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Research article

Hub Number of Incidence and Power Graph

Abolfazl Bahmani[1], Ozra Naserian[2,*], Mohammad Reza Ghaemi[1], Mohammad Reza Bonabifard[1]

[1] Department of Mathematics, University of Zanjan, Zanjan, Iran

[2] Department of Mathematics, Zanjan Branch, Islamic Azad University, Zanjan, Iran

* **Corresponding author(s):** o.naserian@iau.ir

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Abstract

In graph theory, a set $H \subseteq V(G)$ is defined as a hub set if every pair of non-adjacent vertices outside H can be interconnected by a path that exclusively traverses through the internal vertices contained in H. The hub number of a graph G refers to the minimal cardinality of such a hub set, providing crucial insights into the structural connectivity of the graph. This paper delves into the exploration of the hub number across various graph structures, specifically focusing on incidence graphs and square graphs, both of which possess unique characteristics impacting their connectivity properties. We establish theoretical bounds for the hub numbers of these graphs, facilitating a clearer understanding of their structural complexities. Furthermore, we derive explicit values for the hub numbers of several special types of graphs, including path graphs, star graphs and complete graphs. Through rigorous analysis and evaluation, this study contributes to the broader field of connectivity in graphs by not only identifying the hub numbers for specific examples but also by proposing methodologies for their computation. These findings have important implications for applications in network design and graph optimization, enhancing the utility of hub sets in practical scenarios.

Keywords: Hub set, Connected dominating set, Incidence graph, Square graph

Mathematics Subject Classification (2020): 05C69, 05C76

1 Introduction

Let $G = (V(G), E(G))$ be a graph with a vertex set $V(G)$ of order $v = |V(G)|$ and an edge set $E(G)$. An edge e in G is denoted by its two endpoints, such as $e = \{u, v\}$, in other words, e connects two vertices u and v of G. For a fixed vertex $v \in V(G)$, the number of edges that have v as one of their endpoints is called the degree of v and is denoted by $deg(v)$.

A *subgraph* H of a graph G is a graph whose vertex set and edge set are subsets of those of G. If S is a subset of $V(G)$, then the subgraph of G induced by S, denoted by $G[S]$, is a graph with vertex set S that includes all edges of G connecting vertices of S. A *path* P_n of order n is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *cycle* of order n, denoted by C_n, is a non-trivial path with n vertices such that its first and last vertices are the same and no other vertex is repeated. A *Hamiltonian path (cycle)* of a graph G is a path (cycle) that visits each vertex of G exactly once. A *Hamiltonian graph* is a graph that possesses a Hamiltonian cycle. A graph with no cycle is called *acyclic*. A graph is *connected* if, for each pair of vertices, there exists at least one path that joins them. A tree is a connected acyclic graph. A *star graph* of order n, denoted by S_n, is a tree with one vertex of degree $n-1$ and the other $n-1$ vertices having degree 1. A rooted tree is a tree T with a specified vertex v, called the root of T. A *wheel graph* of order n, denoted by W_n, is a simple graph $(n \geq 4)$, formed by connecting a central vertex to all vertices of C_{n-1}. A *complete graph* of order n, denoted by K_n, is a graph such that all possible pairs of vertices are connected. The *distance* between two vertices u and v in a graph G, denoted by $d_G(u,v)$, is the length of shortest path between u and v. A homomorphism of a graph G into another graph H is a mapping $f : V(G) \longrightarrow V(H)$ such that $(f(u), f(v)) \in E(H)$ for all $(u,v) \in E(G)$. A bijective homomorphism is called an isomorphism. The *eccentricity* of a vertex v is defined as $max\{d_G(u,v) : u \in V(G)\}$.The *Radius* of graph G, denoted by $r(G)$, and the *diameter* of graph G, denoted by $d(G)$, are the minimum and maximum eccentricities over the set of all vertices of G, respectively. A subgraph H of G is called an *isometric subgraph*, if for every pair of vertices u and v in H, we have $d_H(u,v) = d_G(u,v)$. In a graph G, a set $S \subseteq V(G)$ is called *dominating set* if every vertex not in S has a neighbor in S. In this paper, we follow the terminology given in [5].

Definition 1. *[6] An incidence graph of a graph G, denoted by $I(G)$, is defined such that its vertex set is*

$$V(I(G)) = \{(v,e) : v \in V(G), e \in E(G), where\ v\ is\ incident\ to\ e \in G\}.$$

In $I(G)$, a pair of vertices (u,e) and (v,f) (where (u,e), $(v,f) \in V(I(G))$) forms an edge of $I(G)$ if only if at least one case of following conditions holds:

a) $u = v$,

b) $e = f$,

c) $(u,v) = e$,

d) $(u,v) = f$.

As an example, the incidence graph of C_3 is given below:

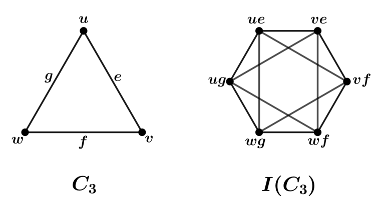

Figure 1. Incidence graph of cycle C_3

Definition 2. *[1] Let G be a graph. The square graph of G, denoted by G^2, is a graph with the vertex set $V(G)$ where two vertices u and v are adjacent if and only if $d_G(u,v) \leq 2$. In the same way, the k-th power graph G, denoted by the G^k, is a graph with vertex set $V(G)$ in which two vertices u and v are adjacent if and only if $d_G(u,v) \leq k$.*

Clearly that if a graph has diameter d, then its d-th power is a complete graph. The following is an example for G^2:

Let $H \subseteq V(G)$ and $u, v \in V(G)$. An H-path between u and v is defined as a path where all intermediate vertices are from H. (This definition includes the degenerate cases, where the path consists of a single edge uv or a single vertex u if $u = v$; which are referred to as trivial H-paths.) A set $H \subseteq V(G)$ is called a hub set of G if it satisfies the property that, for any $u, v \in V(G) \setminus H$, there exists an H-path in G connecting u and v. The hub number of G, denoted by $h(G)$, is defined as the minimum size of a hub set in G.The connected hub number $h_c(G)$ of a connected graph G is the smallest order of a connected subgraph H of G such that any two non-adjacent vertices of $G \setminus H$ are joined in G by a path with all internal vertices in H. The connected hub number of G, denoted by $h_c(G)$, is the minimum size of a connected hub set in G [2], [3]. First, we review some known necessary results.

Theorem 1. *[3] For a connected graph G, $h(G) \leq \gamma_c(G)$.*

Theorem 2. *[4] If G is a graph with diameter $d(G)$, then $d(G) - 1 \leq h(G)$.*

Theorem 3. *[4] G is a complete graph if only if $h(G) = 0$.*

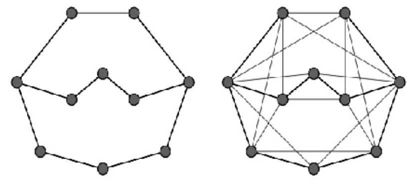

Figure 2. The square of G

2 Hub Number of Incidence Graphs

Let T be a tree and G be a graph. In this section, we establish some relations between the two parameters $h(T)$ and $h(I(T))$, present some bounds for $h(I(G))$ and $h(G^2)$. In particular, we compute the hub number of some incidence graphs.

Theorem 4. *If G is a simple graph with maximum degree $\Delta(G)$ and $I(G)$ has order m, then $h(I(G)) \leq m - \Delta(G)$.*

Proof. Suppose v is a vertex of graph G with maximum degree $\Delta(G)$ and the edges $e_1, e_2, e_3, \ldots, e_\Delta$ be located on incident to it. The set

$$S = \{(v, e_1), (v, e_2), \ldots, (v, e_\Delta)\}$$

form a cluster in the graph $I(G)$. It is clear that, the set $H = V(I(G)) \setminus S$ is a hub set for $I(G)$. So, we have

$$h(I(G)) \leq m - |S| = m - \Delta(G).$$

□

Lemma 1. *Let T be a tree, then $I(T)$ has a subtree T' such that $T \simeq T'$ and $V(T')$ is a connected dominating set for $I(T)$.*

Proof. Consider a rooted tree T and label the vertices and edges as follows, label the root as ${v_1}^0$ and the vertices adjacent to ${v_1}^0$ as the first level adjacent to the root, denoted as ${v_1}^1, {v_2}^1, \ldots$. Similarly, the vertices in the i-th level are labeled ${v_1}^i, {v_2}^i, \ldots$. By this labeling, edges connecting vertices between consecutive levels and the edge connecting ${v_t}^i$ and ${v_j}^{i-1}$ is labeled as ${e_{jt}}^i$. Now consider the mapping $f : V(T) \longrightarrow V(I(T))$ defined by:

$$f({v_j}^i) = \begin{cases} ({v_j}^i, {e_{lj}}^i) & \text{if } i \neq 0, j \neq 1 \\ ({v_1}^0, {e_{11}}^1) & \text{if } i = 0, j = 1 \end{cases}$$

where l is the index number of the vertex ${v_l}^{i-1}$ in level $i-1$ connecting ${v_j}^i$. This mapping is an injective graph homomorphism, so the image of T under f in $I(T)$ is a tree, denoted by T', and we have $T \simeq T'$. We need to show that $V(T')$ is a dominating set for $I(T)$. Suppose $x = ({v_s}^i, {e_{ts}}^i) \notin V(T')$ is an arbitrary vertex of $V(I(T))$. Consider $y = ({v_s}^i, {e_{ls}}^i)$ as the image of the vertex ${v_s}^i \in T$ under f. Clearly, $y = ({v_s}^i, {e_{ls}}^i) \in V(T')$, and since x and y are adjacent, $V(T')$ is a connected dominating set for $I(T)$. □

Theorem 5. *If T is a tree, then $h(I(T)) \leq n$.*

Proof. By the previous lemma and Theorems 1, we have

$$h(I(T)) \leq n.$$

□

Theorem 6. *If G is a graph of order n and has a Hamiltonian path, then $h(I(G)) \leq n$.*

Proof. Suppose G has a Hamiltonian path S in the form
$v_1, v_2, v_3, \ldots, v_{n-1}, v_n$. For each $1 \leq i \leq n-1$, $e_i = \{v_i, v_{i+1}\}$. We define a function $f : V(G) \longrightarrow V(I(G))$ as follows:

$$f(v_i) = \begin{cases} (v_i, e_i) & 1 \leq i \leq n-1, \\ ((v_n, e_{n-1})) & i = n. \end{cases}$$

By f, $I(G)$ has a path R corresponding to S in the form of

$$(v_1,e_1),(v_2,e_2),\dots,(v_{n-1},e_{n-1})(v_n,e_{n-1}).$$

Since the vertices of $I(G)$ are of the form of (v_i,e_w), $(1\leq i\leq n)$ it is clear that each vertex of $I(G)$ is adjacent to at least one vertex from the path R. Therefore, the vertices of R form a connected dominating set for $I(G)$ and consequently, by Theorem 1 we have, $h(I(G))\leq |V(R)|=n$. □

Theorem 7. *Suppose that $G={P_n}^2$ and $n\geq 3$, then*

$$h_c(G)=h(G)=[\frac{n}{2}]-1.$$

Proof. Let P_n is a path of order n that

$$V(P_n)=\{v_1,\dots,v_n\},\ E(P_n)=\{e_i=\{v_i,v_{i+1}\}\mid 1\leq i\leq n-1\}.$$

It is clear that the maximum distance between any two vertices in G is distance between v_1 and v_n. Therefore $d(G)=[\frac{n}{2}]$ and according to Theorem 2, we have $[\frac{n}{2}]-1=d(G)-1\leq h(G)$. If n is even, we consider the set

$$S=\{v_3,v_5,v_7\dots,v_{n-1}\},$$

and if n is odd, we consider the set

$$S=\{v_3,v_5,v_7\dots,v_{n-2}\}.$$

It can be easily seen that the set S is a connected dominating set for the graph G. According to Theorem 1,

$$h(G)\leq |S|=[\frac{n}{2}]-1.$$

Consequently, we obtain $h_c(G)=h(G)=[\frac{n}{2}]-1$. □

Theorem 8. *If $n\geq 3$ and $G=I(P_n)$, then $h(G)=n-2$.*

Proof. It is easy to see that $I(P_n)=P^2_{2n-2}$. Thus, by previous theorem, we have

$$h(I(P_n))=h(P^2_{2n-2})=[\tfrac{2n-2}{2}]-1=n-2.$$

□

Theorem 9. *If $G=I(S_n)$ and $n\geq 4$, then*

a) $h_c(G)=h(G)=1,$

b) $h(S_n^2)=0.$

Proof.

a) Let S_n be a star graph of order n such that $V(S_n)=\{v_1,v_2,\dots,v_{n-1},v\}$, $deg(v_i)=1$ (for $1\leq i\leq n-1$), $deg(v)=n$ and $E(S_n)=\{e_1,e_2,\dots,e_{n-1}\}$ that $e_i=\{v,v_i\}$. It is clear that, $S=\{(v,e_1)\}$ is a connected dominating set for G. Therefore, according to the Theorem 1, we have $h(G)\leq\mid S\mid=1$. On the other hand, since the graph G is not complete, by Theorem 3, $h(G)\geq 1$. So $h_c(G)=h(G)=1$.

b) Since the graph S_n^2 is a complete graph, by Theorem 3, $h(S_n^2)=0$.

□

Theorem 10. *If $G=I(K_n)$ and $n\geq 4$, then $h_c(G)=h(G)\leq n-1$.*

Proof. Suppose K_n is a complete graph of order n. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(K_n) = \{e_1, e_1, \ldots, e_p\}$ where $p = \frac{n(n-1)}{2}$. Assume that the edge $e_1, e_2, \ldots, e_{n-1}$ are incident to vertex v_1. Consider the set

$$S = \{(v_1, e_1), (v_1, e_2), (v_1, e_3), \ldots, (v_1, e_{n-1})\}.$$

This set S is a dominating set for G. Since each vertex v_i (for $2 \leq i \leq n$) is connected to v_1 by at least one edge, thus every vertex in the incidence graph $I(K_n)$ is adjacent to at least one vertex in S. By Theorem 1, we conclude that

$$h(G) \leq | S | = n - 1.$$

□

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