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Approximation Solution to the Heat Conduction Equation in a Rectangular Channel Influenced by Airflow using the Chebyshev Pseudo-Spectral Method

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Abstract

In this paper, the approximation of the solution to the one-dimensional convection-diffusion equation is studied as a model for heat transfer in a rectangular channel influenced by airflow. First, the convection-diffusion equation is derived using basic principles of conservation, including convection and diffusion. Then, the existence and uniqueness of the solution to this equation are briefly discussed, taking into account the properties of the convective and diffusive coefficients, boundary conditions, and initial conditions. Numerical solution methods for the problem have been explored by researchers, leading to various approaches. As examples, [7–10, 14] used the Chebyshev pseudo-spectral method for spatial discretization and the fourth-order Runge-Kutta (RK4) method for temporal discretization to solve this equation. The numerical characteristics of this method, including accuracy, stability, and convergence rate, are analyzed using eigenvalue analysis of the system and stability regions. The results obtained include temperature distribution, absolute error, and a three-dimensional analysis of the temperature distribution in space and time. Additionally, the impact of the time step on the stability of the numerical method has been investigated, and it is shown that the proposed method can achieve desirable accuracy and stability with proper parameter adjustments. This study confirms the effectiveness of the Chebyshev pseudo-spectral method in solving dynamic problems such as heat transfer and related applications providing a foundation for using this method in more complex problems.

Keywords: Convection-Diffusion equation, Existence and uniqueness of solution, Chebyshev Pseudo-Spectral method, Numerical accuracy and stability

Mathematics Subject Classification (2020): 65-XX, 65Mxx, 65M06

1 Introduction

Partial differential equations play a crucial role in modeling physical, chemical, and biological phenomena. Among these equations, the convection-diffusion equation is of particular importance because it can describe the processes of mass, heat, or energy transfer in various environments. This equation combines two fundamental transport mechanisms: convection (displacement) and diffusion (molecular dispersion), and it has widespread applications in fluid dynamics, atmospheric flows, heat transfer, and even biomedicine. The precise formulation of this equation requires an understanding of the fundamental principles of conservation and the analysis of transport flows.

Moreover, the existence and uniqueness of solutions to partial differential equations is a fundamental topic in applied mathematics, ensuring that modeling leads to stable and reliable solutions.

The analytical solution of the convection-diffusion equation is often challenging or even impossible due to its nonlinear nature and the complexity of boundary conditions. Therefore, numerical methods have been proposed as powerful tools for solving these equations. Numerical methods for approximating the solution of the convection-diffusion equation, due to its inherent complexities, such as the interplay between the two processes with distinct characteristics, are often associated with challenges. One of the most important challenges is numerical instability, which arises from the mismatch between time and spatial discretization methods. Additionally, in real-world problems, there is a need for high accuracy in computing derivatives and reducing computational costs, which has led to the development of more advanced methods for solving this equation. Initially, Fick (1855) introduced his law of diffusion, which forms the foundation for modeling diffusion processes in various materials. This law states that the rate of diffusion of a substance depends on the concentration gradient and is directly used in convection-diffusion equations [2]. In 1822, Fourier introduced the heat conduction equation, which provided a framework for heat transfer in materials and later influenced the formulation of convection-diffusion equations. This equation, modeling heat transfer in a solid body, became the basis for developing similar equations in other fields [13]. In the 1940s, Crank and Nicolson introduced their semi-implicit numerical method for solving diffusion equations, which, due to its stability and high accuracy, was also applied to solve convection-diffusion equations. This method allows for more accurate solutions of partial differential equations by reducing numerical errors [4]. In 2005, Morton and Mayers presented analyses on the stability and convergence of numerical methods for solving partial differential equations, including the convection-diffusion equation. They demonstrated how the choice of numerical methods and boundary conditions can impact solution accuracy and stability [17].

In the realm of mathematical analysis, Gilbarg and Trudinger (1977) used Sobolev spaces and weak form methods to provide proofs for the existence and uniqueness of solutions to partial differential equations, including the convection-diffusion equation. This work became a foundation for more precise analyses in boundary and initial value problems [6]. In the 1990s, Shyy, in his book, examined various numerical methods for solving convection-diffusion equations. During this period, stabilization methods and accurate solutions for convection-dominated problems were developed, which were particularly useful in complex geometrical issues [20]. In recent decades, numerical methods such as Meshfree Methods and Smoothed Particle Hydrodynamics (SPH) have been introduced to solve convection-diffusion equations for problems with complex geometries [16]. These methods have found wide applications in solving problems with intricate geometries and in numerical simulations in engineering.

Lastly, in recent years, machine learning and neural networks have gained attention for solving partial differential equations, particularly convection-diffusion equations. These methods provide faster and more accurate solutions to complex problems using data-driven models [18]. These advancements reflect significant progress in understanding and numerically solving the convection-diffusion equation, and future research will continue to improve methods and algorithms for emerging problems.

In this research, the convection-diffusion equation is first derived based on physical principles, and then the conditions for the existence and uniqueness of its solution are analyzed. Subsequently, using numerical methods such as the Chebyshev pseudo-spectral method, a solution strategy for this equation is proposed, and the accuracy and stability of the method are examined. In recent years, spectral methods have become one of the popular tools for the numerical solution of differential equations due to their high accuracy and rapid convergence rates. Among these methods, the Chebyshev pseudo-spectral method, based on Chebyshev polynomials and Chebyshev nodes, performs exceptionally well in approximating spatial derivatives. This method calculates the first and second derivatives with high accuracy using the Chebyshev derivative matrix, making it highly suitable for problems with continuous boundaries and smooth conditions.

The objective of this paper is to provide a comprehensive framework for the mathematical and numerical analysis of the convection-diffusion equation and to explore its applications in practical problems. Specifically, the heat diffusion equation in a rectangular channel subjected to air flow is approximated using the Chebyshev pseudo-spectral method.

2 Problem Definition and Equation Derivation

In modeling phenomena involving the transfer of mass, heat, or any conserved quantity in a physical environment, the convection-diffusion equation serves as a key tool. This equation is derived based on the principles of mass conservation and combines two fundamental processes: diffusion (which depends on the concentration gradient) and convection (which depends on fluid motion). The primary objective is to derive this equation from these fundamental principles (the law of conservation) and provide a framework for the initial and boundary conditions, which are essential for the analytical or numerical solution of the equation.

Theorem 1 (Law of Conservation of Mass (or Energy)). *[5, 12, 19] Let C (x, t) be a scalar function that is discontinuous and dependent on space $x \in \Omega \subseteq \mathbb{R}^n$ and time $t \geq 0$ .representing a quantity (such as concentration or temperature) in a physical system. If is a compact region with boundary, the law of conservation of mass in its integral form is expressed as:*

$$\frac{d}{dt}\int_{\Omega} = -\int_{\partial\Omega} J \cdot n\mathrm{d}s + \int_{\Omega} s\mathrm{d}x,$$

*where **J** is the fluxand and **n** is the unit normal vector to the boundary $\partial\Omega$ and **S** is the rate of generation or destruction of the quantity per unit volume.*
This law represents the balance of the quantity C inside the domain Ω considering both the influx/outflux through the boundary $\partial\Omega$ and any sources or sinks inside the domain.

Lemma 1 (Differential Form of the Conservation Law). *[5, 12, 19] By applying the Gauss-Divergence theorem, the law of conservation of mass can be easily converted into its differential form:*

$$S = \frac{\partial c}{\partial t} + \nabla \cdot J,$$

where:
S represents the source term (rate of generation or destruction),
$\frac{\partial c}{\partial t}$ is the time rate of change of the quantity C,
$\nabla \cdot J$ is the divergence of the flux J.

Theorem 2 (Modeling of Total Flux). *[12, 19] Assume that the total flux J consists of two components:*
Convective flux: $J_{conv} = u \cdot C$ where $u \in \mathbb{R}^n$ is the fluid velocity vector.
Diffusive flux: $J_{diff} = -D\nabla C$ where D is the diffusion coefficient and ∇C represents the gradient of C.
By combining these two, the total flux is defined as follows:

$$J = D\nabla C - uC.$$

Theorem 3 (Advection-Diffusion Equation). *[5, 12, 19] By substituting into the mass conservation equation, the advection-diffusion equation is derived as follows:*

$$S = \frac{\partial c}{\partial t} + \nabla \cdot (uC) - \nabla \cdot (D\nabla C).$$

Lemma 2 (Special Case). *[5, 12, 19] If the fluid flow u is steady and incompressible $(D \cdot u) = 0$, and D is assumed to be a homogeneous and constant coefficient, the equation simplifies to:*

$$\frac{\partial c}{\partial t} + u \cdot \nabla C = D \cdot \nabla^2 C + s.$$

This is the simplified form of the advection-diffusion equation under these specific conditions.

Definition 1 (Definition of Boundaries and Initial Conditions). *To solve this equation, the following boundary and initial conditions are defined:*

- ***Initial Condition:*** *Specifies the concentration distribution at the initial time $t = 0$, $C(x,0) = C_0(x)$, where $C_0(x)$ is the given initial concentration.*

- ***Boundary Conditions:*** *Depending on the physical problem, different types of boundary conditions can be applied:*

  – ***Dirichlet Condition:*** *Specifies the concentration at the boundary:*

$$C(x,t) = C_b(x,t), \quad \forall x \in \Omega.$$

  – ***Neumann Condition:*** *Specifies the flux at the boundary:*

$$\frac{\partial c}{\partial n} = h(x,t).$$

– ***Robin Condition:*** *A combination of Dirichlet and Neumann conditions:*

$$ac + b\frac{\partial c}{\partial n} = h(x,t).$$

These conditions ensure a well-posed problem for solving the advection-diffusion equation.

3 Existence and Uniqueness of the Advection-Diffusion Equation

One of the fundamental aspects of analyzing partial differential equations (PDEs), such as the advection-diffusion equation, is examining the conditions for the existence and uniqueness of solutions. This analysis ensures that the proposed mathematical model accurately represents the behavior of the physical system. To achieve this, mathematical tools such as existence and uniqueness theorems (e.g., Banach's fixed-point theorem or energy methods) are employed.

The objective of this section is to explore the necessary and sufficient conditions for the existence and uniqueness of the solution within the framework of the advection-diffusion equation. This exploration considers the specific form of the equation, the problem domain, and the boundary and initial conditions. Proving the existence and uniqueness of solutions for the advection-diffusion equation requires precise mathematical tools rooted in the theory of PDEs and functional analysis. This topic has been the focus of research by various scholars, and references [1, 3] offer further studies on the matter.

In the following, we outline a general approach to proving the existence and uniqueness of the solution:

3.1 Weak Formulation

To begin, we convert the advection-diffusion equation into its weak form. This formulation is particularly useful in mathematical analysis and numerical methods, such as the finite element method. For test functions $v \in H1(\Omega)$, the weak form is expressed as follows:

$$\int_{\Omega} \frac{\partial c}{\partial t} v \mathrm{d}x + \int_{\Omega} (u \cdot \nabla c) v \mathrm{d}x - \int_{\Omega} D\nabla C \cdot \nabla v \mathrm{d}x = \int_{\Omega} Sv \mathrm{d}x,$$

where:

- H1(Ω) is the Sobolev space of functions with square-integrable derivatives.
- The first term represents the time evolution of the concentration.
- The second term accounts for advection (transport by fluid flow).
- The third term represents diffusion, using integration by parts to ensure a well-posed weak formulation.

This weak form provides a foundation for applying the theory of Hilbert spaces, which is essential in proving the **existence** and **uniqueness** of solutions.

3.2 Proof Techniques

3.2.1 Existence

By formulating the problem in the Hilbert space $H_1(\Omega)$, we can use functional analysis techniques, such as:

- **Lax-Milgram theorem:** for linear equations, this theorem guarantees the existence of a solution in a Hilbert space, provided that the operator is bilinear.
- **Galerkin Method:** by approximating the solution in a finite-dimensional space, a sequence of solvable problems is created. Then, using density theorems such as Banach-Alaoglu and Rellich-Kondrachov compactness, the convergence of the sequence to a weak solution is proven.
- **Energy estimates:** by defining a suitable energy function and examining its bounds, the existence of a solution for the equation can be confirmed.

Using these tools, we analyze the properties of the weak solution and derive sufficient conditions for its existence and uniqueness.

3.2.2 Uniqueness

Uniqueness is typically proven through the maximum principle or monotonicity properties.

- **Maximum Principle:** For advection and diffusion equations, it can be shown that the value of the solution at any point cannot exceed the boundary or initial values.
- **Energy Gradient Analysis:** By demonstrating that the difference between two hypothetical solutions is zero, uniqueness is proven.

3.2.3 Boundary and Initial Conditions

For proving existence and uniqueness, boundary and initial conditions are important:

- **Dirichlet Conditions:** For fixing the value on the boundary.
- **Neumann Conditions:** For fixing the flux on the boundary.
- **Robin Conditions:** A combination of Dirichlet and Neumann conditions. For each type of condition, the proof must show that the solution fits within the framework of the physical problem.

3.2.4 General Existence and Uniqueness Theorem

If the following conditions are satisfied:

- The diffusion coefficient $D > 0$ in Ω
- The velocity vector u is continuous and has bounded divergence.
- The source function S and boundary conditions are continuous and bounded.

Then the convection-diffusion equation has a weak solution $C \in H_1(\Omega)$that is unique.

3.2.5 Advanced Proof Tools

For more complex cases (such as nonlinear boundary conditions), the following methods may be used:

- **Fixed Point Theory:** This is applied to prove the existence of solutions in nonlinear problems. Fixed-point theorems (such as Banachs Fixed Point Theorem or Schauders Fixed Point Theorem) are often used in the context of existence proofs.
- **Semigroup Theory:** This method is used to analyze time-dependent problems, especially in parabolic partial differential equations like the convection-diffusion equation. It provides a framework for proving the existence and uniqueness of solutions in infinite-dimensional spaces.
- **Monotonicity Methods:** Monotonicity Methods: These methods are used to prove uniqueness in nonlinear problems. If the operator associated with the problem is monotonic, one can use these methods to demonstrate that a solution exists and is unique, often leveraging the monotonicity to establish convergence.

These advanced methods allow for handling more complex scenarios and are particularly useful when traditional techniques do not suffice.

4 Numerical Methods for Solving Convection-Diffusion Equations

Due to the complexities involved in solving the equation accurately, the development of efficient and stable numerical methods for solving it is of great importance. The numerical solution methods for the problem have been the subject of research, leading to various numerical methods. For example, in [7, 8], one-dimensional convection-diffusion differential equations with partial derivatives are introduced. In the next step, among the numerical methods for solving the convection-diffusion equation, the Chebyshev spectral method is selected, and its numerical solution, stability, and accuracy are examined [8]. Then, in this study, an example of approximating the solution to the heat diffusion equation in a rectangular channel under the influence of air flow using the Chebyshev spectral method is presented. The numerical results show that the proposed method is capable of solving the convection-diffusion equation with high accuracy and suitable numerical stability. The results indicate that the combination of the Chebyshev spectral method and the RK4 time discretization achieves the desired accuracy and numerical stability. Additionally, this study demonstrates the effectiveness of this method in solving similar problems, such as modeling atmospheric flows or mass transfer in industrial systems.

5 Implementation of the Chebyshev Spectral Method for Approximating the Solution of the Convection-Diffusion Equation

The Chebyshev spectral method is a powerful numerical technique for solving partial differential equations (PDEs) that offers high accuracy and rapid convergence rates. This method is based on Chebyshev nodes and Chebyshev polynomials, and due to the use of the Chebyshev derivative matrix, it has the capability to accurately approximate spatial derivatives. The steps for using this method to approximate the solution of the convection-diffusion equation are generally outlined as follows:

5.1 Spatial Discretization using the Chebyshev Derivative Matrix

We consider the one-dimensional convection-diffusion equation subject to the following initial and boundary conditions:

$$\begin{aligned} &\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=r\frac{\partial^2 u}{\partial x^2}, && 0<x<L, \qquad t>0,\\ &u(x,0)=f(x), && 0\le x\le L,\\ &u(0,t)=g_0(t), && t\ge 0,\\ &u(L,t)=g_1(t), && t\ge 0, \end{aligned} \tag{1}$$

where $u(x,t)$ is the dependent variable (such as temperature or concentration), c is the convective velocity, r is the diffusion coefficient, and L is the length of the domain. We focus on a semi-discrete method obtained by discretizing equation (1) with respect to the spatial variable using the Chebyshev spectral method. Below, the first-order $(n+1)\times(n+1)$ Chebyshev derivative matrix associated with the collocation points (in space) is denoted by D.

$$0=x_0<x_1<x_2<\cdots<x_n, \qquad x_j=[1-\cos(\frac{j\pi}{n})] \qquad j=0,1,\dots,n. \tag{2}$$

The main feature of these nodes is their non-uniform distribution, which leads to a higher density of points near the boundaries. This distribution of nodes (collocation points) enhances the approximation accuracy, especially at the boundaries.

Now, the Chebyshev derivative matrix D is used for approximating the first and second-order derivatives.

$$(DU)_j\approx\left.\frac{\partial u}{\partial x}\right|_{x_j} \qquad (D^2u)_j=\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_j} \tag{3}$$

It is worth mentioning that the Chebyshev derivative matrix D is the main tool for numerically approximating first and second-order derivatives in the Chebyshev spectral method. This matrix is obtained using Chebyshev nodes and the orthogonality properties of Chebyshev polynomials. The advantages of the Chebyshev derivative matrix D include exponential convergence, the lack of requirement for uniform grids, and its application in solving PDEs.

5.2 The Semi-Discrete Model

If we disregard the approximation error and represent $v_i(t)$ as an approximation to $u(x_i,t)$ by substituting the Chebyshev derivative matrix D into the convection-diffusion equation (1), the semi-discrete system is obtained as follows:

$$\begin{aligned} &\frac{\mathrm{d}v}{\mathrm{d}t}=Av+b(t),\\ &V(0)=[f(x_1),\cdots,f(x_n)]. \end{aligned} \tag{4}$$

In this equation,

- The system matrix resulting from spatial discretization $A = \gamma D2 - CD$.
- B(t)Time-dependent boundary condition vector.
- V(t) Vector of function values at internal grid points.

5.3 Time Integration (Time Discretization)

The fourth-order Runge-Kutta (RK4) method is used for time integration:

$$v^{n+1} = v^n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{5}$$

where k_1, k_2, k_3, k_4, are obtained from the intermediate stages of the (RK4) method.

6 Stability of the Method

Numerical stability is an important aspect when solving partial differential equations (PDEs), and it must be carefully considered during the design and implementation of numerical methods. In the Chebyshev spectral method, stability depends on the time step, the eigenvalues of the system matrix A, and the stability region of the time integration method. By discretizing the convection-diffusion equation spatially using the Chebyshev derivative matrix, the system matrix $A = \gamma D_2 - CD$ is obtained, which approximates the first and second derivatives. The eigenvalues λ_i of this matrix determine the spatial behavior of the system. For numerical stability, the scaled eigenvalues $z = \lambda \Delta t$ must lie within the stability region of the time integration method.

In this study, the fourth-order Runge-Kutta (RK4) method is used for time discretization. The stability region of this method is defined as a disk-like region in the complex plane, and the scaled eigenvalues $z_i = \lambda \Delta t$ should lie within this region. The stability condition for the numerical method depends on the proper choice of the time step, ensuring that $\lambda_i |\Delta t|$ does not exceed the radius of the stability region.

In the stability analysis of this paper, the eigenvalues of matrix A were computed, and the position of the scaled eigenvalues relative to the RK4 stability region was examined. The results show that a suitable choice of time step ensures that all scaled eigenvalues remain inside the stability region, maintaining system stability. However, excessively large time steps may cause the scaled eigenvalues to move outside the stability region, leading to numerical instability, particularly when convection or diffusion components dominate.

Finally, it can be concluded that the Chebyshev spectral method, combined with RK4 time integration, offers high numerical accuracy and stability when an appropriate time step is chosen. The analysis of eigenvalues and the stability region is an essential part of designing this numerical method, ensuring stable behavior and reliable results. For further reading on this subject, refer to [8].

7 Numerical Example

The heat transfer problem in a rectangular channel with a length of $L = 1$ meter is considered. The governing equation for this problem is the one-dimensional convection-diffusion equation, given as follows:

$$\begin{aligned} &\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = r\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < 1,\ t > 0, \\ &T(x,0) = \sin(\pi x), \qquad 0 \le x \le 1, \\ &T(0,t) = g_0(t), \qquad t \ge 0, \\ &T(1,t) = g_1(t), \qquad t \ge 0, \end{aligned} \tag{6}$$

where $T(x,t)$ is the temperature as a function of space and time and $u = 2$ is the flow velocity and $\alpha = 0.01$ is the thermal diffusivity coefficient. The objective of this problem is to compute the temperature distribution $T(x,t)$ over the interval $[0, L]$ and analyze its behavior over time t using the Chebyshev pseudo-spectral method. In this study, the one-dimensional convection-diffusion equation is used as a heat transfer model in a channel and is solved using the Chebyshev pseudo-spectral method combined with fourth-order Runge-Kutta (RK4) time integration. The approximation analysis of the solution is presented in three aspects: accuracy, stability, and physical behavior of the results.

In the first part, the accuracy of the Chebyshev pseudo-spectral method is very high due to the use of Chebyshev nodes and the Chebyshev differentiation matrix. A comparison of the numerical results with the analytical solution $T_{\text{exact}}(x,t) = e^{-\alpha\pi^2 t}\sin(\pi x)$ shows an extremely small absolute error: $|T_{\text{exact}}(x,t) - T_{\text{numeric}}(x,t)| =$ error at $t = 0$ and absolute error is less than 10^{-4}. This high accuracy is due to the exponential convergence of the Chebyshev pseudo-spectral method and the effectiveness of the fourth-order Runge-Kutta method in time integration.

In the second part, the numerical stability of the method is examined. The analysis of the eigenvalues of the system matrix A and the scaled values $z = \lambda\Delta t$ shows that the time step $\Delta t = 0.01$ lies within the stability region of the RK4 method. All values of z are located inside the stability region of RK4 (a disk-like region in the complex plane). This analysis ensures that the numerical method is stable, with no abnormal oscillations or divergence observed in the results.

In the third part, the physical behavior of the problem is analyzed. The temperature distribution $T(x,t)$ is computed at different time instances $t = 0, 0.025, 0.05, 0.1$. The results show that at t=0, the temperature is distributed according to the initial condition of the problem, $T(x,0) = \sin(\pi x)$. As time progresses, the temperature decreases and approaches the boundary conditions of the problem. The temperature reduction over time is due to the combined effect of convection and diffusion, which aligns with physical expectations. The 3D plot of T(x,t) clearly illustrates the temperature decrease over time and its convergence to the boundary conditions. The number of Chebyshev nodes n=20 is sufficient for this problem, yielding a very small error. The choice of time step $\Delta t = 0.01$ ensures a balance between stability and computational efficiency.

7.1 Analysis of the Graphs

Here, we analyze the four graphs generated in the corresponding code. These graphs include the temperature distribution over time, the absolute error, the stability region for the Runge-Kutta method, and the three-dimensional temperature distribution.

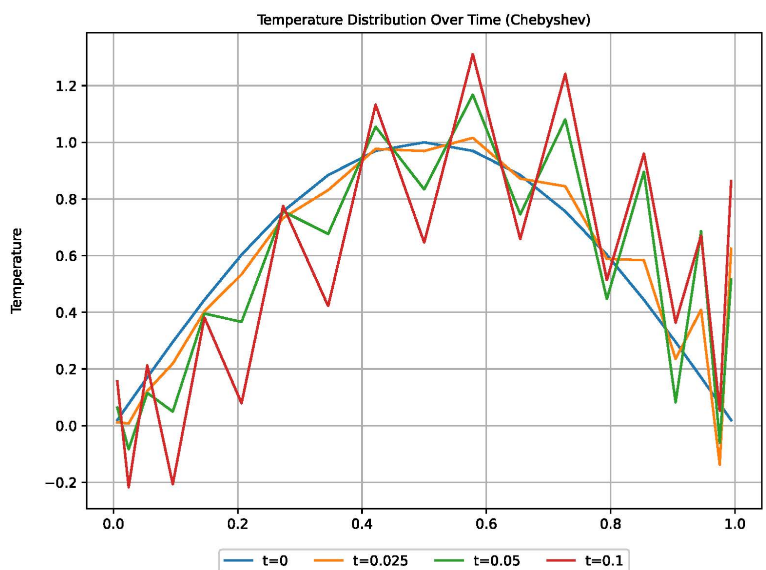

Figure 1. Temperature Distribution Over Time.

In Figure 1 shows the changes in temperature over the specified time interval. The horizontal axis (x) represents the spatial position in the domain, ranging from 0 to L (domain length), and the vertical axis (y) represents the temperature at different spatial points in the domain at different times.The different lines in the graph represent the temperature changes at various times ($t = 0$, $t = 0.025$, $t = 0.05$, $t = 0.1$). From the graph, it can be concluded that the temperature gradually changes over time, with these changes varying depending on the spatial location. These changes are caused by the effects of homogenization, diffusion, or convection in space and time, which are naturally modeled by the partial differential equation (PDE).

In Figure 2 shows the absolute error between the computational results (numerical solution) and the exact results (analytical solution). The horizontal axis (x) still represents the spatial position, and the vertical axis (y) represents the absolute error. This graph is specifically

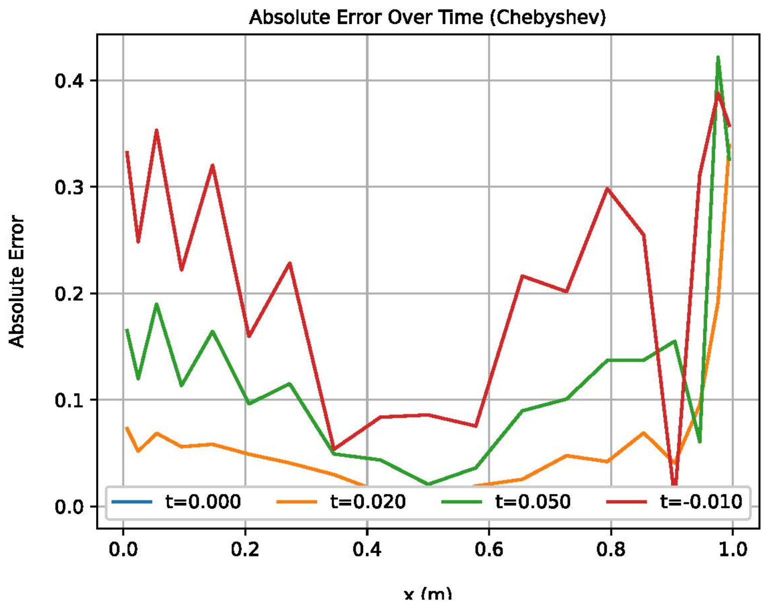

Figure 2. Absolute Error Over Time.

used to assess the accuracy of the numerical method employed.The graph shows that the error changes over time, and initially, the error may be larger, but it decreases as time progresses. This indicates the good performance of the numerical method, which gradually approaches a more accurate solution.

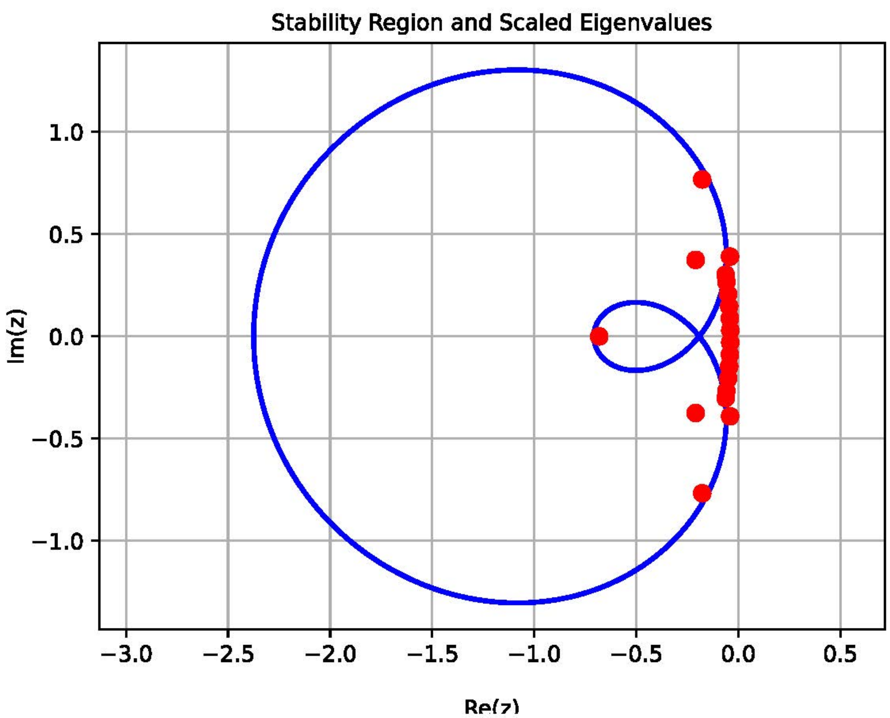

Figure 3. Stability Region for the Runge-Kutta Method.

In Figure 3 shows the stability region of the Runge-Kutta (RK4) method. The horizontal axis (Re(z)) represents the real part of the complex number, and the vertical axis (Im(z)) represents the imaginary part of the complex number. The stability region of the Runge-Kutta method is represented by the blue curve, which expresses the circular region of numerical stability in solving differential equations. In this graph, the eigenvalues of the system matrix are shown as red dots inside the stability region. These values represent the behavior of the system matrix at different time steps. If these eigenvalues fall outside the stability region, the numerical method may become unstable.

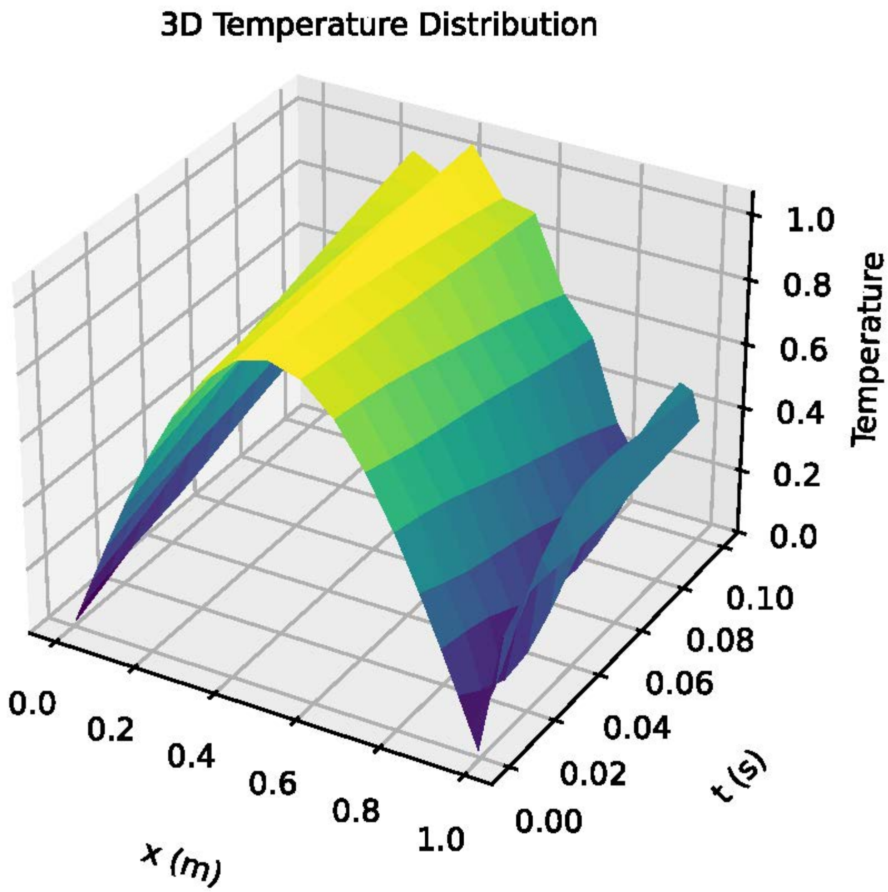

Figure 4. Three-Dimensional Temperature Distribution.

The Figure 4 is a three-dimensional plot that shows the temperature distribution in space and time. The horizontal axis (x) represents the spatial position. The vertical axis (y) represents time in the interval $(0, t_max)$. The third vertical axis (z) represents the temperature at different spatial points in the domain and at different times.

The graph clearly shows the variations in temperature over time and throughout the spatial domain. It can be observed that the temperature distribution gradually changes over time, and the temperature at different spatial points changes smoothly. This model clearly demonstrates the effects of heat transfer and similar physical phenomena.

8 Conclusion

In this paper, numerical solutions to heat transfer problems were addressed using Chebyshev spectral methods and the Runge-Kutta time integration technique. The main focus was on analyzing the temperature distribution over time and evaluating the performance of the numerical method in comparison to an exact analytical solution. The results showed that the numerical method gradually approached the exact solution, with error decreasing over time, confirming the accuracy and reliability of the chosen method. The use of Chebyshev spectral methods provided high accuracy in solving the spatial components of the problem, while the Runge-Kutta method offered an efficient and stable time solution. Moreover, stability analysis based on the eigenvalues of the system matrix and the stability region of the Runge-Kutta method indicated that the chosen time step and spatial discretization ensured numerical stability throughout the simulation. The absolute error analysis revealed the convergence behavior of the numerical solution, showing that as the simulation progressed, the error decreased. This indicates that the chosen method is appropriate for solving such problems with adequate accuracy. Finally, the 3D surface plot of the temperature distribution in space and time comprehensively displayed the dynamics of the heat transfer process.

In conclusion, the combination of Chebyshev spectral methods and Runge-Kutta time integration provides an effective and accurate approach to solving heat transfer problems, offering valuable insights into the system's behavior while maintaining numerical stability and accuracy. Future work may extend this method to more complex problems involving nonlinearity or multidimensional domains and enhance its performance.

Authors' Contributions

The contributions of each author to this study are as follows: Esmaeil Yousefi, conceptualized the research design, Mehdi Azhini, conducted data analysis, and Amir Mohammad Iranbodi, contributed to the literature review.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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