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Research article

Solving the Fractional HIV Model using Bell Polynomials and the Tau Method

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Abstract

This paper introduces a Tau-based numerical approach utilizing Bell polynomials to solve a fractional HIV model involving $CD4^+T$ cells. The model comprises a system of three fractional nonlinear ordinary differential equations. The method begins with deriving the operational matrix for the fractional derivative of Bell polynomials. Next, any function within the space $L^2[0,1]$ is approximated using Bell polynomials, and these approximations are substituted into the model equations. The resulting expressions are used to compute Chebyshev collocation points within the interval $[0,1]$. By applying the Tau method along with the boundary conditions, the problem is converted into a system of algebraic equations that can be solved using standard numerical techniques such as Newton's method. An example is provided to illustrate the accuracy and effectiveness of the proposed method.

Keywords: Ordinary differential equations, Bell polynomials, Tau method, Operational matrix, HIV infection

Mathematics Subject Classification (2020): 34A08, 41A10, 65L05

1 Introduction

Most phenomena observed in the real world can be represented using mathematical models. Analyzing and solving these models can provide valuable insights for decision-makers who continuously seek effective strategies to address current global issues. One of the most significant threats to humanity in recent decades is acquired immune deficiency syndrome (AIDS), which is caused by the human immunodeficiency virus (HIV). HIV infects and damages vital cells in the body, especially a particular type of white blood cell called $CD4^+T$ cells. These cells play a crucial role in the immune system by defending the body and assisting other T cells in combating infections. Despite extensive research and the development of various treatment methods over the years, a permanent cure for HIV/AIDS remains undiscovered, and no vaccine has been developed. Nevertheless, early diagnosis and prompt treatment can slow or prevent disease progression. Tracking the levels of infected and uninfected $CD4^+T$ cells is essential for evaluating HIV progression and optimizing treatment plans. Mathematical models are indispensable tools in this domain, as their solutions can improve our understanding of the disease, aid in its management, and support the development of effective therapies. Numerous models have been proposed to study the dynamics of HIV infection (see, for example, [2–6, 8–12]). The most renowned HIV model was developed by Perlson et al. [9] and [12], and many subsequent models have been based on this foundational framework. This model is described by a system of three nonlinear ordinary differential equations, as

outlined in [1].

$$\begin{cases} \frac{dT(t)}{dt} = p - \alpha T(t) + r\,T(t)\left(1 - \frac{T(t)+I(t)}{T_{max}}\right) - k\,V(t)\,T(t), \\ \frac{dI(t)}{dt} = k\,V(t)\,T(t) - \beta I(t), \\ \frac{dV(t)}{dt} = N\beta\,I(t) - \gamma V(t), \end{cases} \tag{1}$$

with initial conditions

$$T(0) = T_0 \ , \ I(0) = I_0 \ , \ V(0) = V_0, \quad 0 \le t \le l < \infty. \tag{2}$$

The dependent variables T, I, and V represent the uninfected CD4$^+$T cells, infected CD4$^+$T cells, and free HIV virus particles in the bloodstream at time t, respectively. The term $\left(1 - \frac{T(t)+I(t)}{T_{\max}}\right)$ models the logistic growth of healthy CD4$^+$T cells, while $kV(t)T(t)$ captures the rate at which healthy CD4$^+$T cells become infected by HIV. It is important to note that the proliferation of infected CD4$^+$T cells is not considered in this model. The parameters used are detailed in Table 1.

The fractional form of the proposed system is considered as follows

$$\begin{cases} D^{v_1}\ T(t) = p - \alpha T(t) + r\,T(t)\left(1 - \frac{T(t)+I(t)}{T_{max}}\right) - k\,V(t)\,T(t), \\ D^{v_2}\ I(t) = k\,V(t)\,T(t) - \beta I(t), \\ D^{v_3}\ V(t) = N\beta\,I(t) - \gamma V(t), \end{cases} \tag{3}$$

with the same initial conditions (2), Where $0<v_1, v_2, v_3\leq 1$ and D is the derivative operator of the Caputo fraction. The remaining sections

Table 1. Definition of the parameters in model (1) [8].

Parameters	Definition
p	The generation rate of uninfected CD4$^+$T cells in the body
α	Turnover rate of uninfected CD4$^+$T cells
r	Rate of cells' duplication through the process of mitosis when they are stimulated by antigen and mitogen
T_{max}	Maximal of CD4$^+$T cells
k	The infection rate of CD4$^+$T cells by HIV virus
β	Turnover rate of infected CD4$^+$T cells
N	The number of virus particles produced by each infected CD4$^+$T cell during its life time
γ	Turnover rate of HIV virus particles

of this paper are structured as follows: Section 2 provides an introduction, where Bell polynomials and the fractional derivative operational matrix are introduced along with their key properties. Section 3 details the Tau method and its application within this study. In Section 4, a numerical example with $M = 1$ is solved, and the results are shown both numerically and graphically. Finally, Section 5 offers the concluding remarks.

2 Preliminaries

In this section, some basic concepts and definitions are provided.

2.1 Bell Polynomials

Definition 1. *It is widely recognized that the Bell polynomials are characterized by the following generating function. [13]*

$$e^{t(e^x-1)} = \sum_{i=0}^{\infty} B(t)\,\frac{x^i}{i!}. \tag{4}$$

Note that

$$\begin{aligned} e^{(s+t)(e^x-1)} &= e^{s(e^x-1)}e^{t(e^x-1)} \\ &= \left(\sum_{l=0}^{\infty} B_l(s)\frac{x^l}{l!}\right)\left(\sum_{m=0}^{\infty} B_m(t)\frac{x^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n} B_l(s)B_{n-l}(t), \end{aligned} \tag{5}$$

and

$$\mathrm{e}^{(s+t)(\mathrm{e}^x-1)} = \sum_{n=0}^{\infty} \mathrm{B}_n(s+t)\frac{x^n}{n!}. \tag{6}$$

Therefore, using equations (5) *and* (6)*, the binomial identity is derived as follows.*

$$\mathrm{B}_n(\mathrm{s}+\mathrm{t}) = \sum_{l=0}^{n} \binom{n}{l} \mathrm{B}_l(s)\mathrm{B}_{n-l}(t), \qquad n \geq 0. \tag{7}$$

Definition 2. *Stirling numbers of the second kind are defined as follow*

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n, \qquad k = 0,1,\ldots,n. \tag{8}$$

By expanding the function on the right side of Eq. (4), we will obtain

$$e^{t(e^x-1)} = \sum_{k=0}^{\infty} \frac{t^k}{k!}(e^x-1)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!}k! \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!} = \sum_{k=0}^{\infty} \left(\sum_{k=0}^{n} S(n,k)t^k\right)\frac{x^n}{n!}.$$

This suggests that, based on Eq. (4), we are able to express it as follows

$$B_n(t) = \sum_{k=0}^{n} S(n,k)t^k, \ \ n = 1,2,\ldots. \tag{9}$$

For approximate purposes, with a given M, the Bell vector forms a basis for the space $L^2[0,1]$ as

$$B(t) = [B_0(t), B_1(t), \ldots, B_M(t)]^T. \tag{10}$$

Based on Eq. (9), the Bell vector can be derived from the following matrix representation

$$\mathrm{B(t)} = \begin{pmatrix} S(0,0) & 0 & \cdots & 0 \\ S(1,0) & S(1,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S(M,0) & S(M,1) & \cdots & S(M,M) \end{pmatrix} \begin{pmatrix} 1 \\ t \\ \vdots \\ t^M \end{pmatrix}$$

In other words

$$B(t) = S\Phi(T), \tag{11}$$

where Φ is the power basis vector $\Phi(t) = [1,t,\ldots,t^M]$ and S is the Stirling number matrix, which is

$$S_{ij} = \begin{cases} S(i,j), & i \geq j, \\ 0, & i < j. \end{cases}$$

Any $u(t) \in L^2(0,1)$ can be expressed using a Bell basis vector in the following form

$$u(t) = \sum_{i=0}^{M} u_i B_i(t) = U^T B(t) = U^T S\Phi(t),$$

where u_i is the vector of Bell polynomial coefficients.

The advantage of using Bell polynomials is that they can be used in power bases, which makes mathematical calculations simple and inexpensive. Also, the inclusion of Stirling numbers as coefficients leads to a better approximation in the desired model.

2.2 Fractional Derivative Operational Matrix

Based on the definition of Caputo's fractional derivative, we can show that

$$D^{\alpha}(\Phi(t)) = D^{\alpha}\left(\begin{pmatrix} 1 \\ t \\ \vdots \\ t^{M} \end{pmatrix}\right) = \left(\begin{pmatrix} D^{\alpha}(1) \\ D^{\alpha}(t) \\ \vdots \\ D^{\alpha}(t^{M}) \end{pmatrix}\right) \tag{12}$$

We will have

$$D^{\alpha}(t^{M}) = \begin{cases} \frac{\Gamma(M+1)}{\Gamma(M+1-\alpha)}, & M \geq \lceil \alpha \rceil, \\ 0, & M < \alpha, \end{cases}$$

where the gamma function is of the form

$$\Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha+1} e^{-x} dx.$$

Therefore, for $0 < \alpha < 1$, Eq. (12) is rewritten as follows

$$D^{\alpha}(\Phi(t)) \begin{pmatrix} 0 \\ \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} \\ \vdots \\ \frac{\Gamma(M+1)}{\Gamma(M+1-\alpha)} t^{M-\alpha} \end{pmatrix} = t^{-\alpha} \begin{pmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} & \cdots & \frac{\Gamma(M+1)}{\Gamma(M+1-\alpha)} \end{pmatrix} \begin{pmatrix} 1 \\ t \\ \vdots \\ t^{M} \end{pmatrix}$$

So

$$D^{\alpha}(\Phi(t)) = t^{-\alpha} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(M+1)}{\Gamma(M+1-\alpha)} \end{pmatrix} \Phi(t)$$

Finally

$$D^{\alpha}(\Phi(t)) = F^{\alpha}\Phi(t), \tag{13}$$

where F^{α} is the operational matrix of the fractional derivative of order α. Assuming $u(t) = U^{T}B(T)$, we can write to determine $D^{\alpha}(u(t))$ based on $u(t)$

$$D^{\alpha}(u(t)) = D^{\alpha}(U^{T}B(t)) = D^{\alpha}(U^{T}S\Phi(t)) = U^{T}SD^{\alpha}(\Phi(t)),$$

eventually, Using Eq. (13), We have

$$D^{\alpha}(u(t)) = U^{T}SF^{\alpha}\Phi(t). \tag{14}$$

There are several reasons for using operational matrices to approximate functions, including:

- simplification of calculations: matrices provide a structural way to express derivative and integral operators, which makes calculations simple and fluid.
- facilitation of problem solving: these matrices make it easier to solve complex problems, especially in problems related to fractional calculus, due to the ability to express problems in matrix form.
- ease of implementation: using operational matrices in implementing algorithms and software related to solving mathematical and physical problems is easier than other methods.
- application in fractional calculus: operational matrices are widely used in fractional calculus problems, in which functions of non-integer order are examined.
- optimization in optimization problems: matrices are used to represent constraints and objective functions and help to solve the problem optimally.

3 Tau Method

In the Tau method [7], by substituting the approximation of functions into the problem, we obtain the residual expression $(R_i(t))$, And we expect that $R_i(t) \approx 0$. By substituting M Chebyshev collocation points in the interval [0,1] which are calculated by the formula

$$t_i = \frac{1}{2}(1+\cos(\frac{(2i+1)\pi}{2M})), \quad i = 0,1,\ldots,M-1, \tag{15}$$

and using the boundary conditions as preconditions of the Tau method, the problem is transformed into a system of algebraic equations. Approximating the functions $T(t)$, $I(t)$, and $V(t)$ by the basis of Bell polynomials as

$$T(t) = C_1^T S\Phi(t),$$

$$I(t) = C_2^T S\Phi(t),$$

$$V(t) = C_3^T S\Phi(t),$$

and also by applying fractional derivatives to these functions we will have

$$D^{v_1}T(t) = C_1^T SF^{v_1}\Phi(t) = A_1^T\Phi(t), \tag{16}$$

$$D^{v_2}I(t) = C_2^T SF^{v_2}\Phi(t) = A_2^T\Phi(t), \tag{17}$$

$$D^{v_3}V(t) = C_3^T SF^{v_3}\Phi(t) = A_3^T\Phi(t). \tag{18}$$

Substituting Eqs. (16), (17), and (18) into model (3), we have

$$\begin{cases} R_1(t) = p - \alpha C_1^T S\Phi(t) + rC_1^T S\Phi(t)(1-\frac{1}{T_{max}}(C_1^T + C_2^T)S\Phi(t)) - k(C_1^T S\Phi(t))(C_2^T S\Phi(t)) - C_1^T SF^{v_1}\Phi(t), \\ R_2(t) = k(C_1^T S\Phi(t))(C_2^T S\Phi(t)) - \beta C_2^T S\Phi(t) - C_2^T SF^{v_2}\Phi(t), \\ R_3(t) = n\beta C_2^T S\Phi(t) - \gamma C_3^T S\Phi(t) - C_3^T SF^{v_3}\Phi(t) \end{cases}$$

By substituting M Chebyshev collocation points(Eq.(15)) in the above relations, we arrive at a system consisting of $3M$ equations and $3M+1$ unknowns, which are the Bell coefficient vectors C_1, C_2 and C_3. With the help of boundary conditions, the other 3 required equations are satisfied.

$$\Phi(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow S\Phi(0) = \begin{pmatrix} S(0,0) \\ S(1,0) \\ \vdots \\ S(M,0) \end{pmatrix} = S_1$$

S_1 means the first column of matrix S. So we have

$$T(0) = C_1^T S\Phi(0),$$

$$I(0) = C_2^T S\Phi(0),$$

$$V(0) = C_3^T S\Phi(0).$$

Considering the boundary conditions of the problem, which are in the form:

$$T(0) = T_0, \quad I(0) = I_0, \quad V(0) = V_0.$$

We will have

$$R_4 = C_1^T S_1 - T_0 \approx 0,$$

$$R_5 = C_2^T S_1 - I_0 \approx 0,$$

$$R_6 = C_3^T S_1 - V_0 \approx 0.$$

By applying Newton's method to solve the system of nonlinear equations consisting of $\{R_1(t_i), R_2(t_i), R_3(t_i), R_4, R_5, R_6\}$, the coefficient vectors C_1, C_2, and C_3 are determined. Subsequently, using the relations $T(t) = C_1^T B(t)$, $I(t) = C_2^T B(t)$, and $V(t) = C_3^T B(t)$, the desired functions are derived in terms of Bell polynomials.

4 Numerical Example

In this section, we will solve the model under consideration for $M = 1$. Therefore, we have

$$\Phi(t) = [1,t]^T$$

and

$$B(t) = \begin{pmatrix} S(0,0) & 0 \\ S(1,0) & S(1,1) \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}$$

Assuming that

$$T(t) = \begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} = T_1 + T_2 t.$$

In a similar way we will have

$$I(t) = I_1 + I_2 t, \;\; V(t) V_1 + V_2 t.$$

We assume that $v_1 = v_2 = v_3 = 1$:

$$D^{v_1} T(t) = \begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t^{-v_1} \begin{pmatrix} 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-v_1)} \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} \to DT(t) = T_2$$

Similarly, for $v_2 = v_3 = 1$ we have

$$DI(t) = I, \quad DV(t) = V_2.$$

By substituting the above expression into the desired model, we have

$$R_1(t) = p - \alpha(T_1 + T_2 t) + r(T_1 + T_2 t)(1 - \frac{1}{T_{max}}(T_1 + I_1 + (T_2 + I_2)t) - k(T_1 + T_2 t)(V_1 + V_2 t)) - T_2,$$

$$R_2(t) = k(T_1 + T_2 t)(V_1 + V_2 t)) - \beta(I_1 + I_2 t) - I_2,$$

$$R_3(t) = n\beta(I_1 + I_2 t) - \gamma(V_1 + V_2 t)) - V_2.$$

By placing point $t_0 = \frac{1}{2}(1 + \cos(\frac{\pi}{2})) = \frac{1}{2}$, in these relations and simplifying the expressions, as well as by setting the model parameters, we have

$$\begin{cases} R_1(\frac{1}{2}) = 0.018T_1 - 0.991T_2 - 0.002I_1 - 0.001I_2 - 0.0027(T_1 + \frac{1}{2}T_2)(V_1 + \frac{1}{2}V_2) + 3.1, \\ R_2(\frac{1}{2}) = 0.0027(T_1 + \frac{1}{2}T_2)(V_1 + \frac{1}{2}V_2) - 1.15I_2 - 0.3I_1, \\ R_3(\frac{1}{2}) = 3I_1 + 1.5I_2 - 2.4V_1 - 2.2V_2. \end{cases}$$

Using the initial conditions $I(0) = 0$ and $V(0) = T(0) = 0.1$, we have

$$R_4 = \begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0.1 \to R_4 = T_1 - 0.1.$$

Likewise

$$R_5 = I_1 - 0, \quad R_6 = V_1 - 0.1.$$

By forming the system, we have

$$\begin{cases} R_1(\frac{1}{2}) = 0, \\ R_2(\frac{1}{2}) = 0, \\ R_3(\frac{1}{2}) = 0, \\ R_4 = 0 \to T_1 = 0.1, \\ R_5 = 0 \to I_1 = 0, \\ R_6 = 0 \to V_1 = 0.1. \end{cases}$$

By substituting the values R_4, R_5, R_6 into R_1, R_2, R_3, we write

$$\begin{cases} 3.1 - 0.991T_2 - 0.0027(\frac{1}{2}T_2 + 0.1)(\frac{1}{2}V_2 + 0.1) - 0.001I_2 = 0, \\ 0.0027(\frac{1}{2}T_2 + 0.1)(\frac{1}{2}V_2 + 0.1) - 1.15I_2 = 0, \\ 1.5I_2 - 2.2V_2 - 0.24 = 0. \end{cases}$$

By solving the above nonlinear system, the remaining 3 unknowns, namely T_2, I_2, and V_2, are determined, so

$$T_2 = 0.018, \quad I_2 = 2.54, \quad V_2 = 1.28.$$

And finally

$$T(t) = \begin{pmatrix} 0.1 & 0.018 \end{pmatrix} T(t) = \begin{pmatrix} 1 \\ t \end{pmatrix} = 0.1 + 0.018t$$

$$I(t) = 2.54t \ , \ V(t) = 0.1 + 1.28t$$

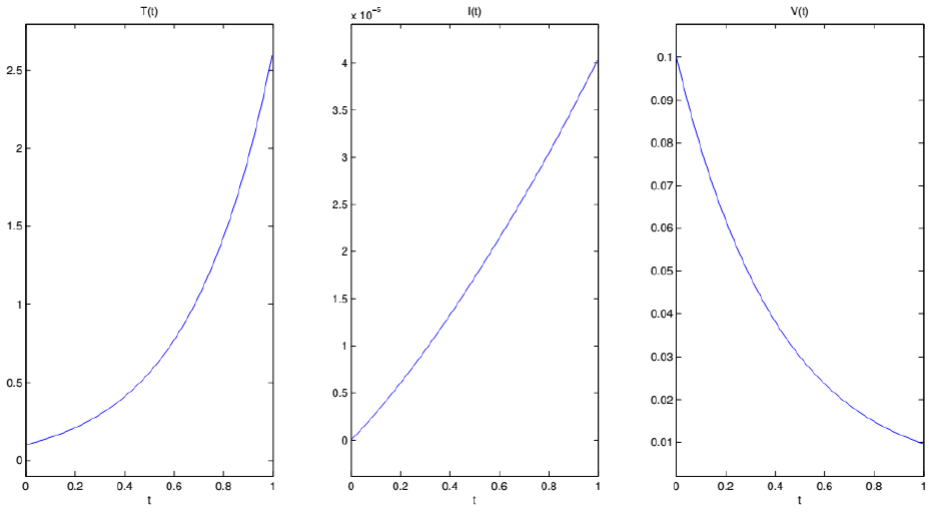

Figure 1. The approximate solutions $T(t)$, $I(t)$ and $V(t)$ for $v = 1$ and $M = 15$.

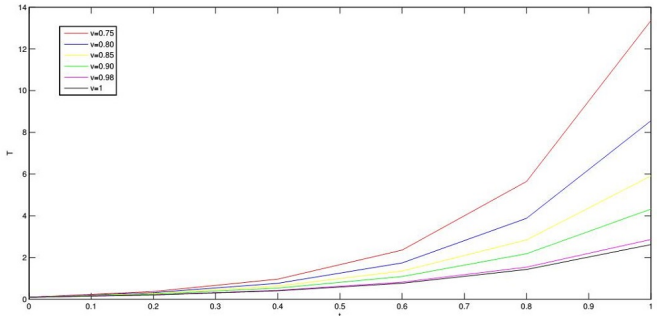

Figure 2. The approximate solution for $T(t)$ for different values of v.

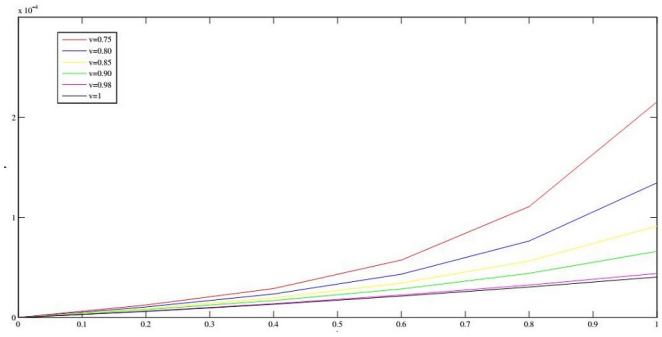

Figure 3. The approximate solution for $I(t)$ for different values of v

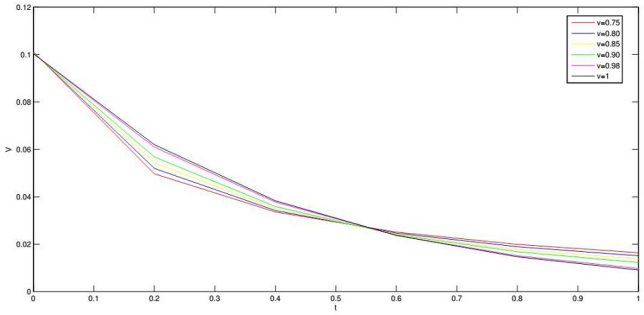

Figure 4. The approximate solution for $V(t)$ for different values of v

5 Conclusion

The fractional HIV infection model was solved utilizing the Tau method based on Bell polynomials. This approach involved initially introducing Bell polynomials, then deriving the operational matrix of the fractional derivative of order α, and subsequently approximating the model functions with Bell polynomials. This process led to a system of linear equations that can be efficiently solved using traditional techniques such as Newton's method. To demonstrate the simplicity, efficiency, and accuracy of the proposed method, a numerical example with $M = 1$ was provided. As shown in Figure 1, the accuracy of the solutions improves as M increases. Increasing the value of M along with the time interval results in even more precise outcomes. The graphs of $T(t)$, $I(t)$, and $V(t)$ for various values of v are displayed in Figures 2, 3, and 4, respectively. A notable advantage of our method is its straightforward computational process. We believe that the core concept introduced in this study will be further applied to solve similar nonlinear problems.

Authors' Contributions

All authors have the same contribution.

Data Availability

All data generated or analyzed during this study are included in this published article.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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