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On the Dynamics of the Family $ax^d(x-1)+x$

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Abstract In this paper we consider the dynamics of the real polynomials of degree $d+1$ with a fixed point of multiplicity $d \geq 2$. Such polynomials are conjugate to $f_{a,d}(x) = ax^d(x-1)+x,\ a \in \mathbb{R}\setminus\{0\},\ d \in \mathbb{N}$. Our aim is to study the dynamics $f_{a,d}$ in some special cases.

Keywords Multiplicity · Chaotic · Polynomial

Mathematics Subject Classification (2010) 37E05 · 37D05 · 37D45

1 Introduction

The study of the dynamics of polynomials was begun by investigation of the dynamics of the quadratic family, x^2+c. By decreasing the parameter c, the behavior of this system becomes more complicated and after the appearance of sequential period doubling bifurcations, it becomes chaotic on some invariant subset of $\mathbb{R}$. See [3,5–7].

By increasing the degree of a polynomial, we expect that the behavior of the system becomes more complicated, since the behavior of the fixed points and the critical points play major role in determining the behavior of a system. For example it can be refereed to [2] in the real case, [4,8,9,11] in the complex cases and [10] for more general cases.

In [1], a family of the cubic polynomials are considered that have a fixed point of multiplicity two. These polynomials are conjugate to $ax^2(x-1)+x,\ a \in$

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$\mathbb{R} \setminus \{0\}$. The purpose of this paper is to consider the general form of this family, namely $f_{a,d}(x) = ax^d(x-1) + x,\ a \in \mathbb{R} \setminus \{0\},\ d \geq 3$. When d is an even number, the behavior of $f_{a,d}$ is completely similar to $f_{a,2}$. This subject is discussed in section 2. However when d is an odd number, the behavior is completely different. We study it in section 3.

2 d is an even number

In this section, we suppose $d \geq 4$ is an even number. The case $d = 2$ has been investigated in [1]. We will show that the behavior of $f_{a,d}$ in this case, is similar to $f_{a,2}$. Indeed, many of the results are similar and they will be modified, if necessary. To do this, we need the following lemmas. These lemmas show that the properties of $f_{a,d}$ are similar to $f_{a,2}$.

Lemma 1 *Let* $a > 0$*, then* $f_{a,d}$ *is increasing on* $(-\infty,\ 0) \cup (1,\ \infty)$*.*

Proof We have

$$f''_{a,d}(x) = adx^{d-2}\Big((d+1)x - (d-1)\Big). \tag{1}$$

So $f'_{a,d}$ is decreasing on the interval $(-\infty,\ 0)$ and increasing on the interval $(1,\ \infty)$. Thus for $x > 1$, $f'_{a,d}(x) > f'_{a,d}(1) = a + 1 > 0$ and for $x < 0$, $f'_{a,d}(x) > f'_{a,d}(0) = 1 > 0$. Therefore $f_{a,d}$ is increasing on $(-\infty,\ 0) \cup (1,\ \infty)$. □

Lemma 2 *Let* $a > 0$*, then the equation* $ax^d - ax^{d-1} + 1 = 0$ *has no solution if* $a < d(\frac{d}{d-1})^{d-1}$*, has only one solution if* $a = d(\frac{d}{d-1})^{d-1}$ *and has exactly two distinct solutions if* $a > d(\frac{d}{d-1})^{d-1}$*. Morever the solutions belong to the interval* $(0,\ 1)$*.*

Proof Let $H(x) = ax^d - ax^{d-1} + 1$, then H has a minimum in $\frac{d-1}{d}$. So H is decreasing on $(-\infty,\ \frac{d-1}{d})$ and increasing on $(\frac{d-1}{d},\ \infty)$. Also we have $H(0) = H(1) = 1$ and $H(\frac{d-1}{d}) = a(\frac{d-1}{d})^{d-1}(\frac{-1}{d}) + 1$. Now, the assertion holds easily. □

Lemma 3 *Let* $a < 0$*, then* $f_{a,d}(x) = 0$ *has two non-zero solutions* x_0 *and* x_1 *such that* $x_0 < 0 < 1 < x_1$*.*

Proof Let $H(x) = ax^d - ax^{d-1} + 1$. Then $\lim_{x\to\pm\infty} H(x) = -\infty$ and also $H(0) = H(1) = 1$. Therefore H has two solutions, one is in the interval $(-\infty,\ 0)$ and the other is in the interval $(1,\ \infty)$. □

Lemma 4 *Let* $a < 0$*,* $f'_{a,d}(x) = 0$ *has two solutions* $c_0,\ c_1$ *such that* $x_0 < c_0 < 0 < c_1 < x_1$*.*

Proof Let the non-zero solutions of $f_{a,d}(x) = 0$ are $x_0 < 0 < x_1$. Note that 0 is a solution of $f_{a,d}(x) = 0$, too. So there are $c_0 \in (x_0,\ 0)$ and $c_1 \in (0,\ x_1)$ such that $f'_{a,d}(c_0) = f'_{a,d}(c_1) = 0$. By employing the properties of $f''_{a,d}(x)$, one can prove that $f'_{a,d}(x) = 0$ has no other solution. □

What will be presented, determine the dynamics of the family $f_{a,d}(x) = ax^d(x-1)+x$ where d is an even number. They are proved easily and are similar to the case $d=2$ in [1].

Theorem 1 *Let $f_{a,d}(x) = ax^d(x-1)+x,\ a>0$, then*

1. *if $x \notin [0,1]$, then $\lim_{n\to\infty} |f^n_{a,d}(x)| = \infty$;*
2. *if $a \leq d(\frac{d}{d-1})^{d-1}$, and $x \in (0,1)$, then $\lim_{n\to\infty} f^n_{a,d}(x) = 0$;*
3. *if $a > d(\frac{d}{d-1})^{d-1}$, then the interval [0,1) is the union of a countable number of intervals whose points have orbits that converge to 0 or $-\infty$.*

Proof Note that $f_{a,d}(x) > x$ if and only if $x>1$. So by Lemma 1, for $x \in (-\infty,\ 0) \cup (1,\ \infty)$, $\{|f^n_{a,d}(x)|\}$ is an unbounded increasing sequence. So part 1 holds.
For part 2, note that if $0 < a \leq d(\frac{d}{d-1})^{d-1}$ and $0<x<1$, then $0 \leq f_{a,d}(x) < x < 1$, so the decreasing sequence $\{f^n_{a,d}(x)\}$ converges to 0. Thus the assertion in part 2 holds.
And finally, if $a > d(\frac{d}{d-1})^{d-1}$, then by Lemma 2 the equation $f_{a,d}(x) = 0$ has two non zero solutions in the interval $(0,\ 1)$. The rest of the proof is similar to Theorem 1.1 in [1]. $\square$

Theorem 2 *Suppose $f_{a,d}(x) = ax^d(x-1)+x,\ a<0$, then there exist a negative periodic point p_0 of period 2, a sequence of closed intervals $\{J_n\}_{n\geq 0}$, and a sequence of open intervals $\{I_n\}_{n\geq 0}$ such that*

$$(p_0, f_{a,d}(p_0)) = (\cup_{n\geq 0} I_n) \bigcup (\cup_{n\geq 0} J_n),$$

$f^n_{a,d}(I_n) = I_0$, $f^n_{a,d}(J_n) = J_0$, and moreover for every n the orbit of any point of the interval J_n converges to 0.

Note that we should write $p_0(a,d)$ since it depends on a and d. However, for simplicity, we omit them.
The family $\{f_{a,d}\}$ undergoes also a period-doubling bifurcation at the parameter $a=-2$ for the first time, and by decreasing a, we have a sequential period doubling bifurcations until for the first time the orbit of c_1 is attracted to 0. This happens when $f_{a,d}(c_1) = x_1$ where c_1 and x_1 are the same as that offered at Lemma 4. With the notations of Lemma 4 and Theorem 2, we have:

Theorem 3 *Suppose $f_{a,d}(x) = ax^d(x-1)+x,\ a<0$ and $f_{a,d}(c_1) \in J_1$, then $f_{a,d}$ is chaotic on $\Lambda = \{x \in [0,\ x_1] :\ f^n_{a,d}(x) \in [0,\ x_1];\ \forall n \geq 1\}$.*

3 *d* is an odd number

In this section, we suppose that $d \geq 3$ is an odd number. The lemmas that are subsequently presented, show that in this case, as in the previous case where d was even, the properties of $f_{a,d}$ are same and independent of d.

Lemma 5 *Let $a > 0$, then $f'_{a,d}$ is increasing on $(-\infty,\ 0) \cup (\frac{d-1}{d+1},\ \infty)$ and decreasing on $(0,\ \frac{d-1}{d+1})$.*

Proof Note that in the case $a > 0$ and d odd, the relation (1) shows that $f''_{a,d}(x) > 0$ if $x \in (-\infty,\ 0) \cup (\frac{d-1}{d+1},\ \infty)$ and $f''_{a,d}(x) < 0$ if $(0,\ \frac{d-1}{d+1})$. □

Lemma 6 *Let $a > 0$ and d be odd, then (see Figure 1)*

1. *the solutions of $f_{a,d}(x) = 1$ are 1 and $\sqrt[d]{\frac{-1}{a}}$.*
2. *the equation $f'_{a,d}(x) = 0$ has exactly one negative solution such that it is greater than $\sqrt[d]{\frac{-1}{a}}$. This point is a local minimum for $f_{a,d}$. Also, it has at most two positive solutions. Moreover, if $a > (\frac{d+1}{d-1})^{d-1}$, then it has two distinct positive solutions in $(0,\ 1)$ such that one of them is the local maximum point and the other is the local minimum point of $f_{a,d}$, if $a = (\frac{d+1}{d-1})^{d-1}$, then it has one positive solution in $(0,\ 1)$ such that it is an inflection point of $f_{a,d}$, and the otherwise $f'_{a,d}(x) = 0$ has no positive solution.*
3. *the equation $f_{a,d}(x) = 0$ has only one solution in the interval $(-\infty,\ 0)$.*

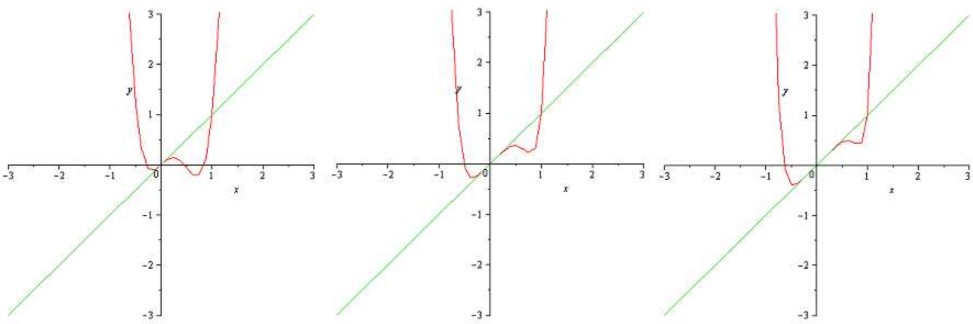

Fig. 1 The graphs of $f_{9,3}$, $f_{9,5}$ and $f_{9,7}$, respectively from the left to the right

Proof The proof of (1) is obvious. For part (2), note that $f'_{a,d}(0) = 1$ and $\lim_{x\to-\infty} f'_{a,d}(x) = -\infty$. So $f'_{a,d}(x)$ has at least one solution $c_0 < 0$. Lemma 5 shows that this solution is unique. Since $f'_{a,d}(x)$ is increasing on $(-\infty,\ 0)$, so $\sqrt[d]{\frac{-1}{a}} < c_0$. To prove the second assertion, note that by Lemma 5, $f'_{a,d}(x) > f'_{a,d}(\frac{d-1}{d+1}) = (-a)(\frac{d-1}{d+1})^{d-1} + 1$ for $x \in (0,\ \infty)$. Now if $a < (\frac{d+1}{d-1})^{d-1}$, then $f'_{a,d}(x) > 0$ for $x \in (0,\ \infty)$, if $a = (\frac{d+1}{d-1})^{d-1}$, then $f'_{a,d}(\frac{d-1}{d+1}) = 0$. Finally if $a > (\frac{d+1}{d-1})^{d-1}$, then $f'_{a,d}(\frac{d-1}{d+1}) < 0$, so $f'_{a,d}(x) = 0$ has two solutions such that one of them is in the interval $(0,\ \frac{d-1}{d+1})$ and the other is in the interval $(\frac{d-1}{d+1},\ \infty)$.

Now, let c_0 be the negative solution of $f'_{a,d}(x) = 0$. According to part (2), $f_{a,d}(c_0) < f_{a,d}(0) = 0$. Thus the assertion in part (3) holds since $f_{a,d}(\sqrt[d]{\frac{-1}{a}}) = 1 > 0$ and $f_{a,d}$ is decreasing on $(-\infty,\ c_0)$. $\square$

By solving the equations, $f_{a,d}(x) = 0$ and $f'_{a,d}(x) = 0$, we have:

Lemma 7 *If* $a > \frac{d^d}{(d-1)^{d-1}}$ *(*$a = \frac{d^d}{(d-1)^{d-1}}$, $a < \frac{d^d}{(d-1)^{d-1}}$*), then* $f_{a,d}(x) = 0$ *has two positive solutions (only one positive solution, no positive solution).*

In the case $a < 0$ we have:

Lemma 8 *Let* $a < 0$*, then (see Figure 2)*

1. *if* $x < 0$*, then* $f_{a,d}(x) < 0$.
2. *the equation* $f'_{a,d}(x) = 0$ *has only one solution* c*, where* $\frac{d-1}{d+1} < c < 1$.
3. *the equation* $f_{a,d}(x) = 0$ *has exactly two solutions.*

Proof Part (1) is obvious. To prove part (2), note that $f'_{a,d}$ is decreasing on $(-\infty,\ 0) \cup (\frac{d-1}{d+1},\ \infty)$ and increasing on $(0,\ \frac{d-1}{d+1})$ and also, $\lim_{x\to\infty} f'_{a,d}(x) = -\infty$ and $f'_{a,d}(0) = 1$. For part (3), let $H(x) = ax^d - ax^{d-1} + 1$. So $H(1) = 1$ and $\lim_{x\to\infty} H(x) = -\infty$. Therefore the equation $f_{a,d}(x) = 0$ has at least one positive solution. It is unique, since $f'_{a,d}(x) = 0$ has only one solution. $\square$

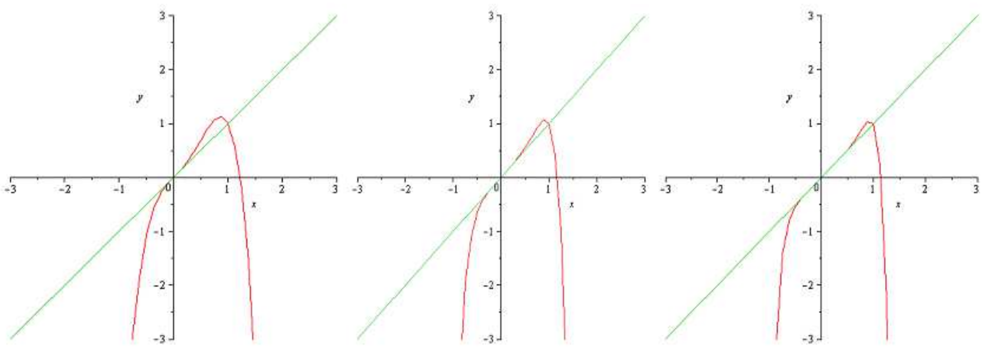

Fig. 2 The graphs of $f_{-3,3}$, $f_{-3,5}$ and $f_{-3,7}$, respectively from the left to the right

The following theorems are about the dynamics of $f_{a,d}$, where d is an odd number.

Theorem 4 *Let* $f_{a,d}(x) = ax^d(x-1)+x$*,* $a > 0$*, and* d *be an odd number, then*

1. *for* $x \in (-\infty,\ \sqrt[d]{\frac{-1}{a}}) \cup (1,\ \infty)$*, the orbit of* x *tends to* ∞.
2. *for* $x \in (\sqrt[d]{\frac{-1}{a}},\ 1)$*, the orbit of* x *tends to 0 provided*

(i) $a \leq \frac{d^d}{(d-1)^{d-1}}$ *or,*

(ii) $a > \frac{d^d}{(d-1)^{d-1}}$, *and* $f_{a,d}(c_2) \geq t_0$, *where* $c_0 < c_1 < c_2$ *are the critical points,* $x_0 < 0 < x_1 < x_2$ *are the solutions of* $f_{a,d}(x) = 0$, $t_0 < 0$ *and* $f_{a,d}(t_0) = x_1$.

Theorem 5 *Let* $f_{a,d}(x) = ax^d(x-1) + x$, $a < 0$, *and* d *be an odd number, then*

1. *the family* $\{f_{a,d}\}$ *undergoes a period-doubling bifurcation at the parameter* $a = -2$ *for the first time.*
2. *if* $f_{a,d}(c) > x_0$, *where* c *is the unique critical point and* x_0 *is non-zero solution of* $f_{a,d}(x) = 0$, *then* $f_{a,d}$ *is chaotic on*

$$\Lambda = \{x \in [0,\ x_0]\ \ f_{a,d}(x) \in [0,\ x_0] \text{ for } \quad \text{all } n \geq 1\}.$$

With the notations of Theorem 4, we conjecture:

Conjecture 1 Let $f_{a,d}(x) = ax^d(x-1) + x$, $a > 0$, and d be an odd number,

1. and let $\sqrt[d]{\frac{-1}{a}} < f_{a,d}(c_2) < t_0$, then $f_{a,d}$ has some periodic points.
2. if $f_{a,d}(c_2) \leq \sqrt[d]{\frac{-1}{a}}$ and $\Lambda = \mathbb{R} \setminus \{x \in \mathbb{R} :\ \lim_{n\to\infty} f_{a,d}(x) = 0 \text{ or } \infty\}$, then the restriction $f_{a,d}$ on Λ is chaotic.

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