



A Pseudo-Spectral Approach to Solving the Fractional Cable Equation Using Lagrange Polynomials

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Abstract

This paper focuses on addressing the fractional cable equation through the pseudo-spectral method, providing an innovative approach for solving problems in fractional calculus. The proposed method utilizes Lagrange polynomials at Chebyshev points to approximate spatial derivatives, ensuring high computational precision. One of the noteworthy features of this study is the time discretization scheme introduced, which is unconditionally stable and boasts a convergence order of $\mathcal{O}(\tau^2)$. Such stability is crucial for solving time-dependent fractional differential equations, especially when long-term simulations are required. Pseudo-spectral methods are known for their exponential accuracy, and the approach detailed in this paper exemplifies this capability. To demonstrate the robustness and reliability of the proposed technique, several numerical examples are presented. These examples highlight the method's efficiency in achieving highly accurate solutions with minimal computational effort. By leveraging spectral techniques, this research offers insights into solving fractional differential equations with greater precision, paving the way for future advancements in this field. The outcomes underscore the proposed method's effectiveness and emphasize its potential applicability to a broad range of scientific and engineering problems.

Keywords: Pseudo-spectral method, Fractional cable equation, Lagrange polynomials, Chebyshev points, Fractional derivatives

Mathematics Subject Classification (2020): 65M70, 35R11

1 Introduction

Over the last few decades, fractional differential equations (FDEs) have been widely used in various models, including those in signal processing, traffic flow, diffusion processes, and fluid dynamics [1–4]. Additionally, non-integer calculus has gained attention in diverse fields, such as mathematical biology, electrochemistry, and various physical phenomena, including relative stress and strain in elastic or viscoelastic materials, as well as Hooke's law [4–7].

The electric cable equation models the propagation of electrical signals along a cylindrical cable, such as a nerve axon or dendrite. When there is damage to the membrane (e.g., a broken membrane or defect), the membrane's electrical properties, such as resistance, capacitance, and ion flow dynamics, are altered. These changes must be incorporated into the equation [8].

The electric cable equation with a broken membrane can be written as:

$$\frac{\partial V(x,t)}{\partial t} = \frac{1}{C_m} \left(-I_{\text{ion}}(x,t) - I_{\text{leak}}(x,t) + \frac{1}{R_m(x)} \frac{\partial^2 V(x,t)}{\partial x^2} \right), \quad (1)$$

where:

- $V(x,t)$ is the **membrane potential** at position x and time t . The membrane potential represents the difference in electric potential across the membrane of the cable (axon or dendrite).
- C_m is the **membrane capacitance** per unit length of the cable. The membrane capacitance is related to the physical properties of the membrane, including its thickness and surface area.
- $I_{\text{ion}}(x,t)$ is the **ionic current**, representing the flow of ions through the membrane, which is governed by the ion channels and their conductances. The current is typically modeled using the *Nernst-Planck equation* for ion diffusion or the *Hodgkin-Huxley model* for voltage-gated ion channels. For a single ion type, the ionic current can be expressed as:

$$I_{\text{ion}}(x,t) = g_{\text{ion}}(x,t)(V(x,t) - E_{\text{ion}}),$$

where $g_{\text{ion}}(x,t)$ is the ion conductance, and E_{ion} is the Nernst equilibrium potential for the ion.

- $I_{\text{leak}}(x,t)$ is the **leak current**, representing the ion flow through a damaged or "broken" membrane. This could result from membrane defects or pathological conditions like injury or disease. The leak current increases in regions of the membrane where the resistance is reduced due to damage. The leak current can be modeled as:

$$I_{\text{leak}}(x,t) = \frac{V(x,t)}{R_{\text{leak}}(x,t)},$$

where $R_{\text{leak}}(x,t)$ is the **leak resistance** at position x , which may decrease in damaged regions.

- $R_m(x)$ is the **membrane resistance** at position x , which may vary along the cable, especially where the membrane is damaged. The membrane resistance represents the resistance of the intact portions of the membrane to ion flow. It is inversely related to the membrane conductance. When the membrane is damaged (e.g., a "broken" membrane), the local membrane resistance $R_m(x)$ may decrease, leading to higher leak currents. In the case of a broken membrane, the resistance $R_m(x)$ could vary spatially and decrease in the affected areas. It may be represented as a spatially varying function:

$$R_m(x) = R_{m0} \cdot f(x),$$

where R_{m0} is the baseline membrane resistance and $f(x)$ accounts for the local damage to the membrane. For instance, $f(x)$ could be a scaling function that reduces the resistance in the damaged region.

- $\frac{\partial^2 V(x,t)}{\partial x^2}$ is the **second spatial derivative** of the membrane potential, describing how the potential spreads along the length of the cable. It accounts for the spread of electrical signals in space, and the form of this term depends on the geometry of the cable. The second derivative reflects the diffusion of the signal along the cable.

When there is a breakage or defect in the membrane, the resistance $R_m(x)$ typically decreases in the affected areas, leading to a higher leak current $I_{\text{leak}}(x,t)$ and altered signal propagation characteristics. This modification is often modeled by introducing spatially varying coefficients for $R_m(x)$ and $I_{\text{leak}}(x,t)$.

Thus, a more specific form of the cable equation with a broken membrane could look like:

$$\frac{\partial V(x,t)}{\partial t} = \frac{1}{C_m} \left(-I_{\text{ion}}(x,t) - \frac{V(x,t)}{R_{\text{leak}}(x)} + \frac{1}{R_m(x)} \frac{\partial^2 V(x,t)}{\partial x^2} \right). \quad (2)$$

Here, the membrane resistance $R_m(x)$ and leak resistance $R_{\text{leak}}(x)$ may vary depending on the spatial location of the cable, with the damaged regions having lower resistance.

To address the issue of irregular ion diffusion caused by factors such as the heterogeneity of neuronal tissue or ion sequestration by buffering proteins, it is necessary to modify the standard Cable equation [9, 10]. Similar to the way the fractional Fokker-Planck equation incorporates anomalous transport in an external field, we must consider the following fractional Cable equation:

$$\frac{\partial V(x,t)}{\partial t} = \frac{1}{C_m} \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left(-I_{\text{ion}}(x,t) - \frac{V(x,t)}{R_{\text{leak}}(x)} + \frac{1}{R_m(x)} \frac{\partial^2 V(x,t)}{\partial x^2} \right), \quad (3)$$

where $\partial^{1-\alpha}/\partial t^{1-\alpha}$ with $0 < \alpha < 1$ denotes the Riemann-Liouville fractional derivative operator defined by

$$\frac{\partial^{1-\alpha} u(x,t)}{\partial t^{1-\alpha}} = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,s)}{(t-s)^{1-\alpha}} ds, \quad 0 < \alpha < 1,$$

where $\Gamma(\cdot)$ is the Gamma function [4, 7, 11].

On the other hand, given that the neuronal membrane is made of a different material than the neuron itself, it is anticipated that the membrane would exhibit distinct diffusive characteristics. This requires the introduction of a unique fractional exponent to account for the unusual diffusion across the membrane. Consequently, a more general fractional Cable equation, incorporating two fractional derivatives [9], is formulated:

$$\frac{\partial V(x,t)}{\partial t} = \frac{1}{C_m} \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} \left(-I_{\text{ion}}(x,t) - \frac{V(x,t)}{R_{\text{leak}}(x)} \right) + \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left(\frac{1}{R_m(x)} \frac{\partial^2 V(x,t)}{\partial x^2} \right), \quad (4)$$

where $0 < \beta < 1$. If we express the membrane potential relative to the equilibrium potential, specifically letting u represent $-I - V/R$, and rescale the time and spatial variables accordingly [9]:

$$\hat{t} = \frac{t}{C_m R_m}, \quad \hat{x} = \frac{x(C_m R_m)^{\alpha/2}}{\sqrt{R_{\text{leak}}/R_m}}, \quad (5)$$

then we obtain the following fractional Cable equation (FCE) in a concise form

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \nabla^2 u(x,t) - \kappa \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} u(x,t), \quad (6)$$

where $\kappa = (C_m R_m)^{-\beta}$.

In this paper, for computational purposes, we consider the above FCE equation with an additional right-hand side function, as in [12–16]:

$$\frac{\partial u(x,t)}{\partial t} = K_1 \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \nabla^2 u(x,t) - K_2 \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} u(x,t) + f(x,t), \quad (7)$$

with initial condition

$$u(x,0) = \psi(x), \quad x \in \Omega, \quad (8)$$

and homogeneous essential boundary condition.

The article [17] introduces a modified cable equation designed to more accurately simulate the behavior of neuronal membranes at high frequencies. Experimental data reveal that neuronal electrical activity at these frequencies follows a specific pattern that traditional models fail to explain. By incorporating non-ideal capacitors into the cable equation, the authors developed a model that better aligns with experimental results and examined its effects on signal transmission and voltage attenuation in neurons. Similarly, Wang et al [18] presents another modification of the cable equation, integrating the influence of transverse polarization of neuronal membranes in response to transverse electric fields. The authors demonstrate that transverse polarization has a limited effect on neuronal activation thresholds, particularly in unmyelinated axons exposed to short pulses and low-gradient fields. The modified model more accurately represents membrane behavior across different time scales and enhances the coupling between electric fields and neurons compared to the conventional cable equation. Additionally, the study underscores that linear membrane models with fixed activation thresholds may not reliably capture the effects of transverse polarization.

Liu and his colleagues have introduced two finite difference numerical schemes to solve the fractional cable equation, examining their stability and convergence [14]. Similarly, Lin and his team have developed a numerical scheme grounded in spectral finite difference methods to address the cable equation with a fractional derivative [9]. Morillo and Yast have presented an implicit numerical scheme for the fractional cable equation, including an error analysis, in their work [19]. Additionally, other authors have demonstrated that the approximate

solution converges to the real solution with a convergence speed of $\mathcal{O}(\tau + h^{\ell+1})$ and have proposed an improved finite element scheme with the same convergence order to solve the fractional cable equation [20]. Moreover, Karatai has implemented a finite difference scheme inspired by Nicholsons crank concept for solving the fractional cable equation, as detailed in reference [12].

Over the years, various numerical methods have been developed for solving partial differential equations, which can be broadly categorized into global and local approaches. Spectral and pseudo-spectral methods fall under the global approximation category. These methods gained prominence in the 1970s for their effectiveness in solving differential equations, particularly for problems defined on regular domains. When the input data is smooth, spectral methods excel by providing high accuracy and requiring relatively less computational memory [21]. In recent years, several innovative methods rooted in these approaches have been introduced. Notable examples include the pseudo-spectral Chebyshev-collocation method [22], Fourier-spectral methods [23], and the Galerkin-type pseudo-spectral method [24], among others.

The numerical results of the examples in Section 4 highlight the exceptional accuracy of the proposed method, a hallmark of spectral techniques. For instance, Table 1 demonstrates an error of approximately $1e-9$ for $n_p = 14$ across the two specified states. Coupled with an execution time of just 0.007, this underscores the remarkable efficiency of the method.

This paper is organized into four sections. In the second section, we detail the method used for discretizing time derivatives. The third section introduces the pseudo-spectral method and its application to the fractional Cable equation, leading to the final discretized form. The fourth section demonstrates the effectiveness of the proposed method through three test problems. Lastly, the conclusion summarizes the findings of the study.

2 Time Discretization Scheme

In this section, we delve into the discretization of the time derivative terms. Due to the fractional nature of the time derivative, we provide several definitions and lemmas to aid in this process. Let $u^k = u(x, t_k)$ represent a grid function on $\Omega \times (0, T)$. We introduce the following notations:

$$\delta_t u^{k+1/2} = \frac{1}{\eta} (u^{k+1} - u^k), \quad \delta_x^2 u^{k+1/2} = \frac{1}{2} (\delta_x^2 u^{k+1} + \delta_x^2 u^k), \quad u^{k+1/2} = \frac{1}{2} (u^{k+1} + u^k), \quad (9)$$

where $\eta = T/N$ is the time step size, and $t_k = k\eta$, $k = 0, 1, \dots, N$.

Lemma 1. [25, 26] Let $v(t)$, ${}_{-\infty}D_t^{\gamma+2}v(t)$ and its Fourier transform belong to $L^1(\mathbb{R})$ and also $0 < \gamma < 1$. Then the following second-order weighted and shifted Grünwald approximation formula holds

$$\frac{\partial^\gamma v(t_n)}{\partial t^\gamma} = \sum_{i=0}^n \mu_\gamma(i) v(t_{n-i}) + \mathcal{O}(\tau^2), \quad (10)$$

in which

$$\mu_\gamma(i) = \begin{cases} \frac{\gamma+2}{2} g_0^\gamma, & i = 0, \\ \frac{\gamma+2}{2} g_i^\gamma - \frac{\gamma}{2} g_{i-1}^\gamma, & i > 0, \end{cases} \quad (11)$$

and

$$g_0^\gamma = 1, \quad g_i^\gamma = \frac{\Gamma(i-\gamma)}{\Gamma(-\gamma)\Gamma(i+1)}, \quad g_i^\gamma = \left(1 - \frac{\gamma+1}{i}\right) g_{i-1}^\gamma, \quad i \geq 1. \quad (12)$$

Lemma 2. [25] Let $v(t)$ be well-defined on the bounded interval $[c, d]$. If $v(c) = v(d) = 0$, the function $v(t)$ can be zero extended for $t < c$ or $t > d$. Then the γ -order left Reimann-Liouville fractional derivative of $v(t)$ at each point t can be approximated by the weighted and shifted Grünwald approximation operators with second-order accuracy as follows

$${}_c D_t^{\gamma+2} v(t) = \frac{\mu_1}{\tau^\gamma} \sum_{i=0}^{\lfloor \frac{t-c}{\tau} \rfloor + p} g_i^{(\gamma)} v(t - (i-p)\tau) + \frac{\mu_2}{\tau^\gamma} \sum_{i=0}^{\lfloor \frac{t-c}{\tau} \rfloor + q} g_i^{(\gamma)} v(t - (i-p)\tau) + \mathcal{O}(\tau^2), \quad (13)$$

in which $\tau = \frac{d-c}{N}$, $\mu_1 = \frac{\gamma-2q}{2(p-q)}$, and $\mu_2 = \frac{2p-\gamma}{2(p-q)}$.

It is worth mentioning that in this paper, we set $(p, q) = (1, -1)$.

Lemma 3. [25, 26] For sequence $\{g_i^\gamma\}_{i=1}^\infty$ that is defined in (13), the following properties hold

$$g_0^\gamma = 1 > 0, \quad g_i^\gamma < 0, \quad (i = 1, 2, \dots), \quad \sum_{i=1}^\infty g_i^\gamma = -1. \quad (14)$$

Lemma 4. [25–27] Let $\{\mu_\gamma(i)\}_{i=1}^\infty$ is introduced in (11). Then for any positive integer N and real vector $(v^0, v^1, \dots, v^N) \in \mathbb{R}^{N+1}$, the following inequality holds

$$\sum_{j=0}^N \sum_{i=0}^j \mu_\gamma(i) (v^{j-i}, v^j) \geq 0. \quad (15)$$

Considering Equation (7) at the points $(x, t_{n+\frac{1}{2}})$ and using the Crank-Nicolson method [28] yields

$$\delta_t u^{k+1/2} = K_1 \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left(\delta_x^2 u^{k+1/2} + \delta_y^2 u^{k+1/2} \right) - K_2 \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} \left(u^{k+1/2} \right) + f^{k+1/2}. \quad (16)$$

Now, by applying relations (10) and (13) to Equation (16), we have

$$\begin{aligned} \delta_t u^{k+1/2} = & \frac{K_1}{2\tau^{1-\alpha}} \left(\sum_{\ell=0}^{K+1} \mu_\alpha(\ell) \left(\delta_x^2 u^{k+1-\ell} + \delta_y^2 u^{k+1-\ell} \right) + \sum_{\ell=0}^K \mu_\alpha(\ell) \left(\delta_x^2 u^{k-\ell} + \delta_y^2 u^{k-\ell} \right) \right) \\ & - \frac{K_2}{2\tau^{1-\beta}} \left(\sum_{\ell=0}^{K+1} \mu_\beta(\ell) u^{k+1-\ell} + \sum_{\ell=0}^K \mu_\beta(\ell) u^{k-\ell} \right) + f^{k+1/2}. \end{aligned} \quad (17)$$

Putting $H_1 = \frac{K_1 \tau^\alpha}{2}$ and $H_2 = \frac{K_2 \tau^\beta}{2}$, we obtain

$$\begin{aligned} u^{k+1} = & u^k + H_1 \left(\sum_{\ell=0}^{K+1} \mu_\alpha(\ell) \left(\delta_x^2 u^{k+1-\ell} + \delta_y^2 u^{k+1-\ell} \right) + \sum_{\ell=0}^K \mu_\alpha(\ell) \left(\delta_x^2 u^{k-\ell} + \delta_y^2 u^{k-\ell} \right) \right) \\ & - H_2 \left(\sum_{\ell=0}^{K+1} \mu_\beta(\ell) u^{k+1-\ell} + \sum_{\ell=0}^K \mu_\beta(\ell) u^{k-\ell} \right) + \tau f^{k+1/2} \end{aligned} \quad (18)$$

In the following, we will prove the stability of the above method. However, before doing so, we will state a remark from [29].

Remark 1. Let φ_n and δ_k be non-negative sequences and $\kappa_0 > 0$, and that the sequence ϑ_n satisfies

$$\begin{cases} \vartheta_0 \leq \kappa_0, \\ \vartheta_n \leq \kappa_0 + \sum_{k=0}^{n-1} \delta_k + \sum_{k=0}^{n-1} \varphi_k \vartheta_k, \quad n \geq 1. \end{cases} \quad (19)$$

If $\kappa_0 \geq 0$ and $\delta_0 \geq 0$, it follows

$$\vartheta_n \leq \left(\kappa_0 + \sum_{k=0}^{n-1} \delta_k \right) \exp \left(\sum_{k=0}^{n-1} \varphi_k \right), \quad n \geq 1.$$

We begin with a brief explanation of stability and, by presenting a theorem, demonstrate that the method described is unconditionally stable. Stability implies that if the initial conditions are altered slightly, the system's response changes proportionally and remains controlled, ensuring that the system does not exhibit erratic or unbounded behavior as a result of small perturbations. This property is critical in many mathematical and physical systems, as it ensures that solutions remain meaningful and reliable under realistic conditions where minor errors or uncertainties are inevitable.

In the context of differential equations, stability signifies that slight changes in the initial conditions lead to minimal variation in the solution of the equation. This means that the solutions are robust to numerical approximations and measurement inaccuracies, which are common in real-world applications. For example, in the numerical solution of partial differential equations, stability guarantees that errors introduced during computation, such as rounding errors or truncation errors, do not grow uncontrollably as the computation progresses. This is particularly important when simulating long-term behavior of systems or when dealing with complex, sensitive models where even small errors could otherwise lead to significant deviations.

Theorem 1. Let $u^n \in H_0^1(\Omega) \cap H^2(\Omega)$. Then, the time-discrete scheme (18) is unconditionally stable.

Proof. First, we derive the weak form of Equation (18) as follows:

$$\begin{aligned} \langle u^{k+1}, v \rangle &= \langle u^k, v \rangle + H_1 \left(\sum_{\ell=0}^{K+1} \mu_\alpha(\ell) \langle \nabla u^{k+1-\ell}, v \rangle + \sum_{\ell=0}^K \mu_\alpha(\ell) \langle \nabla u^{k-\ell}, v \rangle \right) \\ &\quad - H_2 \left(\sum_{\ell=0}^{K+1} \mu_\beta(\ell) \langle u^{k+1-\ell}, v \rangle + \sum_{\ell=0}^K \mu_\beta(\ell) \langle u^{k-\ell}, v \rangle \right) + \tau \langle f^{k+1/2}, v \rangle \end{aligned} \quad (20)$$

where $\langle u^{k+1}, v \rangle = \int_{\Omega} u^{k+1} v d\Omega$ is an inner product. putting $v = u^{k+1}$, we find that

$$\begin{aligned} \langle u^{k+1}, u^{k+1} \rangle &= \langle u^k, u^{k+1} \rangle + H_1 \left(\sum_{\ell=0}^{K+1} \mu_\alpha(\ell) \langle \nabla u^{k+1-\ell}, u^{k+1} \rangle + \sum_{\ell=0}^K \mu_\alpha(\ell) \langle \nabla u^{k-\ell}, u^{k+1} \rangle \right) \\ &\quad - H_2 \left(\sum_{\ell=0}^{K+1} \mu_\beta(\ell) \langle u^{k+1-\ell}, u^{k+1} \rangle + \sum_{\ell=0}^K \mu_\beta(\ell) \langle u^{k-\ell}, u^{k+1} \rangle \right) + \tau \langle f^{k+1/2}, u^{k+1} \rangle \end{aligned} \quad (21)$$

By summing (21) for $k = -1, \dots, N$, we have

$$\begin{aligned} \sum_{k=-1}^N \|u^{k+1}\|_{L^2(\Omega)}^2 &= \sum_{k=-1}^N \langle u^k, u^{k+1} \rangle + H_1 \left(\sum_{k=-1}^N \sum_{\ell=0}^{K+1} \mu_\alpha(\ell) \langle \nabla u^{k+1-\ell}, u^{k+1} \rangle + \sum_{k=-1}^N \sum_{\ell=0}^K \mu_\alpha(\ell) \langle \nabla u^{k-\ell}, u^{k+1} \rangle \right) \\ &\quad - H_2 \left(\sum_{k=-1}^N \sum_{\ell=0}^{K+1} \mu_\beta(\ell) \langle u^{k+1-\ell}, u^{k+1} \rangle + \sum_{k=-1}^N \sum_{\ell=0}^K \mu_\beta(\ell) \langle u^{k-\ell}, u^{k+1} \rangle \right) + \tau \sum_{k=-1}^N \langle f^{k+1/2}, u^{k+1} \rangle \end{aligned} \quad (22)$$

By utilizing the Cauchy-Schwartz inequality

$$\int_{\Omega} |U(x)V(x)| dx \leq \|U\|_{L^2(\Omega)} \|V\|_{L^2(\Omega)}, \quad U, V \in L^2(\Omega),$$

we derive

$$\begin{aligned} \frac{1}{2} \sum_{k=-1}^N \|u^{k+1}\|_{L^2(\Omega)}^2 &- H_1 \left(\sum_{k=-1}^N \sum_{\ell=0}^{K+1} \mu_\alpha(\ell) \langle \nabla u^{k+1-\ell}, u^{k+1} \rangle \right) + H_2 \left(\sum_{k=-1}^N \sum_{\ell=0}^{K+1} \mu_\beta(\ell) \langle u^{k+1-\ell}, u^{k+1} \rangle \right) \\ &\leq \frac{1}{2} \sum_{k=-1}^N \|u^k\|_{L^2(\Omega)}^2 + H_1 \left(\sum_{k=-1}^N \sum_{\ell=0}^K \mu_\alpha(\ell) \langle \nabla u^{k-\ell}, u^{k+1} \rangle \right) - H_2 \left(\sum_{k=-1}^N \sum_{\ell=0}^K \mu_\beta(\ell) \langle u^{k-\ell}, u^{k+1} \rangle \right) + \tau \sum_{k=-1}^N \langle f^{k+1/2}, u^{k+1} \rangle \end{aligned} \quad (23)$$

Next, by using Lemma 4, we get

$$\sum_{k=-1}^N \sum_{\ell=0}^{K+1} \mu_\alpha(\ell) \langle \nabla u^{k+1-\ell}, u^{k+1} \rangle \geq 0, \quad (24)$$

$$\sum_{k=-1}^N \sum_{\ell=0}^{K+1} \mu_\beta(\ell) \langle u^{k+1-\ell}, u^{k+1} \rangle \geq 0, \quad (25)$$

$$- \sum_{k=-1}^N \sum_{\ell=0}^{K+1} \mu_\alpha(\ell) \langle \nabla u^{k-\ell}, u^{k+1} \rangle \leq 0, \quad (26)$$

$$\sum_{k=-1}^N \sum_{\ell=0}^{K+1} \mu_\beta(\ell) \langle u^{k-\ell}, u^{k+1} \rangle \leq 0. \quad (27)$$

Consequently, by applying relations (24)-(27), Equation (23) can be reformulated as follows:

$$\frac{1}{2} \sum_{k=-1}^N \|u^{k+1}\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \sum_{k=-1}^N \|u^k\|_{L^2(\Omega)}^2 + \tau \sum_{k=-1}^N \langle f^{k+1/2}, u^{k+1} \rangle \quad (28)$$

Alternatively, it can be rewritten in the following equivalent form

$$\|u^{k+1}\|_{L^2(\Omega)}^2 \leq \|u^k\|_{L^2(\Omega)}^2 + \tau \sum_{q=-1}^K \|f^{q+1/2}\|_{L^2(\Omega)}^2 + \tau \sum_{q=-1}^K \|u^q\|_{L^2(\Omega)}^2 \quad (29)$$

Finally, by applying Remark 1, we obtain

$$\begin{aligned}
 \|u^{k+1}\|_{L^2(\Omega)}^2 &\leq \left(\|u^0\|_{L^2(\Omega)}^2 + \tau \sum_{q=-1}^K \|f^{q+1/2}\|_{L^2(\Omega)}^2 \right) \exp \left(\sum_{q=-1}^K \tau \right) \\
 &\leq \left(\|u^0\|_{L^2(\Omega)}^2 + \tau \sum_{q=-1}^K \|f^{q+1/2}\|_{L^2(\Omega)}^2 \right) \exp(K\tau) \\
 &\leq \left(\|u^0\|_{L^2(\Omega)}^2 + K\tau \max_{0 \leq q \leq K} \|f^q\|_{L^2(\Omega)}^2 \right) \exp(K\tau) \\
 &\leq \left(\|u^0\|_{L^2(\Omega)}^2 + T \max_{0 \leq q \leq K} \|f^q\|_{L^2(\Omega)}^2 \right) \exp(T),
 \end{aligned} \tag{30}$$

This demonstrates that the discretized scheme exhibits unconditional stability. $\square$

3 The Pseudo-Spectral Method

In this section, we offer a brief overview of pseudo-spectral methods. For more in-depth information on both spectral and pseudo-spectral methods, please refer to reference [21]. The selection of bases for the approximation space is vital in pseudo-spectral methods. In our approach, we use Lagrange polynomials for an appropriate approximation of the solution, ensuring spectral accuracy or exponential convergence. It's noteworthy that we apply the pseudo-spectral method solely for spatial variable discretization. Moreover, we outline the implementation of this method specifically for the two-dimensional case.

Suppose $\Omega \subseteq \mathbb{R}^2$ is a bounded domain, and n_p is a positive integer. We consider Chebyshev points as nodal points:

$$x_{ij} = (\cos(i\pi/n_p), \cos(j\pi/n_p)), \quad i, j = 0, 1, \dots, n_p. \tag{31}$$

Furthermore, we represent the solutions of the fractional Cable equation (7) and the right-hand side function f as a linear combination of Lagrange polynomials:

$$u_{n_p}(x) = \sum_{i,j=1}^{n_p-1} u_{ij}(t) \ell_{ij}(x), \quad u_{ij}(t) := u(x_{ij}, t), \tag{32}$$

$$f_{n_p}(x, t) = \sum_{i,j=1}^{n_p-1} f_{ij}(t) \ell_{ij}(x), \quad f_{ij}(t) := f(x_{ij}, t). \tag{33}$$

Here, ℓ_{ij} represents the Lagrange polynomials, which are defined as

$$\ell_{ij}(x) = \ell_i(x) \ell_j(x), \quad i, j = 0, 1, \dots, n_p,$$

where

$$\ell_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n_p} \left(\frac{x - x_k}{x_i - x_k} \right), \quad i = 0, 1, \dots, n_p.$$

Note that $\ell_i(x)$ belongs to $\mathbb{P}_{n_p}$ (polynomials of degree $\leq n_p$) and satisfies the Kronecker Delta property

$$\ell_i(x_j) = \delta_{ij}, \quad i, j = 0, 1, \dots, n_p.$$

We can calculate the second-order derivatives of $\ell_{ij}(\cdot)$ with respect to x and y at Chebyshev nodal points (31) in the following manner:

$$\begin{aligned}
 \frac{\partial^2}{\partial x^2} \ell_{ij}(x_{rs}) &= \ell_i''(x_r) \ell_j(y_s) = [D_{n_p}^2]_{ir} \delta_{js}, \\
 \frac{\partial^2}{\partial y^2} \ell_{ij}(x_{rs}) &= \ell_i(x_r) \ell_j''(y_s) = \delta_{ir} [D_{n_p}^2]_{js},
 \end{aligned}$$

where $x_{rs} = (x_r, y_s)$, $r, s = 0, 1, \dots, n_p$, and $D_{n_p}^2$ denotes the second-order derivative matrix at Chebyshev points [30]. By substituting relations (32)-(33) into Equation (18), we obtain:

$$\begin{aligned} U_{n_p}^m = & U_{n_p}^{m-1} + H_1 \left(\sum_{\ell=0}^m \sum_{i,j=1}^{n_p-1} \mu_\alpha(\ell) \Delta \ell_{ij}(x_{rs}) U_{rs}^{m-\ell} + \sum_{\ell=0}^{m-1} \sum_{i,j=1}^{n_p-1} \mu_\alpha(\ell) \Delta \ell_{ij}(x_{rs}) U_{rs}^{m-\ell-1} \right) \\ & - H_2 \left(\sum_{\ell=0}^m \sum_{i,j=1}^{n_p-1} \mu_\beta(\ell) U_{rs}^{m-\ell} + \sum_{\ell=0}^{m-1} \sum_{i,j=1}^{n_p-1} \mu_\beta(\ell) U_{rs}^{m-\ell-1} \right) + \tau \sum_{i,j=1}^{n_p-1} F_{rs}^{m-1} \end{aligned} \quad (34)$$

where

$$\Delta \ell_{ij}(x_{rs}) = [D_{n_p}^2]_{ir} \delta_{js} + \delta_{ir} [D_{n_p}^2]_{js}, \quad r, s = 0, 1, \dots, n_p. \quad (35)$$

We observe that U_{n_p} and F_{n_p} are vectors of dimension n_p .

4 Numerical Outcomes

In this section, we conduct numerical investigations to validate and evaluate the effectiveness of the method described in the previous sections. The method is tested on three different problems in one, two, and three spatial dimensions. We thoroughly analyze the results for each case. All algorithms are implemented using MATLAB software. It is important to note that all results were obtained using a system equipped with an 11th Gen Intel(R) Core(TM) i5 processor (2.40 GHz) and 8 GB of RAM. To measure the accuracy of the numerical results, we use the L_2 and L_∞ error norms. Additionally, we calculate the numerical convergence rate using the following formula:

$$C - \text{order} = \frac{\log\left(\frac{e(i-1)}{e(i)}\right)}{\log 2}$$

where $e(i)$ represents the corresponding error of the numerical result for $\eta(i)$.

Test Problem 1

Examine the nonlinear coupled FCE (7) over the domain $\Omega = [0, 1]$, with parameters $K_1 = K_2 = 1$ and the exact solution [31]

$$u(x, t) = t^{2+\alpha+\beta} x^{12} (1-x)^{12} \sin(\pi x).$$

Consequently, the right-hand side function f can be derived as

$$\begin{aligned} f(x, t) = & (\alpha + \beta + 2) t^{1+\alpha+\beta} x^{12} (1-x)^{12} \sin(\pi x) + x^{10} (1-x)^{10} \\ & \times \left[\sin(\pi x) (\pi^2 x^2 (1-x)^2 - 552x^2 + 552x - 132) - 24\pi x \cos(\pi x) (2x^2 - 3x + 1) \right] \\ & \times \left[\frac{\Gamma(\alpha + \beta + 3)}{\Gamma(2\alpha + \beta + 2)} t^{1+2\alpha+\beta} \right] + \left[\frac{\Gamma(\alpha + \beta + 3)}{\Gamma(\alpha + 2\beta + 2)} t^{1+\alpha+2\beta} \right] t^{2+\alpha+\beta} x^{12} (1-x)^{12} \sin(\pi x). \end{aligned}$$

Table 1 presents the L_∞ and L_2 errors, as well as the numerical convergence order, for $T = 1$ under two conditions: $\alpha = 0.7$, $\beta = 0.8$ and $\alpha = 0.4$, $\beta = 0.8$.

The findings presented in Table 1 indicate that as n_p increases, the error diminishes, leading to more favorable outcomes. Also, as can be seen from the CPU-time column in this table, the execution speed of the presented method is very high and the execution times are less than one second even for large n_p . Figure 1 illustrates the L_∞ errors as functions of n_p : on the left, for $\beta = 0.8$ with varying values of α , and on the right, for $\alpha = 0.2$ with different values of β at $T = 1$.

Test Problem 2

We now examine the two-dimensional case of the nonlinear FCE (7) over the domain $\Omega = [0, 1]^2$ with $K_1 = K_2 = 1$. The exact solution is as follows [16]:

$$u(x, y, t) = t^2 \sin(\pi x) \sin(\pi y).$$

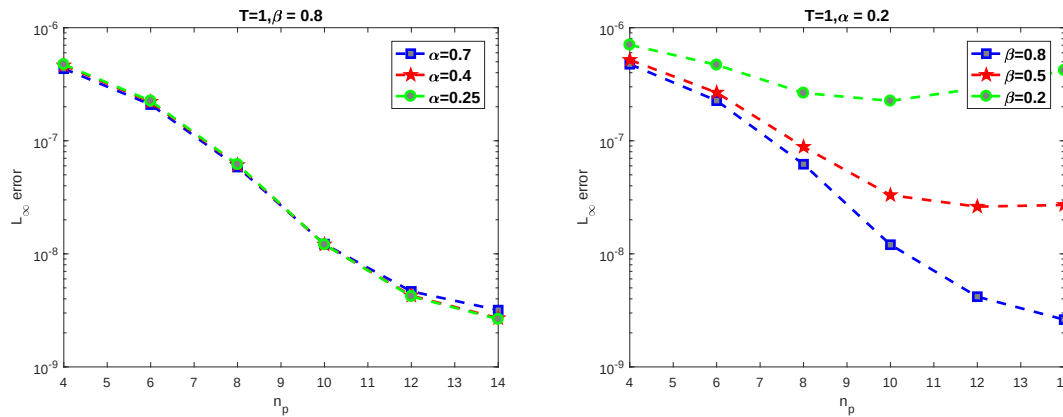


Figure 1. The L_∞ errors plotted as functions of n_p : on the left for $\beta = 0.8$ with various values of α , and on the right for $\alpha = 0.2$ with different values of β at $T = 1$ (Test problem 1).

Table 1. The L_∞ and L_2 errors, C-orders and CPU-times at $T = 1$ for Test problem 1.

n_p	η	$\alpha = 0.7, \beta = 0.8$			$\alpha = 0.4, \beta = 0.8$			CPU-time
		L_∞	L_2	C-order	L_∞	L_2	C-order	
4	1/4	$4.33e-07$	$5.00e-07$	—	$4.64e-07$	$5.35e-07$	—	$1.4e-03$
6	1/6	$2.09e-07$	$2.69e-07$	1.05	$2.21e-07$	$2.87e-07$	1.06	$2.7e-03$
8	1/8	$5.85e-08$	$7.17e-08$	1.84	$6.10e-08$	$7.59e-08$	1.85	$2.8e-03$
10	1/10	$1.22e-08$	$1.25e-08$	2.26	$1.20e-08$	$1.26e-08$	2.33	$3.8e-03$
12	1/12	$4.65e-09$	$4.80e-09$	1.38	$4.26e-09$	$4.48e-09$	1.50	$4.6e-03$
14	1/14	$3.18e-09$	$3.42e-09$	0.55	$2.69e-09$	$3.18e-09$	0.66	$7.2e-03$

As a result, the right-hand side function f can be obtained from this solution as

$$f(x, y, t) = \left[2t + \frac{2t^{1+\beta}}{\Gamma(2+\beta)} + \frac{4t^{1+\alpha}}{\Gamma(2+\alpha)} \right] \sin(\pi x) \sin(\pi y).$$

In Table 2, the L_∞ and L_2 errors, along with the order of numerical convergence, are presented at $T = 1$ for two scenarios ($\alpha = 0.1, \beta = 0.3$ and $\alpha = 0.2, \beta = 0.7$). As shown in Table 2, the errors decrease exponentially with a reduction in η . The execution times for the relevant programs are also provided in the last column of these tables. It is noted that the accuracy of the results improves with an increase in n_p . Remarkably, excellent results have been achieved even with relatively small values of n_p . To further assess the accuracy of the proposed method, Figure 2 plots the maximum errors for various values of α while keeping β constant, and vice versa. This figure highlights the exponential decrease in errors, a characteristic feature of spectral methods. Finally, Figure 3 displays the graphs of both the exact and numerical solutions for $\alpha = 0.1, \beta = 0.3$, demonstrating their close correspondence, and a log-log plot of the L_∞ errors as a function of η for two cases ($\alpha = 0.25, \beta = 0.45$ and $\alpha = 0.65, \beta = 0.85$) at $T = 1$.

To compare the numerical results, Tables 3 and 4 present the L_∞ errors and numerical convergence rates of the EFG method from reference [16] alongside the results of our method for two cases $\alpha = 0.1, \beta = 0.3$ and $\alpha = 0.4, \beta = 0.8$. It is evident that the method proposed in this paper demonstrates significantly better performance in terms of both output accuracy and convergence rate.

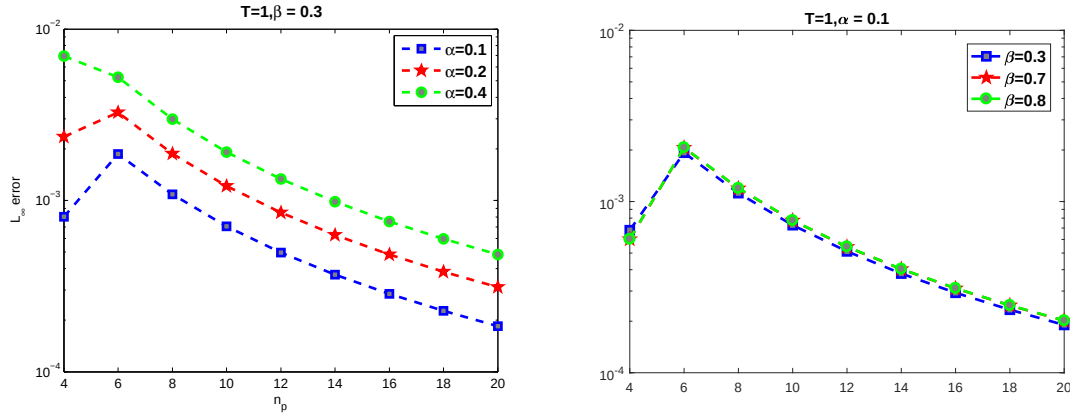
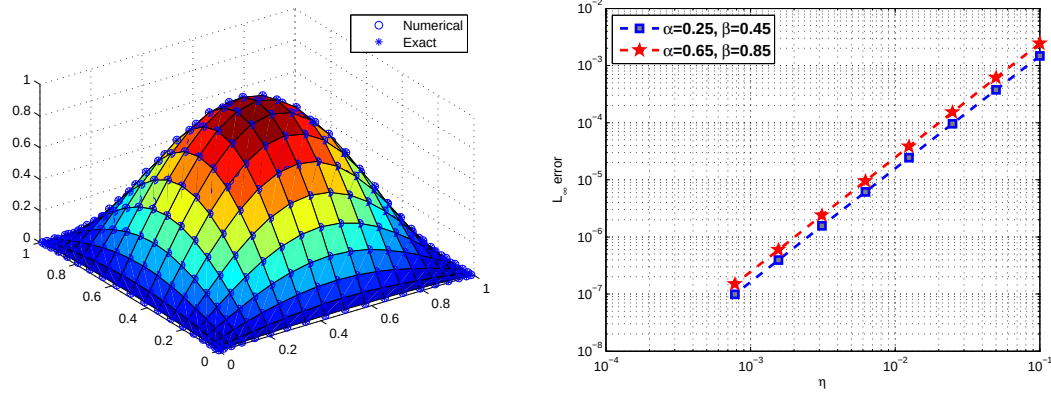
Test Problem 3

For the three-dimensional case, consider the nonlinear FCE (7) in the following manner:

$$\frac{\partial u(\mathbf{x}, t)}{\partial t} = \left(\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \right) \Delta u(\mathbf{x}, t) - \left(\frac{\partial^{1-\beta}}{\partial t^{1-\beta}} \right) u(\mathbf{x}, t) + \left[2t + \frac{2t^{1+\beta}}{\Gamma(2+\beta)} + \frac{6t^{1+\alpha}}{\Gamma(2+\alpha)} \right] \sin(\pi x) \sin(\pi y) \sin(\pi z).$$

Table 2. The L_∞ and L_2 errors, C-orders and CPU times at $T = 1$ for Test problem 2.

n_p	η	$\alpha = 0.1, \beta = 0.3$			$\alpha = 0.2, \beta = 0.7$			CPU-time
		L_∞	L_2	C-order	L_∞	L_2	C-order	
4	1/4	$6.80e-04$	$1.60e-03$	—	$2.71e-03$	$5.38e-03$	—	$2.4e-02$
6	1/6	$1.92e-03$	$4.09e-03$	-1.50	$3.45e-03$	$7.27e-03$	-0.34	$1.1e-02$
10	1/10	$7.27e-04$	$2.53e-03$	1.40	$1.27e-03$	$4.45e-03$	1.43	$2.0e-02$
16	1/16	$2.93e-04$	$1.63e-03$	1.31	$5.09e-04$	$2.83e-03$	1.32	$1.9e-01$
20	1/20	$1.90e-04$	$1.32e-03$	0.62	$3.28e-04$	$2.28e-03$	0.63	$4.9e-01$

**Figure 2.** The L_∞ errors plotted as functions of n_p : on the left for $\beta = 0.3$ with various values of α , and on the right for $\alpha = 0.1$ with different values of β at $T = 1$ (Test problem 2).**Figure 3.** (left) Numerical and exact solutions for $\alpha = 0.1, \beta = 0.3$ and $n_p = 20$ at $T = 1$. (right) log-log plot of the L_∞ errors as a function of η and $n_p = 10$ at $T = 1$ (Test problem 2).

Consider the exact solution

$$u(x, y, t) = t^2 \sin(\pi x) \sin(\pi y) \sin(\pi z),$$

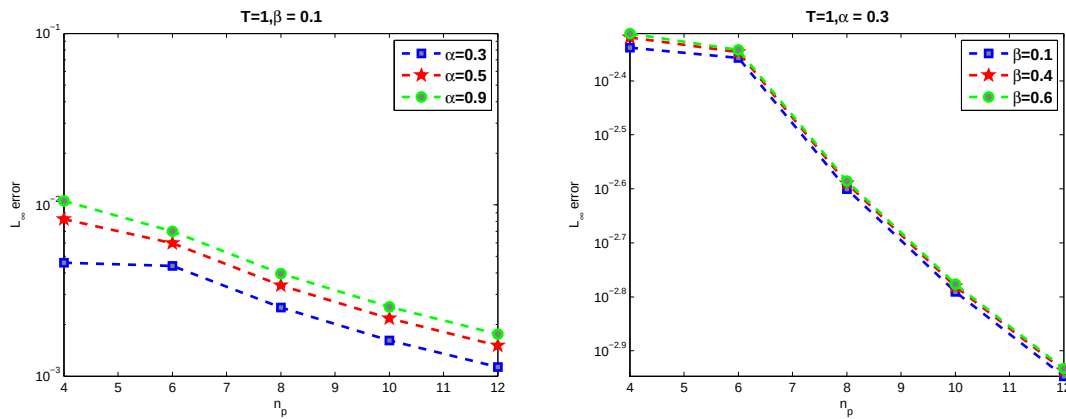
under homogeneous initial and boundary conditions. As in the previous section, Table 5 presents the L_∞ and L_2 errors along with the numerical convergence order at $T = 1$ for two cases ($\alpha = 0.3, \beta = 0.1$ and $\alpha = 0.5, \beta = 0.4$). Table 5 indicates that as η is decreased, the error diminishes and the exponential accuracy improves. The corresponding execution times for the relevant programs are also listed. These results once again confirm the method's accuracy. Additionally, Figures 4 show the maximum errors for various α values with fixed β and vice versa. These graphs clearly demonstrate the exponential error reduction, characteristic of spectral methods.

Table 3. The L_∞ errors and C-orders at $T = 1$ for $\alpha = 0.1$, $\beta = 0.3$ (Test problem 2).

n_p	η	EFG method [16]		Our method	
		L_∞	C-order	L_∞	C-order
4	1/16	$1.00e-01$	—	$4.66e-03$	—
8	1/64	$9.85e-03$	3.35	$1.96e-05$	7.89
16	1/1024	$7.75e-04$	3.67	$1.32e-06$	3.89

Table 4. The L_∞ errors and C-orders at $T = 1$ for $\alpha = 0.4$, $\beta = 0.8$ (Test problem 2).

n_p	η	EFG method [16]		Our method	
		L_∞	C-order	L_∞	C-order
4	1/16	$1.95e-01$	—	$4.14e-03$	—
8	1/64	$1.67e-02$	3.39	$4.91e-05$	6.39
16	1/1024	$1.15e-03$	3.74	$3.11e-06$	3.98

**Figure 4.** The L_∞ errors plotted as functions of n_p : on the left for $\beta = 0.1$ with various values of α , and on the right for $\alpha = 0.3$ with different values of β at $T = 1$ (Test problem 3).**Table 5.** The L_∞ and L_2 errors, C-orders and CPU-times at $T = 1$ for Test problem 3.

n_p	η	$\alpha = 0.3, \beta = 0.1$			$\alpha = 0.5, \beta = 0.4$			CPU-time
		L_∞	L_2	C-order	L_∞	L_2	C-order	
4	1/4	$4.59e-03$	$9.92e-03$	—	$8.35e-03$	$1.60e-02$	—	$7.3e-03$
6	1/6	$4.39e-03$	$1.34e-02$	0.06	$6.04e-03$	$1.83e-02$	0.46	$3.3e-02$
8	1/8	$2.51e-03$	$1.17e-02$	0.81	$3.42e-03$	$1.59e-02$	0.81	$1.5e-01$
10	1/10	$1.61e-03$	$1.05e-02$	0.63	$2.20e-03$	$1.42e-02$	0.64	$1.1e+00$
12	1/12	$1.12e-03$	$9.64e-03$	0.51	$1.53e-03$	$1.30e-02$	0.52	$7.3e+00$
14	1/14	$8.34e-04$	$8.96e-03$	0.43	$1.13e-03$	$1.21e-02$	0.44	$4.3e+01$

5 Conclusions

In this paper, we developed a pseudo-spectral method utilizing Lagrange polynomials to solve the fractional cable equation. We implemented a finite difference scheme for time derivatives and a pseudo-spectral method for spatial discretization. As shown in the previous section, the method exhibits a decreasing error trend and maintains exponential accuracy across all dimensions. Additionally, we demonstrated the method's unconditional stability.

In summary, our proposed method offers spectral accuracy, unique basis functions, proven effectiveness, unconditional stability, and exponential accuracy. By emphasizing these distinctions, we present our research as a significant advancement in solving nonlinear time fractional equations.

For future research, we recommend the following directions:

- Investigate other types of time or spatial fractional PDEs or physical models that might benefit from the pseudo-spectral method.
- Examine the method's applicability to more complex geometries or boundary conditions.
- Explore hybrid approaches that combine the pseudo-spectral method with other numerical methods (e.g., finite element methods, finite difference methods).

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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