{cnt}(\textit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) = \cap_{Q\in\mathbf{Q}} \cup_{A\in\mathbf{A}} \cup_{G\in\mathbf{G}} \{X \in \mathbb{K}^{n\times n} : A^*X + XA + Q = XGX\}.$$

Proof By exploiting the matrices $\mathbf{A}^{\forall}, \mathbf{A}^{\exists}, \mathbf{G}^{\forall}, \mathbf{G}^{\exists}, \mathbf{Q}^{\forall}$ and $\mathbf{Q}^{\exists}$ and Definition 31, we can rewrite the $\alpha\beta\gamma$-solution set to ICARE (1.2) as

$$\begin{aligned}
&\Sigma_{\alpha\beta\gamma}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) = \{X \in \mathbb{K}^{n\times n} : ((\forall A' \in \mathbf{A}^{\forall})(\forall G' \in \mathbf{G}^{\forall})(\forall Q' \in \mathbf{Q}^{\forall}) \\
&(\exists A'' \in \mathbf{A}^{\exists})(\exists G'' \in \mathbf{G}^{\exists})(\exists Q'' \in \mathbf{Q}^{\exists}) \\
&((A' + A'')^* X + X(A' + A'') + (Q' + Q'') = X(G' + G'')X))\} \\
&= \bigcap_{A' \in \mathbf{A}^{\forall}} \bigcap_{G' \in \mathbf{G}^{\forall}} \bigcap_{Q' \in \mathbf{Q}^{\forall}} \{X \in \mathbb{K}^{n\times n} : ((\exists A'' \in \mathbf{A}^{\exists})(\exists G'' \in \mathbf{G}^{\exists})(\exists Q'' \in \mathbf{Q}^{\exists}) : \\
&((A' + A'')^* X + X(A' + A'') + (Q' + Q'') = X(G' + G'')X))\} \\
&= \bigcap_{A' \in \mathbf{A}^{\forall}} \bigcap_{G' \in \mathbf{G}^{\forall}} \bigcap_{Q' \in \mathbf{Q}^{\forall}} \bigcup_{A'' \in \mathbf{A}^{\exists}} \bigcup_{G'' \in \mathbf{G}^{\exists}} \bigcup_{Q'' \in \mathbf{Q}^{\exists}} \\
&\{X \in \mathbb{K}^{n\times n} : ((A' + A'')^* X + X(A' + A'') + (Q' + Q'') = X(G' + G'')X)\},
\end{aligned}$$

which establishes formula (3.1). The rest of the proof is straightforward.

Succeeding theorem asserts that the fundamental theorem for characterization of the AE-solution sets of an interval linear system ($x \in \Sigma_{\alpha\beta}(\mathbf{A}, \mathbf{b})$ if and only if $\mathbf{A}^{\forall} X - \mathbf{b}^{\forall} \subseteq \mathbf{b}^{\exists} - \mathbf{A}^{\exists} X$) [27, Theorem 3.4] does not hold any more for ICAREs. Remark 35 shows the reasons why this occurs.

Theorem 34 $X \in \Sigma_{\alpha\beta\gamma}(\mathit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$ *if and only if*

$$\begin{aligned}
&\{A'^* X + XA' + Q' - XG'X : A' \in \mathbf{A}^{\forall}, G' \in \mathbf{G}^{\forall}, Q' \in \mathbf{Q}^{\forall}\} \subseteq \\
&\{-(A''^* X + XA'' + Q'' - XG''X) : A'' \in \mathbf{A}^{\exists}, G'' \in \mathbf{G}^{\exists}, Q'' \in \mathbf{Q}^{\exists}\}.
\end{aligned} \tag{3.2}$$

Proof As the proof of Theorem 33, one could recompose the definition of $\Sigma_{\alpha\beta\gamma}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$ in the following equivalent form:

$$\begin{aligned}
&\Sigma_{\alpha\beta\gamma}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) = \{X \in \mathbb{K}^{n\times n} : \\
&((\forall A' \in \mathbf{A}^{\forall})(\forall G' \in \mathbf{G}^{\forall})(\forall Q' \in \mathbf{Q}^{\forall})(\exists A'' \in \mathbf{A}^{\exists})(\exists G'' \in \mathbf{G}^{\exists})(\exists Q'' \in \mathbf{Q}^{\exists}) \\
&((A' + A'')^* X + X(A' + A'') + (Q' + Q'') = X(G' + G'')X))\}.
\end{aligned} \tag{3.3}$$

Thus, for all $A' \in \mathbf{A}^{\forall}$, $G' \in \mathbf{G}^{\forall}$ and $Q' \in \mathbf{Q}^{\forall}$, there exist some $A'' \in \mathbf{A}^{\exists}$, $G'' \in \mathbf{G}^{\exists}$ and $Q'' \in \mathbf{Q}^{\exists}$ such that

$$A'^* X + XA' + Q' - XG'X = -(A''^* X + XA'' + Q'' - XG''X). \tag{3.4}$$

Now, let $X \in \Sigma_{\alpha\beta\gamma}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$ and $D \in \{A'^* X + XA' + Q' - XG'X : A' \in \mathbf{A}^{\forall}, G' \in \mathbf{G}^{\forall}, Q' \in \mathbf{Q}^{\forall}\}$, so one can find some matrices like $A'_1 \in \mathbf{A}^{\forall}, G'_1 \in \mathbf{G}^{\forall}, Q'_1 \in \mathbf{Q}^{\forall}$ for which $D = A'^*_1 X + XA'_1 + Q'_1 - XG'_1 X$. Now, (3.3) implies that for adequate matrices $A''_1 \in \mathbf{A}^{\exists}, G''_1 \in \mathbf{G}^{\exists}, Q''_1 \in \mathbf{Q}^{\exists}$, $D = -(A''^*_1 X + XA''_1 + Q''_1 - XG''_1 X)$ holds. Thus, (3.2) is achieved. Now, assume that (3.2) holds. According to (3.3), it is sufficient to show that for all $A' \in \mathbf{A}^{\forall}, G' \in \mathbf{G}^{\forall}, Q' \in \mathbf{Q}^{\forall}$, there exist $A'' \in \mathbf{A}^{\exists}, G'' \in \mathbf{G}^{\exists}, Q'' \in \mathbf{Q}^{\exists}$ such that

$$(A' + A'')^* X + X(A' + A'') + (Q' + Q'') - X(G' + G'')X = 0.$$

Thus (3.2) yields that for these fixed $A' \in \mathbf{A}^{\forall}, G' \in \mathbf{G}^{\forall}, Q' \in \mathbf{Q}^{\forall}$ we should have $A'' \in \mathbf{A}^{\exists}, G'' \in \mathbf{G}^{\exists}, Q'' \in \mathbf{Q}^{\exists}$ for which (3.4) is true.

Remark 35 *Only the weak equality*

$$\mathbf{D}Y = \Box\{DY \ : \ D \in \mathbf{D}\} \tag{3.5}$$

is valid for any $\mathbf{D} \in \mathbb{IK}^{n\times k}$ *and any* $Y \in \mathbb{K}^{k\times m}$ *unless when* $m = 1$ *[21, Proposition 3.1.4]. Therefore, we have*

$$\begin{aligned} &\mathbf{A}^*X + X\mathbf{A} + \mathbf{Q} - X\mathbf{G}X \supseteq \\ &\Box\left\{ A^*X + XA + Q - XGX = 0 : A \in \mathbf{A}, G \in \mathbf{G}, Q \in \mathbf{Q} \right\}. \end{aligned} \tag{3.6}$$

Indeed, (3.6) *is a special case of the definition of the interval arithmetic operations.*

Two characterizations of some solution sets are provided by Theorem 36 and Theorem 37.

Theorem 36 *If* $X \in \Sigma_{\exists\exists\gamma}(\mathit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$ *then*

$$-\mathbf{Q}^{\forall} \subseteq \mathbf{A}^*X + X\mathbf{A} + \mathbf{Q}^{\exists} - X\mathbf{G}X. \tag{3.7}$$

Proof Let $X \in \Sigma_{\exists\exists\gamma}(\mathtt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$. Then, (3.2) leads to

$$\{Q' : Q' \in \mathbf{Q}^{\forall}\} \subseteq \{-(A''^*X + XA'' + Q'' - XG''X) : A'' \in \mathbf{A}, G'' \in \mathbf{G}, Q'' \in \mathbf{Q}^{\exists}\}.$$

Moreover, Lemma 24 part 1 and (3.6) yields

$$\begin{aligned} &\mathbf{Q}^{\forall} = \Box\{Q' : Q' \in \mathbf{Q}^{\forall}\} \subseteq \\ &\Box\{-(A''^*X + XA'' + Q'' - XG''X) : A'' \in \mathbf{A}, G'' \in \mathbf{G}, Q'' \in \mathbf{Q}^{\exists}\}. \end{aligned}$$

Similarly to (3.6) we can deduce

$$\mathbf{Q}^{\forall} \subseteq -(\mathbf{A}^*X + X\mathbf{A} + \mathbf{Q}^{\exists} - X\mathbf{G}X).$$

Besides, for any two interval(s) (quantities) $\mathbf{x}$ and $\mathbf{y}$ we have $\mathbf{x} \subseteq \mathbf{y} \Longleftrightarrow -\mathbf{x} \subseteq -\mathbf{y}$ [4]. Therefore, we conclude that $-\mathbf{Q}^{\forall} \subseteq \mathbf{A}^*X + X\mathbf{A} + \mathbf{Q}^{\exists} - X\mathbf{G}X$.

Theorem 37 *If*

$$\mathbf{A}^*X + X\mathbf{A} + \mathbf{Q}^{\forall} - X\mathbf{G}X \subseteq -\mathbf{Q}^{\exists}, \tag{3.8}$$

then $X \in \Sigma_{\forall\forall\gamma}(\mathit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$.

Proof Suppose (3.8) holds. Since

$$\begin{aligned} &\{A^*X + XA + Q' - XGX \mid A \in \mathbf{A}, G \in \mathbf{G}, Q' \in \mathbf{Q}^{\forall}\} \subseteq \\ &\Box\{A^*X + XA + Q' - XGX \mid A \in \mathbf{A}, G \in \mathbf{G}, Q' \in \mathbf{Q}^{\forall}\}, \end{aligned}$$

we can write

$$\begin{aligned} &\Box\{A^*X + XA + Q' - XGX : A \in \mathbf{A}, G \in \mathbf{G}, Q' \in \mathbf{Q}^{\forall}\} \subseteq \\ &\mathbf{A}^*X + X\mathbf{A} + \mathbf{Q}^{\forall} - X\mathbf{G}X \subseteq -\mathbf{Q}^{\exists} = \{-Q'' : Q'' \in \mathbf{Q}^{\exists}\}. \end{aligned}$$

We have thus proved

$$\{A^*X + XA + Q' - XGX : A \in \mathbf{A}, G \in \mathbf{G}, Q' \in \mathbf{Q}^{\forall}\} \subseteq \{-Q'' : Q'' \in \mathbf{Q}^{\exists}\}.$$

That is enough to put $\alpha = \beta = \forall$ and $\mathbf{A}^{\exists} = \mathbf{G}^{\exists} = 0$ and now Theorem 34 concludes $X \in \Sigma_{\forall\forall\gamma}(\mathtt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$.

New characterizations of the AE-solution sets $\Sigma_{\exists\exists\gamma}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$ and $\Sigma_{\forall\forall\gamma}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$ in terms of midpoint and radius matrices are our next conclusions.

Theorem 38

$$\begin{aligned}&\Sigma_{\exists\exists\gamma}(\mathit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) \subseteq\\ &\{X \in \mathbb{K}^{n\times n} : |\operatorname{mid}(\mathbf{A})^* X + X \operatorname{mid}(\mathbf{A}) - X \operatorname{mid}(\mathbf{G}) X + \operatorname{mid}(\mathbf{Q})| \leq\\ &\operatorname{rad}(\mathbf{A})^*|X| + |X| \operatorname{rad}(\mathbf{A}) - |X| \operatorname{rad}(\mathbf{G})|X| + \operatorname{rad}(\mathbf{Q}^{\exists}) - \operatorname{rad}(\mathbf{Q}^{\forall})\}.\end{aligned}$$

Proof Suppose $X \in \Sigma_{\exists\exists\gamma}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$. Then, Theorem 36 together with Lemma 23 part 1 follows

$$\begin{aligned}&|\operatorname{mid}(\mathbf{A}^* X + X\mathbf{A} + \mathbf{Q}^{\exists} - X\mathbf{G}X) - \operatorname{mid}(-\mathbf{Q}^{\forall})| \leqslant\\ &\operatorname{rad}(\mathbf{A}^* X + X\mathbf{A} + \mathbf{Q}^{\exists} - X\mathbf{G}X) - \operatorname{rad}(-\mathbf{Q}^{\forall}).\end{aligned}$$

By repeatedly using Lemma 23 and considering $|X^2| \leqslant |X|^2$, we attain

$$\begin{aligned}&|\operatorname{mid}(\mathbf{A})X + X \operatorname{mid}(\mathbf{A}) - X \operatorname{mid}(\mathbf{G})X + \operatorname{mid}(\mathbf{Q}^{\exists}) + \operatorname{mid}(\mathbf{Q}^{\forall})| \leq\\ &\operatorname{rad}(\mathbf{A})^*|X| + |X| \operatorname{rad}(\mathbf{A}) - |X| \operatorname{rad}(\mathbf{G})|X| + \operatorname{rad}(\mathbf{Q}^{\exists}) - \operatorname{rad}(\mathbf{Q}^{\forall}).\end{aligned}$$

Because of $\operatorname{mid}(\mathbf{Q}^{\exists}) + \operatorname{mid}(\mathbf{Q}^{\forall}) = \operatorname{mid}(\mathbf{Q}^{\exists} + \mathbf{Q}^{\forall}) = \operatorname{mid}(\mathbf{Q})$ the proof is completed.

Theorem 39 *If*

$$\begin{aligned}&|\operatorname{mid}(\mathbf{A})^* X + X \operatorname{mid}(\mathbf{A}) - X \operatorname{mid}(\mathbf{G}) X + \operatorname{mid}(\mathbf{Q})| \leq\\ &\operatorname{rad}(\mathbf{Q}^{\exists}) - \operatorname{rad}(\mathbf{Q}^{\forall}) - \operatorname{rad}(\mathbf{A})^*|X| - |X| \operatorname{rad}(\mathbf{A}) - |X| \operatorname{rad}(\mathbf{G})|X|,\end{aligned} \tag{3.9}$$

then $X \in \Sigma_{\forall\forall\gamma}(\mathit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$.

Proof In view of $\mathbf{Q} = \mathbf{Q}^{\exists} + \mathbf{Q}^{\forall}$, (3.9) is equivalent to

$$\begin{aligned}&|\operatorname{mid}(\mathbf{A}^* X + X\mathbf{A} + \mathbf{Q}^{\forall} - X\mathbf{G}X) - \operatorname{mid}(-\mathbf{Q}^{\exists})| \leq\\ &\operatorname{rad}(-\mathbf{Q}^{\exists}) - \operatorname{rad}(\mathbf{A}^* X + X\mathbf{A} + \mathbf{Q}^{\forall} - X\mathbf{G}X).\end{aligned}$$

Now, Lemma 23 part 1 yields $\mathbf{A}^* X + X\mathbf{A} + \mathbf{Q}^{\forall} - X\mathbf{G}X \subseteq -\mathbf{Q}^{\exists}$ and then Theorem 37 implies that $X \in \Sigma_{\forall\forall\gamma}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$.

4 The United Solution Set of ICARE: characterization and estimation

Apart from the many applications in the so-called scientific computation fields [3], the united solution set is the most common and most straightforward method to define a solution set to an interval equation. So, special attention to this set is not unexpected.

Theorem 41

$$\Sigma_{unt}(\mathit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) \subseteq \{\, X \in \mathbb{K}^{n\times n} : (\mathbf{A}^* X + X\mathbf{A} - X\mathbf{G}X) \cap (-\mathbf{Q}) \neq \emptyset \,\},$$

and

$$\Sigma_{unt}(\mathit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) \subseteq \{\, X \in \mathbb{K}^{n\times n} : 0 \in \mathbf{A}^* X + X\mathbf{A} - X\mathbf{G}X + \mathbf{Q} \,\}.$$

Moreover,

$$\begin{aligned} &X \in \Sigma_{unt}(\mathit{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) \Rightarrow \\ &|\operatorname{mid}(\mathbf{A})^* X + X \operatorname{mid}(\mathbf{A}) - X \operatorname{mid}(\mathbf{G}) X + \operatorname{mid}(\mathbf{Q})| \\ &\leq \operatorname{rad}(\mathbf{A})^* |X| + |X| \operatorname{rad}(\mathbf{A}) - |X| \operatorname{rad}(\mathbf{G}) |X| + \operatorname{rad}(\mathbf{Q}). \end{aligned} \tag{4.1}$$

Proof Suppose $X \in \Sigma_{unt}(\mathtt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$. Therefore for some $A \in \mathbf{A}, G \in \mathbf{G}$ and $Q \in \mathbf{Q}$, the equality $A^*X + XA - XGX = -Q$ holds. Hence, $-Q \in (\mathbf{A}^*X + X\mathbf{A} - X\mathbf{G}X) \cap (-\mathbf{Q})$. On the other hand, $\mathbf{A}^*X + X\mathbf{A} - X\mathbf{G}X \cap (-\mathbf{Q}) \neq \emptyset$ implies $0 \in \mathbf{A}^*X + X\mathbf{A} + \mathbf{Q} - X\mathbf{G}X$ and the first two results are found. (4.1) is also an immediate result of Theorem 38.

Recall that a Hermitian solution X_s (X_a) of CARE (1.1) is called *stabilizing* (resp. *anti-stabilizing*) if all the eigenvalues of the closed loop matrix $A - GX_s$ (resp. $A - GX_a$) have negative (resp. positive) real parts or $A - GX_s$ be *Hurwitz stable*. Such solutions play important roles in applications, therefore conditions that guarantee existence and/or uniqueness of stabilizing and anti-stabilizing solutions are of considerable interest, see for instance [7]. So, in CAREs the interesting solutions are the stabilizing and anti-stabilizing ones. Therefore, it is reasonable to define the *united stable solution set* as

$$\begin{aligned} &\Sigma_{unts}(\mathtt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q}) := \\ &\{X \in \mathbb{K}^{n\times n} | ((\exists A \in \mathbf{A})(\exists G \in \mathbf{G})(\exists Q \in \mathbf{Q}) \\ &A^*X + XA + Q = XGX \text{ and } A - GX \text{ is Hurwitz stable})\}. \end{aligned} \tag{4.2}$$

Note that the techniques presented in this paper can be adapted with minor sign changes to anti-stabilizing solutions.

Theorem 42 *A necessary and sufficient condition for the interval continuous-time algebraic Riccati equation* (1.2) *to have a non-empty united stable solution set for all* $\mathbf{Q} \in \mathbb{IC}^{n\times n}$ *is that* $\sigma(\mathbf{A} - \mathbf{G}X) \cap I = \emptyset$ *where* I *denotes the imaginary axis,* $\sigma(\mathbf{A} - \mathbf{G}X) := \{\lambda(A - GX) : \lambda$ *is an eigenvalue for* $A - GX, A \in \mathbf{A}, G \in \mathbf{G}\}$ *denotes the interval spectrum of* $\mathbf{A} - \mathbf{G}X$ *and* $X^* = X$.

Proof First, let us fix $Q_1 \in \mathbf{Q}$ and its corresponding $A_1 \in \mathbf{A}$ and $G_1 \in \mathbf{G}$. Recall the conditions of existence and uniqueness of stabilizing solutions which are given in terms of spectral properties of associated Hamiltonian matrix $A_1 - G_1X$ [7]: "X_s is the stabilizing solution for $A_1^*X + XA_1 + Q_1 - XG_1X = 0$ if and only if $A_1 - G_1X_s$ has no imaginary eigenvalue". This completes the proof.

From this point on, let us suppose that the united solution sets are nonempty and bounded. Our goal in Section 4.1 is to find a *verified* outer estimation (with the narrowest possible interval components) for the united stable solution set because $\Sigma_{unts}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$ is generally not an interval matrix.

4.1 A variation of the Krawczyk method for enclosing $\Sigma_{unts}(\texttt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$

In nonlinear analysis, the Brouwer fixed point theorem is a well-known theorem [22]:

Theorem 43 *Let $\Phi \subseteq \mathbb{R}^N$ be a convex compact set. If $h : \Phi \to \mathbb{R}^N$ is a continuous function that maps Φ into itself, i.e.,*

$$h(\Phi) \subseteq \Phi, \tag{4.3}$$

then h has a fixed point x_* on Φ.

One can skirts the difficulties arising in the practical verification of crucial inclusion (4.3) if the tools for computing interval extensions of functions are available. Furthermore, we should restrict ourselves to consider the domain in the form of interval boxes, that is, requiring $\Phi \in \mathbb{IC}^N$ and also, we should change the exact range of values of the function h over Φ to its outer estimate through interval extension.

Thus, one can derive the solution existence test proposed by Krawczyk which is very popular in the modern interval analysis [10, 13]. The classic Krawczyk method is very costly in verifying CAREs, see again [10, 13] to discover its cost which is at least $\mathcal{O}(n^5)$. It can be modified as follows:

Theorem 44 *(see e.g. [13]) Assume that $h : \Psi \subset \mathbb{C}^N \to \mathbb{C}^N$ is continuous. Let $R \in \mathbb{C}^{N\times N}, \check{x} \in \Psi$ and $\mathbf{z} \in \mathbb{IC}^N$ be such that $\check{x} + \mathbf{z} \subset \Psi$. Moreover, assume that $\mathcal{S} \subset \mathbb{C}^{N\times N}$ is a set of matrices such that the slopes $S(\check{x}, x')$ belong to $\mathcal{S}$ for every $x' \in \check{x} + \mathbf{z} =: \mathbf{x}$. If*

$$\mathcal{K}_h(\check{x}, R, \mathbf{z}, \mathcal{S}) := \{-Rh(\check{x}) + (I_N - RS)z : S \in \mathcal{S}, z \in \mathbf{z}\} \subseteq \operatorname{int}(\mathbf{z}), \tag{4.4}$$

then the function h has a zero x_* in $\check{x} + \mathcal{K}_h(\check{x}, R, \mathbf{z}, \mathcal{S}) \subseteq \mathbf{x}$. Moreover, if $S(y, y') \in \mathcal{S}$ for each $y, y' \in \mathbf{x}$, then x_* is the only zero of h contained in $\mathbf{x}$.

Note that a *slope* S of the function $h : \Psi \subseteq \mathbb{C}^N \to \mathbb{C}^N$ at $(x, y) \in \mathbb{C}^N \times \mathbb{C}^N$ is a mapping from $\Psi \times \Psi$ to $\mathbb{C}^{N\times N}$ such that $h(x) - h(y) = S(x, y)(x - y)$.

Now, assume that

$$F(X) := A^*X + XA + Q - XGX, \quad A \in \mathbf{A}, G \in \mathbf{G}, Q \in \mathbf{Q}, \tag{4.5}$$

and let the point matrix $\operatorname{mid}(\mathbf{A} - \mathbf{G}\check{X})$ has the following computed spectral decomposition:

$$\operatorname{mid}(\mathbf{A}-\mathbf{G}\check{X}) \approx V\Lambda W,\ V, \Lambda, W \in \mathbb{C}^{n\times n},\ \Lambda = \operatorname{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), W \approx V^{-1}.$$

Recall that a matrix function $H : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ is said to be Fréchet differentiable at X_0 if there is some linear transformation $L_H : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ such that for all $E \in \mathbb{C}^{N\times N}$,

$$H(X_0 + E) - H(X_0) - L_H(X_0, E) = o(\|E\|).$$

Also, the unique matrix $K_H(X_0) \in \mathbb{C}^{N^2\times N^2}$ is called the *Kronecker matrix form* of the Fréchet derivative of H at X_0 whenever for any matrix $E \in \mathbb{C}^{N\times N}$, $\mathrm{vec}\,(L_H(A, E)) = K_H(A)\,\mathrm{vec}\,(E)$.

When F is as in (4.5) and $N = n$ then Lemma 21 part 5 turns out that

$$K_F(X) = I_n \otimes (A - GX)^* + (A - GX)^T \otimes I_n,\ X = X^*. \tag{4.6}$$

By the inclusion property of interval arithmetics, $\mathbf{k}_f(\check{x}, R, \mathbf{z}, \mathbf{S}) := -Rf(\check{x}) + (I_N - R\mathbf{S})\mathbf{z}$ is a superset of $\mathcal{K}_f(\check{x}, R, \mathbf{z}, \mathcal{S})$. So, if $\mathbf{k}_f(\check{x}, R, \mathbf{z}, \mathbf{S}) \subset \mathrm{int}(\mathbf{z})$ holds then $\mathcal{K}_f(\check{x}, R, \mathbf{z}, \mathcal{S}) \subseteq \mathrm{int}(\mathbf{z})$ will result.

The typical choice so that the relation (4.4) is likely to hold is taking as $\check{x}$ a good approximation of a zero of f and as R a good approximation of $(K_F(\check{x}))^{-1}$. The following result shows that the Kronecker matrix form of the Fréchet derivative can be used to find an enclosure for the slope in the modified Krawczyk method. We report this conclusion from [13].

Theorem 45 *[13, Theorem 3.8] Let $\mathbf{X}$ be an interval complex matrix, and $\check{X} \in \mathbf{X}$ be Hermitian. Then, the interval matrix $I_n \otimes (A - G\check{X})^* + (A - G\mathbf{X})^T \otimes I_n$ contains the slopes $S(\check{x}, x')$ for each $X' \in \mathbf{X}$ where $\check{x} = \mathrm{vec}\,(\check{x})$ and $x' = \mathrm{vec}\,(X')$.*

So, the next ingredient in applying Krawczyk algorithm, R, can be chosen as

$$R = (V^{-T} \otimes W^*)\Delta^{-1}(V^T \otimes W^{-*}), \quad \Delta := I_n \otimes \Lambda^* + \Lambda^T \otimes I_n,$$

provided that Δ, V and W be invertible. The reason for this choice is that we can factorize $K_F(\check{X})$ by using Lemma 21 so that

$$\begin{aligned} K_F(\check{X}) &= I_n \otimes (A - G\check{X})^* + (A - G\check{X})^T \otimes I_n \\ &= (V^{-T} \otimes W^*)(I_n \otimes (W(A - G\check{X})W^{-1})^* \\ &\quad + (V^{-1}(A - G\check{X})V)^T \otimes I_n)(V^T \otimes W^{-*}), \end{aligned}$$

and therefore $R \approx (K_F(\check{X}))^{-1}$.

Then, the computation of two enclosures, i.e., $\mathrm{vec}(\mathbf{L})$ for $l := -Rf(\check{x})$ and $\mathrm{vec}(\mathbf{U})$ for $u := (I_{n^2} - RS)z$ in $\mathcal{K}_f(\check{x}, R, \mathbf{z}, \mathcal{S})$ can be done using exclusively the matrix-matrix operations, as shown in Lines 1–8 of Algorithm 1 for $\mathrm{vec}\,(\mathbf{L})$ and Lines 10–17 of Algorithm 1 for $\mathrm{vec}\,(\mathbf{U})$. More details for computing these supersets are:

$$\begin{aligned} l &:= -Rf(\check{x}) \\ &= -(V^{-T} \otimes W^*)\Delta^{-1}(V^T \otimes W^{-*})f(\check{x}), \end{aligned}$$

and

$$\begin{aligned} u := (I_{n^2} - RS)z = (I_{n^2} - (V^{-T} \otimes W^*)\Delta^{-1}(V^T \otimes W^{-*}) \\ (I_n \otimes (A - GY)^* + (A - GY^{'})^T \otimes I_n))z \\ = ((V^{-T} \otimes W^*)\Delta^{-1}(\Delta - I_n \otimes (W(A - GY)W^{-1})^* \\ - (V^{-1}(A - GY^{'})V)^T \otimes I_n)(V^T \otimes W^{-*}))z. \end{aligned}$$

The standard method to find an interval vector $\mathbf{z} = \mathrm{vec}(\mathbf{Z})$ that satisfies $\mathbf{k}_f(\check{x}, R, \mathbf{z}, \mathbf{S}) \subset \mathrm{int}(\mathbf{z})$ is an iterative one based on the known ε-*inflation* technique [24]. Indeed, we start from the residual matrix $\mathbf{Z}_0 := \mathbf{F}(\check{\mathbf{X}})$, and proceed alternating successive steps of enlarging this interval with ε-inflation technique.

Note that the evaluation order of an expression in Algorithm 1 is always left to right.

Algorithm 1 A modified Krawczyk algorithm to efficiently compute an interval matrix $\mathbf{X}$ containing $\Sigma_{unts}(\mathtt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$.

Require: Interval matrices $\mathbf{A}, \mathbf{G}$ and $\mathbf{Q}$
Ensure: If successful, this algorithm provides an interval matrix $\mathbf{X}$ containing $\Sigma_{unts}(\mathtt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$ else it reports "**failure**"
Compute with floating point arithmetic a good approximate stabilizing solution $\check{X}$ of CARE $(\mathrm{mid}\,(\mathbf{A}))^* X + X\,\mathrm{mid}\,(\mathbf{A}) + \mathrm{mid}\,(\mathbf{Q}) - X\,\mathrm{mid}\,(\mathbf{G})X = 0$
Compute approximations V, W and Λ for the spectral decomposition of $\mathrm{mid}\,(\mathbf{A} - \mathbf{G}\check{X})$ in floating point
Compute $D := (D_{ij})$ such that $D_{ij} \approx (\bar{\Lambda})_{ii} + (\Lambda)_{jj}$ in floating point
Compute interval matrices $\mathbf{I}_V$ and $\mathbf{I}_W$ containing V^{-1} and W^{-1}, respectively. Return **failure** if this fails, or if D has any zero elements.
$\check{\mathbf{X}} = \langle \check{X}, 0 \rangle$ {To ensure that all operations involving $\check{X}$ are *verified*}
$\mathbf{F} = \mathbf{Q} + \check{\mathbf{X}}\mathbf{A} + (\mathbf{A}^* - \check{\mathbf{X}}\mathbf{G})\check{\mathbf{X}}$ {Gathering $\check{\mathbf{X}}$ in order to reduce the wrapping effects}
$\mathbf{H} = \mathbf{I}_W^* \mathbf{F} V$
$\mathbf{J} = \mathbf{H}./D$
$\mathbf{L} = -W^* \mathbf{J} \mathbf{I}_V$
$\mathbf{Z} = \mathbf{L}$
for $k = 1, 2, \ldots, 10$ **do**
  Set $\mathbf{Z} = \square(0, \mathbf{Z} \cdot \langle 1, 0.1 \rangle + \langle 0, \mathtt{realmin} \rangle)$
  $\mathbf{M} = \mathbf{I}_W^* \mathbf{Z} V$
  $\mathbf{N} = W(A - G\check{X})\mathbf{I}_W$
  $\mathbf{O} = \mathbf{I}_V(A - G(\check{X} + \mathbf{Z}))V$
  $\mathbf{P} = (\Lambda - \mathbf{N})^* \mathbf{M} + \mathbf{M}(\Lambda - \mathbf{O})$
  $\mathbf{Q} = \mathbf{P}./D$
  $\mathbf{U} = W^* \mathbf{Q} \mathbf{I}_V$
  $\mathbf{K} = \mathbf{L} + \mathbf{U}$
  if $\mathbf{K} \subset \mathrm{int}(\mathbf{Z})$ **then**
    Return $\mathbf{X} = \check{X} + \mathbf{K}$
  end if
  $\mathbf{Z} = \mathbf{K}$
end for
Return **failure** {Maximum number of iterations reached}

Theorem 46 *Algorithm 1 requires at most $\mathcal{O}(n^3 \mathtt{s})$ arithmetic operations if the verification succeeds in* `s` *steps.*

Proof Computing $\check{X}$ in Line 1 using, for instance, ordered Schur method followed by one step of Newton refinement in simulated quadruple precision; computing the eigendecompositions in Lines 2 using, for example, the MATLAB command `eig` and also computing the interval matrices $\mathbf{I}_V$ and $\mathbf{I}_W$ using, for instance, `verifylss.m` from INTLAB need $\mathcal{O}(n^3)$ operations. In addition, all the other operations involve only $n \times n$ matrices so they cost again $\mathcal{O}(n^3)$, at most.

As we will show in the examples, our algorithm either terminated after 1 step or failed. So in practice the number of steps can be kept very small.

5 Numerical Examples

In this part, Algorithm 1 is tested in MATLAB 2013a with INTLAB v6 and run on a laptop with 1GB of RAM.

Indeed, the solutions of (1.1) can be put in one-to-one correspondence with certain invariant subspaces of the Hamiltonian matrix

$$H := \begin{bmatrix} A & -G \\ -Q & -A^* \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^* \\ -C^*\tilde{Q}C & -A^* \end{bmatrix} \in \mathbb{C}^{2n\times 2n},$$
$$C \in \mathbb{C}^{p\times n}, B \in \mathbb{C}^{n\times m}, \tilde{Q}^* = \tilde{Q} \in \mathbb{C}^{p\times p}, R^* = R \in \mathbb{C}^{m\times m},$$

that is, X is a solution of (1.1) if and only if

$$H \begin{bmatrix} I_n \\ X \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix} (A - GX).$$

In particular, the columns of the matrix $\left[\begin{smallmatrix} I_n \\ X \end{smallmatrix}\right]$ span an invariant subspace for the matrix H and the eigenvalues of $A - GX$ are a subset of the eigenvalues of H [7]. We refer the reader to the books by Lancaster and Rodman [16] and by Bini, Iannazzo and Meini [7] for details concerning main theoretical properties and numerical solutions.

The coefficient matrices in [6] often come from linear-quadratic control problems:
Minimize

$$J(x_0, u) = \frac{1}{2} \int_0^{+\infty} \big(y(t)^* \tilde{Q} y(t) + u(t)^* R u(t)\big) dt$$

subject to the dynamics

$$\begin{aligned} &\dot{x} = Ax(t) + Bu(t),\ x(0) = x_0, \\ &y(t) = Cx(t). \end{aligned}$$

Following MATLAB commands have been utilized in order to convert the point matrix coefficients to interval ones: the center matrices are just those given in the benchmark [6] and their radius matrices are achieved by the MATLAB commands

```
[A, G, Q] = ChuLM07Carex(i);
IA = midrad(A,1e-5*mig(A));
IG = midrad(G,1e-5*mig(G));
IQ = midrad(Q,1e-5*mig(Q));
```

where IA, IG and IQ are equal to the interval matrices $\mathbf{A}, \mathbf{G}$ and $\mathbf{Q}$, respectively. Note that the radius of disturbance is a positive multiple of $1e-5$ for all matrix coefficients.

The approximate solution of CARE associated to the left half plane required in Line 1 of Algorithm 1 is obtained using the method described in [20] which is the ordered Schur method followed by one step of Newton refinement in simulated quadruple precision.

In all successful examples, we report the maximum radius of the entries of $\mathbf{X}$, `mr`, to show the quality of the resulted enclosure $\mathbf{X}$; the number of iterations executed in Algorithm 1 for the Krawczyk loop, `s`, when $\mathtt{s}_{\max} = 10$ and the total time in seconds, `Time`.

Note that when enclosing $\Sigma_{unts}(\mathtt{CARE}; \mathbf{A}, \mathbf{G}, \mathbf{Q})$, we are seeking an inclusion $\mathbf{X}$ as tight as possible, hence a small `mr`.

We also examine whether $\mathbf{A} - \mathbf{GX}$ is Hurwitz stable using Algorithm 7 in [13] for any successful test. A number 1 in the corresponding `Stability` column confirms the stability property of all solutions contained in the result enclosure $\mathbf{X}$, and a – means that the algorithm had already failed to compute an inclusion.

Below, we have chosen one example from each part of [6] for testing.

Example 51 *[6, Example 5] The first experiment is a parameter-free example of fixed dimension in the benchmark collection for CAREs. Indeed, this is a* 9*th-order continuous state space model of a tabular ammonia reactor. As noted in [6], the underlying model includes a disturbance term which is neglected in [6]. An outer estimation* $\mathbf{X}$ *for the united stable solution set of corresponding ICARE* $\mathbf{A}^*X + X\mathbf{A} + \mathbf{Q} - X\mathbf{G}X = 0$ *will obtain after only one iteration within* 0.060780 *seconds. Also,* $\mathbf{X}$ *has a maximum component-wise radius of 0.0093. The result of stabilizing property is positive, too.*

Example 52 *[6, Example 11] The second example presents a type of algebraic Riccati equation arising in* H_∞*-control problem. This is a parameter-depend problem but of fixed size* 2 *in which the first parameter, i.e.,* ε *is* 1*. Again, assume that there is a perturbation such that one should solve the corresponding ICARE. The maximum component-wise radius of the enclosure* $\mathbf{X}$ *obtained via Algorithm 1 is 3.2631e-04, the total elapsed time is 0.077889 seconds while* `s` *and* `Stability` *are* 1*. For the second parameter, i.e.,* $\varepsilon = 0$ *the verification failed to find an inclusion.*

Example 53 *[6, Example 16] This example is suited to test the behavior of any algorithm for growing problem size. The results are presented at Table 1. As one can see, either Algorithm 1 succeeds after just one step to create an enclosure or it fails (when the size of this example changes from* 10 *to* 1000*). If successful, the verification of the stabilizing property of computed inclusion is also successful.*

Table 1 Results for Example 53

`Problem size`	`mr`	`s`	`Stability`	`Time`
1	4.1215e-05	1	1	0.054154
2	1.7350e-05	1	1	0.055689
3	6.1741e-05	1	1	0.054423
4	5.4894e-05	1	1	0.056061
5	9.9766e-05	1	1	0.056438
6	1.2608e-04	1	1	0.056713
7	1.4439e-04	1	1	0.057836
8	1.5112e-04	1	1	0.058185
9	3.1715e-04	1	1	0.076302
10 : 1000	-	-	-	-

Example 54 *The data of this example come from a system of n integrators connected in series and a feedback controller is supposed to be applied to the n-th system. This is a scalable example with parameters in which the eigenvalues of the Hamiltonian matrix H are the roots of $\lambda^{2n} + (-1)^n qr = 0$, where $\tilde{Q} = q$ and $R = r, q, r \in \mathbb{R}$.*

Table 2 Results for Example 54

`Parameters:`$[n, q, r]$	`mr`	`s`	`Stability`	`Time`
[2,0.001,0.001]	1.4111e-07	1	1	0.078590
[2,0.01,0.01]	1.4111e-06	1	1	0.079697
[2,0.1,0.1]	1.4111e-05	1	1	0.078945
[2,1,1]	1.4111e-04	1	1	0.079624
[2,10,10]	0.0014	1	1	0.078989
[2,100,100]	0.0141	1	1	0.079296
[2,1000,1000]	0.1411	1	1	0.079720

As one can see in all experiments, `mr` and the parameters q and r have changed with a same ratio. In addition, changing these two parameters does not affect the total time, s and `Stability`.

6 Summary

In this note, we first generalized the concept of AE-solution sets to the interval continuous-time algebraic Riccati equation $\mathbf{A}^* X + X\mathbf{A} + \mathbf{Q} - X\mathbf{G}X = 0$.

Then, we proved some analytical results for these solution sets. We then tried to explore a modification of Krawczyk method in order to reduce the computational complexity of obtaining a verified outer estimation for the united stable solution set to cubic, provided that the midpoint of $\mathbf{A} - \mathbf{G}X$ is diagonalizable. One important ingredient is also the interval techniques used to verify. Furthermore, Algorithm 1 uses exclusively matrix-matrix operations and are thus efficient with current implementations of machine interval arithmetic in INTLAB.

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