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Density of the Periodic Points in the Interval Set

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Abstract The dynamical system $(f, \mathbb{R})$ is introduced and some of its properties are investigated. It is proven that there is an invariant set Λ on which the periodic points of f are dense.

Keywords Density · Invariant set · Periodic points · Schwarzian derivative

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1 Introduction

Periodic points are important in the study of discrete dynamical systems. The long term behavior of the orbit of a periodic point is always clear and it can affect the behavior of the nearby points. Determining all the periodic points of a dynamical system explicitly is often impossible. But in a smooth one dimensional discrete dynamical system when Schwarzian derivative is negative, it is possible to find the attracting periodic orbits by following the orbits of the critical points.
In this paper we introduce a class of real functions.Then we describe some of the common properties of this class and show for each function there is an invariant set on which the periodic points are dense.
First we explain the terminologies which are used in this paper.

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Suppose X is a topological space and $f : X \to X$ is a continuous function. (f, X) is called a *discrete dynamical system*. By f^n we mean $\underbrace{f \circ \cdots \circ f}_{n \text{ times}}$.

The point $x_0 \in X$ is called a *periodic point of* f *of period* n if n is the least natural number that $f^n(x_0) = x_0$. If $n = 1$, x_0 is called a *fixed point* of f. From now on we suppose I is an interval and $f : I \to I$ is a C^3 function. The periodic point x_0 is called *attracting (repelling)* if $|(f^n)'(x)| < 1 (> 1)$. The point x_0 is called *neutral* if $|(f^n)'(x)| = 1$.

If $f'(x) \neq 0$ then the *Schwarzian derivative* of f at x, denoted by $Sf(x)$, is

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Functions with negative Schwarzian derivative have some important properties that we will mention in the following.

- The Schwarzian derivative of f^n is negative.
- f' doesn't have positive local minimum or negative local maximum.
- Immediate basin of any attracting periodic orbit contains either a critical point of f or a boundary point of the interval I.
- Each neutral periodic point of f is attracting at least from one side.

See [2] and [4] for more details.

2 The Common Properties of the Function *f*

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^3 function with the following properties:
1. $f(0) = f(1) = 0$,
2. f has just one critical point $c, 0 < c < 1$, such that $f(c) > 1$,
3. $f''(x) < 0,\ x \in \mathbb{R}$.

The following lemmas are easily proved by using the Intermediate Value Theorem and the Mean Value Theorem.

Lemma 1 *Suppose $g(x) = f(x) - x$ and $h(x) = f(x) - 1$. Then the equation $g(x) = 0$ has exactly two solutions and similarly $h(x) = 0$ has also two solutions.*

Lemma 2 *Suppose $p \neq 0$ is the fixed point of f. If $0 < x < p$ then $f(x) > x$ and if $x < 0$ then $f(x) < x$.*

Corollary 1 *With the same assumptions as in Lemma 2, we have $f'(0) > 1$.*

Proof Suppose $0 < x < c$ since $f(x) > x$ then $\frac{f(x)-0}{x-0} = f'(z) > 1$ for some $z \in (0, x)$ and f' is decreasing, therefore if $x < z$ then $f'(x) > f'(z) > 1$.

Proposition 1 *If* $x < 0$ *or* $x > 1$ *then* $\lim_{n\to\infty} f^n(x) = -\infty$.

Moreover, we suppose f satisfies the following conditions.
4. The function f has negative Schwarzian derivative.
Let $\hat{q}, q$ be the solutions of $h(x) = 0$ and $\hat{q} < q$ and the points $\hat{p} < p$ be such that $f(\hat{p}) = f(p) = p$.
5. $f'(\hat{p}) > 1$.
6. $\max\{f(p-q), f(\frac{\hat{q}}{2})\} \leq p$.
The above inequality means $f(p-q) \leq f(\hat{p}) = f(p)$ and $f(\frac{\hat{q}}{2}) \leq f(\hat{p}) = f(p)$. Since f is increasing on $[0,c]$ and decreasing on $[c,1]$, as well as, $\hat{p} \leq \hat{q} \leq c$ then $\frac{\hat{q}}{2} \leq \hat{p}$, and $p - q \leq \hat{p}$.

Now suppose

$$\Lambda = \{x \in [0,1] : f^n(x) \in [0,1], n \in \mathbb{N}\}$$

It can be seen easily that $f(\Lambda) = \Lambda$, so Λ is an invariant set under f.
Our aim is to study the dynamics of $f|_\Lambda$.

3 Density of the Periodic Points

In this section we are going to prove the following theorem.

Theorem 1 *The periodic points of* f *are dense in* Λ.

In order to prove the theorem we need the following lemmas.

Lemma 3 *Suppose* $U \subset [0,\hat{p}]$ *is an open interval in* $[0,1]$ *such that* $U \cap \Lambda \neq \emptyset$. *Then there is* $n \geq 1$ *such that* $f^n(U) \cap [(\hat{p},\hat{q}) \cup (q,p)] \neq \emptyset$.

Proof Suppose $x \in U$ and $0 < x < \hat{p}$ then $f(x) > x$. So there is $n > 0$ such that $f^n(x) \in [\hat{p}, p)$. Since $U \cap \Lambda \neq \emptyset$, then $f^n(U) \not\subset (\hat{q}, q)$.

Lemma 4 *Suppose* $U \subset [0,1]$ *is an open interval that contains* p. *Then there is a positive integer* n *such that* $f^n(U) \supset [0,1]$.

Proof Since $Sf(x) < 0$ and $\{f^n(c)\} \to -\infty$, p is a repelling fixed point and $|f'(p)| > 1$. There is a neighborhood $U' \subset U$ such that $p \in U'$ and if $x \in U' \cup f(U')$ then $f'(x) < -1$. If $x \in U'$ then the sequence $\{f^{2n}(x)\}$ is monotonic and since there is no attracting periodic orbit, for suitable n_0, $f^{2n_0}(x) \geq 1$ or $f^{2n_0}(x) \leq q$. So there is n such that $f^n(U) \supset [0,1]$.

Now suppose the open interval U is a subset of $(\hat{p},\hat{q}) \cup (q,p)$ and $U \cap \Lambda \neq \emptyset$. In this case we want to show that there is $n \geq 2$ such that$f^n(U) \supset [0,1]$. Here we use the method of [1] and [3] in partition the intervals $(\hat{p},\hat{q}) \cup (q,p)$.
Note that $f(q,p) = (p,1)$ and $f(p,1) = (0,p)$, so there is the interval $A_2 \subset (q,p)$ such that $f^2(A_2) = [\hat{p}, p)$, This interval is half-open and half-closed.The subset that its image is $(0,\hat{p})$ is called W_2. $f(0,\hat{p}) = (0,p)$, so there is the interval A_3 such that $f^3(A_3) = [\hat{p}, p)$. BY continuing this process all the subsets

A_n's are constructed. The partition of the interval $(\hat{p}, \hat{q})$ is done similarly. Note that if $A_n = [a_n, b_n)$ then $b_n - a_n \to 0$ as $n \to \infty$. Since $f^n([a_n, b_n)) = [\hat{p}, p)$ and $f^n((q, a_n)) = (0, \hat{p})$ then by the Mean Value Theorem we have

$$\frac{\mid f^n(a_n) - f^n(b_n) \mid}{\mid a_n - b_n \mid} > \frac{\mid p - \hat{p} \mid}{p - q} > 1$$

and

$$\frac{\mid f^n(a_n) - f^n(q) \mid}{\mid a_n - q \mid} > \frac{\mid \hat{p} \mid}{p - q} \geq 1,$$

so for some $c_n \in (a_n, b_n)$ and $d_n \in (q, a_n)$ we have $|(f^n)^{'}(c_n)| > 1$ and $|(f^n)'(d_n)| > 1$.
Since the Schwarzian derivative is negative hence $\mid (f^n)'(a_n) \mid > 1$. Consequently, $\mid (f^n)'(b_n) \mid > 1$. Therefore, if $x \in [a_n, b_n)$ then $\mid (f^n)'(x) \mid > 1$. See [1] for more details.
We use the fact that f^n is expanding on $[a_n, b_n)$ in the proof of the following lemma.

Lemma 5 *Suppose $U \subset (\hat{p}, \hat{q}) \cup (q, p)$ is an open interval such that $U \cap \Lambda \neq \emptyset$. Then there is $n \geq 2$ such that $f^n(U) \supset [0, 1]$.*

The following lemma is useful in proving the existence of a fixed point in an interval.

Lemma 6 *Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and I and J are two closed intervals such that $I \subset J$ and $g(I) \supset J$. Then g has a fixed point in I.*

We can conclude from the above lemmas that for every $x \in \Lambda$ and every neighborhood U of x there is $n > 0$ such that $f^n(U) \supset [0, 1]$ and consequently there is a periodic point of f in U. Therefore, the periodic points of f are dense in Λ.

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