

Yet Another Application of the Theory of ODE in the Theory of Vector Fields

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Abstract In this paper we are supposed to define the θ -vector field on the n -surface S and then investigate about the existence and uniqueness of its integral curves by the Theory of Ordinary Differential Equations. Then the subject is followed through some examples.

Keywords Methods of ordinary differential equations · Location of integral curves · Surfaces of general type

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1 Introduction

Various generalizations of the existence and uniqueness theorems of the integral curves of smooth vector fields have been proposed as well as improvement of the sets that vector fields are defined on them. For example, if $U \subseteq R^n$ is an open subset and $f : U \rightarrow R^{n+1}$ is locally Lipschitz, that is around each point there is a ball on which f is bounded and satisfies Lipschitz condition, then for each $x \in U$ there is a unique $\alpha : I \rightarrow U$ for some interval I containing 0 such that $\alpha'(t) = f(\alpha(t))$ and $\alpha(0) = x$ [4].

If S is a regular surface and $U \subseteq S$ is an open set and χ is a smooth tangent vector field defined on U , then the integral curve, and the similar theorem above extend to the present situation; up to change of R^{n+1} by S [1].

Smooth vector fields on n -dimensional Manifolds are discussed in [2]. The

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straightforward conclusion of this subject which is a consequence of the similar results in R^{n+1} is the existence and uniqueness theorem for the integral curves defined on manifolds.

What is still missing is the concept and properties of a θ -vector field defined on an n -surface, that is, how the integral curve of a given θ -vector field must be related to the S , or what is the differential equation which this kind of vector field must be satisfied in it.

In this paper, we propose a definition of an n -surface S , and a smooth θ -vector field on S , as a generalization of the concepts within the contexts of M. do Carmo and M. Spivak. In the second section, we recall various definitions and theorems related to this subject which will be used later. In the third section, the basic theorems are proved, and at last some examples and conclusions are given respectively in Secs. 4 and 5.

2 Preliminaries

Definition 1 A parameterized curve is a smooth function $\alpha : I \rightarrow R^{n+1}$ for some open interval I , and an n -surface is a non empty subset S of R^{n+1} for some $n \in N$ of the form $S = f^{-1}(c)$ where $f : U \rightarrow R$, U open in R^{n+1} , is a smooth function with the property that $\nabla f(p) \neq 0$ for all $p \in S$.

Definition 2 A vector field χ on an n -surface $S \subseteq R^{n+1}$ is a function which assign to each point $p \in S$ a vector $\chi(p) \subseteq R_p^{n+1}$.

Definition 3 If $\cos \angle(\chi(p), \nabla f(p)) = \sin \theta$ for some $0 \leq \theta \leq \frac{\pi}{2}$ and each $p \in S$, then χ is said a θ -vector field on S .

Definition 4 Let S be an n -surface in R^{n+1} . A function $g : S \rightarrow R^k$ is smooth if it is the restriction to S of a smooth function $\tilde{g} : V \rightarrow R^k$ defined on some open set V consisting S . A smooth vector field is defined similarly.

Theorem 1 Let χ be a smooth vector field on an open set $U \subseteq R^{n+1}$ and let $p \in U$. Then there exists an open interval I containing 0 and a parameterized curve $\alpha : I \rightarrow U$ such that,

1. $\alpha(0) = p$,
2. $\dot{\alpha}(t) = \chi(\alpha(t))$ for all $t \in I$,
3. If $\beta : \tilde{I} \rightarrow U$ is any other parameterized curve in U satisfying 1 and 2, then $\tilde{I} \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

3 Main Results

Theorem 2 Let S be an n -surface, χ be a smooth θ -vector field on S and $p \in S$. Then there exists an open interval I containing 0 and a parameterized

curve $\alpha : I \rightarrow R^{n+1}$ such that $\alpha(0) = p$, $\dot{\alpha}(t) = \chi(\alpha(t))$ for all t such that $\alpha(t) \in S$ and

$$f(\alpha(t)) - f(p) = \sin\theta \int_0^t \|\nabla f(\alpha(t))\| \|\chi(\alpha(t))\| dt \quad (1)$$

for all $t \in I$. Moreover χ has an extension $\tilde{\chi}$ for which α satisfies the following differential equation

$$\dot{\alpha}(t) = \tilde{\chi}(\alpha(t)) - \frac{\tilde{\chi}(\alpha(t)) \cdot \nabla f(\alpha(t))}{\|\nabla f(\alpha(t))\|^2} \nabla f(\alpha(t)) + \sin\theta \frac{\|\tilde{\chi}(\alpha(t))\|}{\|\nabla f(\alpha(t))\|} \nabla f(\alpha(t)) \quad (2)$$

In particular if $\alpha(t) \in S$ then

$$\cos\angle(\dot{\alpha}(t), \nabla f(\alpha(t))) = \sin\theta \quad (3)$$

Proof Since χ is smooth, there exists an open set V containing S and a smooth vector field $\tilde{\chi}$ such that

$$\tilde{\chi}(q) = \chi(q), \forall q \in S \quad (4)$$

Let $f : U \rightarrow R$ and $c \in R$ be such that $S = f^{-1}(c)$, and $\nabla f(q) \neq 0$ for all $q \in U$. Let

$$W = \{q | q \in U \cap V, \nabla f(q) \neq 0\} \quad (5)$$

Then W is an open set containing S and both $\tilde{\chi}$ and f are defined on W . Let Y be the vector field on W defined by

$$Y(q) = \tilde{\chi}(q) - \frac{\tilde{\chi}(q) \cdot \nabla f(q)}{\|\nabla f(q)\|^2} \nabla f(q) + \sin\theta \frac{\|\tilde{\chi}(q)\|}{\|\nabla f(q)\|} \nabla f(q) \quad (6)$$

Then Y is a smooth vector field for which $Y(q) = \tilde{\chi}(q)$ for all $q \in S$ and

$$\nabla f(q) \cdot Y(q) = \sin\theta \|\tilde{\chi}(q)\| \|\nabla f(q)\| \quad (7)$$

for all $q \in W$. According to the Theorem 1, let $\alpha : I \rightarrow W$ be a maximal integral curve of Y through p , i.e., $\alpha(0) = p$, $\dot{\alpha}(t) = Y(\alpha(t))$ for all $t \in I$. Let

$$g(t) = (f \circ \alpha)(t) - \sin\theta \int_0^t \|\nabla f(\alpha(t))\| \|\tilde{\chi}(\alpha(t))\| dt \quad (8)$$

for all $t \in I$, then

$$g'(t) = \nabla f(\alpha(t)) \cdot Y(\alpha(t)) - \sin\theta \|\nabla f(\alpha(t))\| \|\tilde{\chi}(\alpha(t))\| = 0 \quad (9)$$

for all $t \in I$ and

$$g(0) = (f \circ \alpha)(0) = f(p) = c \quad (10)$$

Moreover

$$\begin{aligned} \dot{\alpha}(t) = Y(\alpha(t)) &= \tilde{\chi}(\alpha(t)) - \frac{\tilde{\chi}(\alpha(t)) \cdot \nabla f(\alpha(t))}{\|\nabla f(\alpha(t))\|^2} \nabla f(\alpha(t)) \\ &\quad + \sin\theta \frac{\|\tilde{\chi}(\alpha(t))\|}{\|\nabla f(\alpha(t))\|} \nabla f(\alpha(t)) \end{aligned} \quad (11)$$

Now (4), (7), (9), (10) and (11) imply the Theorem.

Theorem 3 Let S be an n -surface, χ be a smooth tangent vector field on S , and $p \in S$. Then there exists an open interval I containing 0 and a parameterized curve $\alpha : I \rightarrow S$ such that,

1. $\alpha(0) = p$,
2. $\dot{\alpha}(t) = \chi(\alpha(t))$,
3. If $\beta : \tilde{I} \rightarrow S$ is any other parameterized curve in S satisfying $\beta(0) = p$, $\dot{\beta}(t) = \chi(\beta(t))$ for all $t \in \tilde{I}$, then $\tilde{I} \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

Proof Let $\theta = 0$, then χ is a tangent vector field on S and (1) implies that $f \circ \alpha(t) = c$ for all $t \in I$, therefore $\alpha(0) = p$, $\alpha(t) \in S$ and (3) implies that $\dot{\alpha}(t) = \chi(\alpha(t))$ for all $t \in I$. Now let $\beta : \tilde{I} \rightarrow S$ is any other parameterized curve in S satisfying $\beta(0) = p$, $\dot{\beta}(t) = \chi(\beta(t))$ for all $t \in \tilde{I}$ then $\sin\theta = 0$ and β is the integral curve of

$$Y(q) = \chi(q) - \frac{\chi(q) \cdot \nabla f(q)}{\|\nabla f(q)\|^2} \nabla f(q) + \sin\theta \frac{\|\tilde{\chi}(q)\|}{\|\nabla f(q)\|} \nabla f(q) \quad (12)$$

This is a system of $n + 1$ first order ordinary differential equations in $n + 1$ unknowns. So The uniqueness theorem for the solutions of such equations [3], implies the theorem.

Theorem 4 Let $0 \leq \theta \leq \frac{\pi}{2}$, then there exists an n -surface S and a smooth θ -vector field on S .

Proof let \mathcal{C} be the graph of a smooth map $g : R \rightarrow R^+$ and

$$f : R \times R^+ \times \overbrace{R \times \cdots \times R}^{(n-1)\text{times}} \rightarrow R \quad (13)$$

be defined by

$$f(x_1, x_2, \dots, x_{n+1}) = \sqrt{x_2^2 + \cdots + x_{n+1}^2} - g(x_1) \quad (14)$$

Then $S_{\mathcal{C}} = f^{-1}(0)$ is the n -surface of revolution obtained by rotating \mathcal{C} about the x_1 axis. Let $z_i = \frac{x_i}{\sqrt{x_2^2 + \cdots + x_{n+1}^2}}$ for $2 \leq i \leq n + 1$. Define the smooth vector

field χ on $U = R \times R^+ \times \overbrace{R \times \cdots \times R}^{(n-1)\text{times}}$ by

$$\chi(x_1, x_2, \dots, x_{n+1}) = (1, 0, \dots, 0) \quad (15)$$

We show that for any θ there exist a smooth map g such that the n -surface of revolution obtained by rotating \mathcal{C} about the x_1 axis satisfies the theorem. Since the unit normal of $S_{\mathcal{C}}$ is

$$N(x_1, x_2, \dots, x_{n+1}) = \left(\frac{-g'(x_1)}{\sqrt{(g'(x_1))^2 + 1}}, \frac{z_2}{\sqrt{(g'(x_1))^2 + 1}}, \dots, \frac{z_{n+1}}{\sqrt{(g'(x_1))^2 + 1}} \right) \quad (16)$$

it follows from the Definition 3 that $\frac{-g'(x_1)}{\sqrt{(g'(x_1))^2+1}} = \sin\theta$ and hence $g'(x_1) = \pm\tan\theta$. Thus $g(x_1) = \pm x_1\tan\theta + \rho$ for some constant $\rho \in R$. Conversely, Let ℓ be the half line in R^2 with equation $x_2 = cx_1 + \rho$ for some $c, \rho \in R$ such that $x_2 > 0$. Define the function

$$f : R \times R^+ \times \overbrace{R \times \cdots \times R}^{(n-1)\text{times}} \rightarrow R \tag{17}$$

by

$$f(x_1, x_2, \dots, x_{n+1}) = \sqrt{x_2^2 + \cdots + x_{n+1}^2} - cx_1 - \rho \tag{18}$$

for $n \geq 2$. Let $S_\ell = f^{-1}(0)$ and

$$y_i = \frac{x_i}{\sqrt{c^2 + 1}\sqrt{x_2^2 + \cdots + x_{n+1}^2}} \tag{19}$$

for $2 \leq i \leq n+1$. It can be seen that S_ℓ is an n -surface of revolution obtained by rotating ℓ about the x_1 axis with unit normal

$$N(x_1, x_2, \dots, x_{n+1}) = \left(\frac{-c}{\sqrt{c^2 + 1}}, y_2, \dots, y_{n+1} \right) \tag{20}$$

Now the Definition 3 asserts that $\frac{-c}{\sqrt{c^2+1}} = \sin\theta$. Therefore $c = \pm\tan\theta$ and $x_2 = \pm x_1\tan\theta + \rho$.

4 Examples

Example 1 Let C be the n -cylinder

$$C = \{(x_1, x_2, \dots, x_n, x_{n+1}) \in R^n | x_1^2 + x_2^2 + \cdots + x_n^2 = 1\} \tag{21}$$

Let $x \in S$, $0 \leq \theta \leq \frac{\pi}{2}$ and $a > 0$ be such that $a = \sin\theta\sqrt{a^2 + 1}$. Let

$$\chi(x) = (ax_1, \dots, ax_n, \sqrt{x_1^2 + \dots + x_n^2}) \tag{22}$$

Then χ is a smooth θ -vector field on S and the components of its integral curve α satisfies the following $n + 1$ first order ordinary differential equations in $n + 1$ unknowns,

$$\left\{ \begin{array}{l} \alpha'_1(t) = a\alpha_1(t), \\ \alpha'_2(t) = a\alpha_2(t), \\ \vdots \\ \alpha'_n(t) = a\alpha_n(t), \\ \alpha'_{n+1}(t) = \sqrt{\alpha_1^2(t) + \dots + \alpha_n^2(t)} \end{array} \right. \tag{23}$$

The solution of this system of equations through $p = (p_1, p_2, \dots, p_{n+1})$ is the smooth curve α with components

$$\begin{cases} \alpha_1(t) = p_1 e^{at}, \\ \alpha_2(t) = p_2 e^{at}, \\ \cdot \\ \cdot \\ \alpha_n(t) = p_n e^{at}, \\ \alpha_{n+1}(t) = p_{n+1} + \frac{1}{a}(e^{at} - 1)\sqrt{p_1^2 + \dots + p_n^2} \end{cases} \quad (24)$$

A simple calculation shows that $\cos\angle(\alpha'(t), \nabla f(\alpha(t))) = \sin\theta$, $Y = \chi$ and so the integral curve of Y is coming from (24). Moreover,

$$f(\alpha(t)) - f(p) = \sin\theta \int_0^t \|\nabla f(\alpha(t))\| \|\chi(\alpha(t))\| dt = e^{2at} - 1 \quad (25)$$

Example 2 Let S be the $(2n - 1)$ -sphere

$$x_1^2 + x_2^2 + \dots + x_{2n-1}^2 + x_{2n}^2 = 1 \quad (26)$$

Let $x \in S$ and $\chi(x) = (-x_2, x_1, \dots, -x_{2n}, x_{2n-1})$, then χ is a smooth vector field tangent to S . The requirement that a parameterized curve α be an integral curve of χ implies that

$$\begin{cases} \alpha'_1(t) = -\alpha_2(t), \\ \alpha'_2(t) = -\alpha_1(t), \\ \cdot \\ \cdot \\ \alpha'_{2n-1}(t) = -\alpha_{2n}(t), \\ \alpha'_{2n}(t) = -\alpha_{2n-1}(t) \end{cases} \quad (27)$$

The solution of these pair of equations through $p = (p_1, p_2, \dots, p_{2n-1}, p_{2n})$ is

$$\begin{cases} \alpha_1(t) = p_1 \cos t - p_2 \sin t, \\ \alpha_2(t) = p_1 \sin t + p_2 \cos t, \\ \cdot \\ \cdot \\ \alpha_{2n-1}(t) = p_{2n-1} \cos t - p_{2n} \sin t, \\ \alpha_{2n}(t) = p_{2n-1} \sin t + p_{2n} \cos t \end{cases} \quad (28)$$

A simple calculation shows that

$$\alpha_{2k-1}^2(t) + \alpha_{2k}^2(t) = p_{2k-1}^2 + p_{2k}^2 \quad (29)$$

for $k = 1, \dots, n$ and so α with components $\alpha_1, \dots, \alpha_{2n}$ is the maximal integral curve of χ which lies on S .

5 Conclusion

1. For any n -surface $S = f^{-1}(c)$ and χ , a smooth θ -vector field on it, there exists a vector field Y , which the value of f on an arbitrary point of its integral curve is determined by θ, χ and ∇f ,
2. For any $0 \leq \theta \leq \frac{\pi}{2}$ there exists an n -surface S and a smooth vector field χ , which is a θ -vector field on S ,
3. Any smooth tangent vector field on S , has an integral curve which lies on S ,
4. The θ -vector field on an n -surface S through a point $p \in S$, in general is not unique,
5. For any θ -vector field on an n -surface S , there exists a smooth curve α through a point $p \in S$, which satisfies the integral and differential equations (1) and (2),
6. For any $0 \leq \theta \leq \frac{\pi}{2}$ and any 1-surface S in an open set $U \subseteq \mathbb{R}^2$ with smooth function $f : U \rightarrow \mathbb{R}$, satisfies the inequality $\frac{\partial f}{\partial x_2} \cos \theta - \frac{\partial f}{\partial x_1} \sin \theta > 0$ on U , there exists a smooth θ -vector field χ on S . In fact, for $0 < \theta < \frac{\pi}{2}$ let

$$\chi = \left(\left(\frac{\partial f}{\partial x_2} \right)^2 \cos^2 \theta - \left(\frac{\partial f}{\partial x_1} \right)^2 \sin^2 \theta, \|\nabla f\|^2 \sin \theta \cos \theta - \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \right) \quad (30)$$

Then a simple calculation shows that

$$\chi \cdot \nabla f = \left(\left(\frac{\partial f}{\partial x_1} \right)^2 + \left(\frac{\partial f}{\partial x_2} \right)^2 \right) \left(\frac{\partial f}{\partial x_2} \cos \theta - \frac{\partial f}{\partial x_1} \sin \theta \right) \sin \theta \quad (31)$$

$$\|\chi\| \|\nabla f\| = \left(\left(\frac{\partial f}{\partial x_1} \right)^2 + \left(\frac{\partial f}{\partial x_2} \right)^2 \right) \left(\frac{\partial f}{\partial x_2} \cos \theta - \frac{\partial f}{\partial x_1} \sin \theta \right) \quad (32)$$

and so

$$\cos \angle(\chi(p), \nabla f(p)) = \sin \theta \quad (33)$$

The claim in the cases $\theta = 0, \frac{\pi}{2}$ is obvious.

7. Any smooth n -dimensional manifold M , has a smooth trivial tangent vector field. The existence of non zero smooth tangent vector field on the n -dimensional manifold M , can be proved by using the Partition of Unity subordinate to a special cover of M [5, 6].

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