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Numerical Treatment of Geodesic Differential Equations on Two Dimensional Surfaces

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Abstract This paper presents a brief instructions to find geodesics equations on two dimensional surfaces in R^3. The resulting geodesic equations are solved numerically using Computer Program Matlab, the geodesics are displayed through Figures.

Keywords Differential Equations · Differential geometry · Geodesics · Matlab's ODE

Mathematics Subject Classification (2010) 34Gxx · 34G99 · 97Rxx · 97R20 · 97Gxx · 97G99 .

1 Introduction

In an axiomatic approach to geometry we study the properties of points and lines. Most of the theorems in axiomatic geometry deal with the relationships between points and lines. If we are to see how the differential geometry we have been studying is to relate to axiomatic geometry, we need some method for developing an abstract definition of a line. This is different from our axiomatic technique of taking a line as an undefined term. There are various ways in which a straight line in usual Euclidean geometry can be characterized. For instance,it has zero curvature everywhere, all its tangent vectors are parallel, or it is the solution of the simple first order linear differential equation $v''(t) = v_0$.
Neither of these characterizations can be immediately transferred to the case

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of curves within a Riemannian manifold but the following definition is generalizable:A straight line between two points is the curve which minimizes the distance between these points. Since in a Riemannian metric we have the notion of length, we can use this to define what a straight line in a curved space is. Such straight lines are called geodesics. Geometrically, a geodesic on a surface is an embedded simple curve on the surface such that for any two-points on the curve the portion of the curve connecting them is also the shortest path between them on the surface. A different characterization of a geodesic is the following: A curve on a surface is geodesic if and only if the normal vector to the curve is everywhere parallel to the local normal vector of the surface. This goes back to Johann Bernoulli (1697)!
Geodesic on a surface is an intrinsic geometric feature that plays an important role in a diversity of applications. Many geometric operations are inherently related to geodesics. For instance, when a developable surface is flattened into a planer figure (with no distortion), any geodesic on it will be mapped to a straight line in the planer figure [9]. Thus, to flatten an arbitrary non-developable surface with as little distortion as possible, a good algorithm should try to preserve the geodesic curvatures on the surface [1,2]. Geodesic method also finds its applications in computer vision and image processing, such as in object segmentation [5,6,16] and multi-scale image analysis [17, 18]. The concept of geodesic also finds its place in various industrial applications, such as tent manufacturing, cutting and painting path, fiberglass tape windings in pipe manufacturing, textile manufacturing [3,4,10–14,21]. Available approaches for the computation of geodesic curves on surfaces can be classified broadly as analytical reference [8] and numerical [19,15].

2 Main results

t: Unit tangent vector of C at P.
n: Unit normal vector of C at P.
N: Unit surface normal vector of S at P.
u: Unit vector perpendicular to t in the tangent plane defined by N_t.

We can decompose the curvature vector k of C into N component k_n, which is called normal curvature vector, and u component k_g, which is called geodesic curvature vector

$$k = k_n + k_g = -\kappa_n N + \kappa_g u$$

$$\kappa_n = -k \cdot N \ , \quad \kappa_g = k \cdot u$$

Consequently,

$$\kappa_g = \frac{dt}{ds} \cdot (N \times t)$$

Geodesic paths are sometimes defined as shortest path between points on a surface, however this is not always a satisfactory definition.

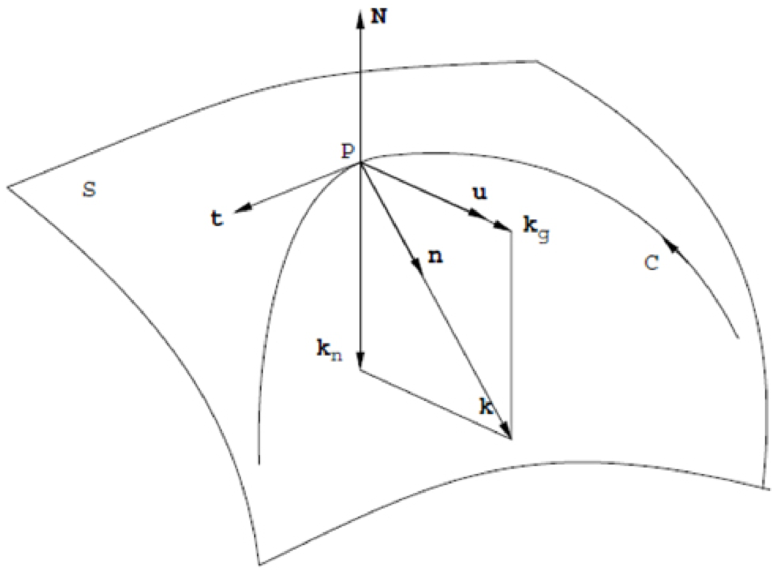


Fig. 1 Definition of geodesic curvature

Definition 1 Geodesics are curves of zero geodesic curvature [20].

The computation of geodesic curves

Let C is a geodesic curve on a surface $r = r\big(u(s), v(s)\big)$. Because the surface normal N has the direction of principal normal n of C. Then, we can write

$$n \cdot r_u = 0 \quad \text{and} \quad n \cdot r_v = 0. \tag{1}$$

The unit tangent vector of the curve C on the surface r is given by

$$\frac{dt}{ds} = r_{uu}\left(\frac{du}{ds}\right)^2 + 2r_{uv}\frac{du}{ds}\frac{dv}{ds} + r_{vv}\left(\frac{dv}{ds}\right)^2 + r_u\left(\frac{d^2u}{ds^2}\right) + r_v\left(\frac{d^2v}{ds^2}\right), \tag{2}$$

Since $kn = dt/ds$, from Eqs. (1) and (2), we obtain

$$(r_{uu} \cdot r_u)\left(\frac{du}{ds}\right)^2 + 2(r_{uv} \cdot r_u)\frac{du}{ds}\frac{dv}{ds} + (r_{vv} \cdot r_u)\left(\frac{dv}{ds^2}\right)^2 + F\left(\frac{d^2v}{ds^2}\right) = 0, \tag{3}$$

$$(r_{uu} \cdot r_v)\left(\frac{du}{ds}\right)^2 + 2(r_{uv} \cdot r_v)\frac{du}{ds}\frac{dv}{ds} + (r_{vv} \cdot r_v)\left(\frac{dv}{ds^2}\right) + G\left(\frac{d^2v}{ds^2}\right) = 0, \tag{4}$$

where $E = r_u \cdot r_u$, $F = r_u \cdot r_v$ and $G = r_v \cdot r_v$ are coefficient of first fundamental form of the surface, and

$$E_u = 2r_{uu} \cdot r_u \ , \quad E_v = 2r_{uv} \cdot r_u \ , \quad F_u = r_{vu} \cdot r_u + r_v \cdot r_{uu} \ ,$$
$$F_v = r_{vv} \cdot r_u + r_v \cdot r_{uv} \ , G_u = 2r_{uv} \cdot r_v \ , \quad G_v 2r_{vv} \cdot r_v \ ,$$
$$E_v - 2F_u = -2r_{uu} \cdot r_v \ , \quad G_u - 2F_v = -2r_u \cdot r_{vv}.$$

By eliminating d^2v/ds^2 from Eq.(3) have,

$$\frac{d^2v}{ds^2} = \left(-\frac{1}{F}\right)(r_{uu} \cdot r_u)\left(\frac{du}{ds}\right)^2 + \left(-\frac{2}{F}\right)(r_{uv} \cdot r_u)\frac{du}{ds}\frac{dv}{ds} + \left(-\frac{1}{F}\right)(r_{vv} \cdot r_u)\left(\frac{dv}{ds}\right)^2 + \left(-\frac{E}{F}\right)\left(\frac{d^2u}{ds^2}\right)$$

Now, using d^2v/ds^2 in Eq(4) obtain,

$$\left(F - \frac{EG}{F}\right)\left(\frac{d^2u}{ds^2}\right) + \left[(r_{uu} \cdot r_v) - \frac{G}{F}(r_{uu} \cdot r_u)\right]\left(\frac{du}{ds}\right)^2 + \left(2(r_{uv} \cdot r_v) - \frac{2G}{F}(r_{uv} \cdot r_u)\right]\frac{du}{ds}\frac{dv}{ds} + \left[(r_{vv} \cdot r_v) - \frac{G}{F}(r_{vv} \cdot r_u)\right]\left(\frac{dv}{ds}\right)^2 = 0$$

Thus

$$\Gamma_{11}^1 = \frac{GE_u + FE_v - 2FF_u}{2(EG - F^2)} \ , \quad \Gamma_{11}^2 = \frac{-E_uF - EE_v + 2EF_u}{2(EG - F^2)}$$
$$\Gamma_{12}^1 = \frac{GE_v - FG_u}{EG - F^2} \ , \quad \Gamma_{12}^2 = \frac{EG_u - FE_v}{EG - F^2}$$
$$\Gamma_{22}^1 = \frac{-GG_u + 2GF_v - FG_v}{2(EG - F^2)} \ , \quad \Gamma_{22}^2 = \frac{FG_u - 2FF_v + EG_v}{2(EG - F^2)}$$

And the geodesic equations become

$$\frac{d^2u}{ds^2} + \Gamma_{11}^1\left(\frac{du}{ds}\right)^2 + \Gamma_{12}^1\frac{du}{ds}\frac{dv}{ds} + \Gamma_{22}^1\left(\frac{dv}{ds}\right)^2 = 0,$$
$$\frac{d^2v}{ds^2} + \Gamma_{11}^2\left(\frac{du}{ds}\right)^2 + \Gamma_{12}^2\frac{du}{ds}\frac{dv}{ds} + \Gamma_{22}^2\left(\frac{dv}{ds}\right)^2 = 0,$$

These two second order differential equations can be rewritten as a system of four first order differential equations [7].

$$\frac{du}{ds} = p$$
$$\frac{dp}{ds} = -\Gamma_{11}^1p^2 - \Gamma_{12}^1pq - \Gamma_{22}^1q^2$$
$$\frac{dv}{ds} = q$$
$$\frac{dq}{ds} = -\Gamma_{11}^2p^2 - \Gamma_{12}^2pq - \Gamma_{22}^2q^2$$

Then using the Matlab function file, we have the numerical solution for the geodesic equations, if that is shown in the following example.

Example

Example 1 bilinear surface $r(u,v) = (u, v, uv)$.

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{22} = \Gamma^2_{22} = 0$$
$$\Gamma^1_{12} = \frac{2v}{1+u^2+v^2}$$
$$\Gamma^2_{12} = \frac{2u}{1+u^2+v^2}.$$

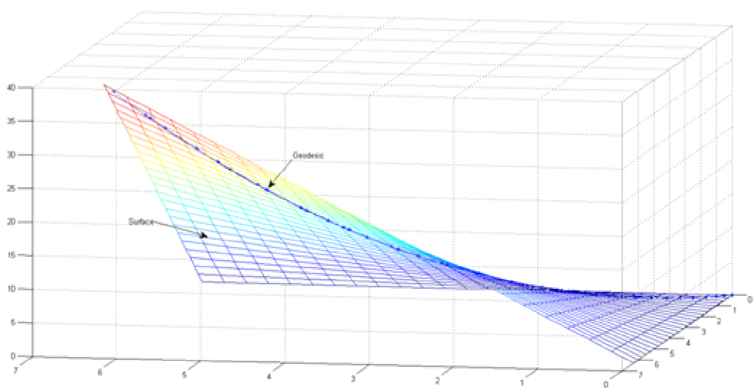
And the geodesic equations become

$$\frac{d^2u}{ds^2} + \frac{2v}{1+u^2+v^2}\frac{du}{ds}\frac{dv}{ds} = 0$$
$$\frac{d^2v}{ds^2} + \frac{2u}{1+u^2+v^2}\frac{du}{ds}\frac{dv}{ds} = 0$$

So

$$\frac{du}{ds} = p$$
$$\frac{dp}{ds} = -\frac{2v}{1+v^2+u^2}pq$$
$$\frac{dv}{ds} = q$$
$$\frac{dq}{ds} = -\frac{2u}{1+v^2+u^2}pq$$

Next, we write a Matlab function file and observe:



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