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Double Barrier Option Pricing Formulas of an Uncertain Stock Model

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Abstract The valuation of options is an essential topic in the financial markets, and barrier options represent a widely utilized category of options that may gain or lose value once the price of the underlying asset hits a specified threshold. A double barrier option includes two barriers, one above and one below the current stock price. It is classified as path dependent due to the fact that the holder's return is influenced by the stock price's breach of these barriers. The double barrier option contract defines three specific payoffs, which are contingent upon whether the upper barrier or lower barrier is breached, or if there is no breach of either barrier throughout the option's duration. In this paper, pricing of the double barrier options when the underlying asset price follows the uncertain stock model is investigated, and also pricing formulas for different types of double barrier options (knock-in and knock-out) are derived by α-paths of uncertain differential equations in the uncertain environment.

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1 Introduction

Within the realm of financial markets, options are considered an exceptional financial instrument, and their pricing remains a pivotal focus in mathematical finance. In contrast, barrier options and vanilla options exhibit similarities, differing primarily in that a barrier option is either activated or deactivated when the underlying asset's price touches with the barrier price before the option's maturity. Barrier options have been available for trading in the over-the-counter (OTC) market since 1967 and have since become the most popular category of exotic options.

Prior methodologies for pricing options have heavily relied on the Black-Scholes model [1] and Merton's [19] option pricing theory, employing stochastic differential equations (SDEs) to represent the price dynamics of the underlying assets.Merton [19] initially introduced the concept of pricing rational options, later expanding his work to include down and out options. Following this, Rich [23] established a framework for pricing barrier options. Subsequently, the focus shifted towards investigating various methodologies for valuing these options. For example, Nouri, Abbasi et al. [20, 21] introduced an enhanced Monte Carlo algorithm aimed at pricing various types of barrier and double barrier options. Meanwhile, [18] employed a Lie-algebraic approach to determine the value of moving barrier options, and [10] conducted an analytical study on the valuation of American double barrier options. [7] conducted an analysis of double barrier option pricing through the application of a regime-switching exponential mean-reverting process. In 2013, Liu [17] proposed that the use of stochastic differential equations to define the stock price process is unsuitable and results in a significant paradox. This perspective is supported by empirical evidence, which indicates that the peak of the distribution of underlying assets exceeds that of the normal probability distribution, while the tails exhibit greater heaviness.

However, numerous empirical studies have demonstrated that the prices of underlying assets do not conform to the principles of probability and randomness. Instead, financial markets are influenced by a combination of randomness and human uncertainty. The degree of investor belief plays a significant role in this, as investors tend to base their decisions on beliefs rather than probabilities. Kahneman [11] pointed out that the variance in beliefs is much more extensive compared to that of frequencies.

In 2004, Cont and Tankov [6] utilized jump-diffusion models to represent uncertainty, illustrating that these models have sophisticated structures beneficial for asset pricing. In 2007, Liu [12] advanced the theory of uncertainty within the framework of uncertain measures, enhancing the modeling of uncertain phenomena by addressing the concept of belief degrees. In 2008, he introduced an uncertain process [13]. Based on it, researchers in [4,16,26,27] developed methods for solving uncertain differential equations (UDEs). Also, Chen and Liu [4] have established the existence and uniqueness theorem pertaining to the solutions of UDEs. Furthermore, Liu [14] demonstrated the stability of UDEs. In 2009, Liu [14] also formulated various equations for option pricing utilizing uncertain stock models. After that, Peng and Yao [22], Yu [29], Chen [3], Yao [28], and Ji and Zhou [8] dedicated significant attention to the exploration of uncertain stock pricing models. In addition, Chen [2] formulated a pricing equation for American options in 2011. In 2020, Jia and Chen [9] disclosed several compelling discoveries related to the pricing formulas of knock-in barrier options, grounded in an uncertain stock pricing model that includes a floating interest rate. Zhou [8] placed significant emphasis on the analysis of uncertain stock pricing models. In a related development, Chen [2] introduced a formula for American option pricing in 2011. In 2020, Jia and Chen [9] revealed several intriguing insights into the pricing formulas for knock-in barrier options, which were based on an uncertain stock pricing model featuring a floating interest rate. That same year, Rong et al. [24] examined pricing formulas for American barrier options, while Yang et al. [25] focused on strategies for assessing the pricing of Asian barrier options in an uncertain environment.

In the following, Section 2 will cover the necessary preliminary information. Then, for as much as uncertain space is more accorded to real decision problems, Section 3 will present the uncertain stock model for stock pricing in uncertain environments. Section 4 will demonstrate the pricing formulas for double barrier options, encompassing both knock-in and knock-out options, in relation to an uncertain stock model. The paper will conclude with a summary in Section 5.

2 Preliminaries

Suppose L be a σ-algebra on a non-empty set Γ(universal set). If $\mathcal{M}$ is a set function $\mathcal{M} : L \to [0,1]$ and it satisfies the following axioms:

1: (Normality axiom) $\mathcal{M}(\Gamma) = 1$;

2: (Subadditivity axiom) For each sequence of events $\{\Theta_j\}$ that can be counted, we have

$$\mathcal{M}(\bigcup_{j=1}^{\infty} \Theta_j) \leq \sum_{j=1}^{\infty} \mathcal{M}(\Theta_j)$$

3: (Duality axiom) $\mathcal{M}(\Theta) + \mathcal{M}(\Theta^c) = 1$ for every event Θ;
Then, (Γ, L) is a measurable space, and the triplet $(\Gamma, L, \mathcal{M})$ is an uncertain space.

Definition 1 [14]. The set function $\mathcal{M}$, which satisfies the above axioms is called an uncertain measure.

5: (Product Axiom) [14]. Let the triple $(\Gamma_k, L_k, \mathcal{M}_k)$, where $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$ and $L = L_1 \times L_2 \times \cdots \times L_k$ be uncertainty spaces for $k = 1, 2, \ldots, n$, the product uncertain measure $\mathcal{M}$ is an uncertain measure on the σ -algebra satisfying

$$\mathcal{M}(\prod_{k=1}^{\infty} \Theta_k) \leq \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\Theta_k)$$

where Θ_k, for $k = 1, 2, \ldots, n$ are arbitrary chosen events from L_k, respectively.

Definition 2 [15]. The uncertainty distribution for an uncertain variable such as η is defined by function $\Psi : \mathbb{R} \to [0, 1]$ that $\Psi(x) = \mathcal{M}\{\eta \leq x\}$.

Definition 3 Following uncertainty distribution is called normal

$$\Psi(x) = \Big(1 + exp\Big(\frac{\pi(e - x)}{\sqrt{3}\sigma}\Big)\Big)^{-1}, \quad x \in \mathbb{R}. \tag{1}$$

If η be an uncertain variable, in this case $\sigma > 0$ and e are real numbers and it is shown by $\mathcal{N}(e, \sigma)$. The normal uncertaintly distribution can be called standard, if e be equal to 0 and σ be equal to 1.
The inverse uncertaintly distribution of η denoted by $\Psi^{-1}(\alpha)$, $\alpha \in (0, 1)$ and the expected value of an uncertain variable η is defined as

$$E[\eta] = \int_0^1 \Psi^{-1}(\alpha) d\alpha \tag{2}$$

Definition 4 [5] Let α be a number between 0 and 1. An uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \tag{3}$$

is said to have an α-path X_t^α if it solves the corresponding ordinary diferential equation (ODE)

$$dX_t^\alpha = f(t, X_t^\alpha)dt + |g(t, X_t^\alpha)|\Psi^{-1}(\alpha)dt, \tag{4}$$

where $\Psi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Psi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} ln \frac{\alpha}{1 - \alpha} \tag{5}$$

Definition 5 [14] Liu process is an uncertain process C_t which have bellow properties:

1) $C_0 = 0$,
2) C_t has independent and stationary increments,
3) almost all sample paths are Lipschitz continuous,
4) all increments $C_{s+t} - C_s$ are normal uncertain variables with expected value 0 and variance t^2.

Theorem 1 *[26] Let X_t be the solution of the UDE eq.(3) and X_t^α be the solution and α-path of ODE eq.(4). Then*

$$\mathcal{M}\{X_t \le X_t^\alpha, \quad \forall t \in [0,T]\} = \alpha,$$
$$\mathcal{M}\{X_t > X_t^\alpha, \quad \forall t \in [0,T]\} = 1-\alpha$$

Theorem 2 *[28] Assume that $\eta_1, \eta_2, \ldots, \eta_m, \ldots, \eta_n$ are independent uncertain variables and $\Psi_1, \Psi_2, \ldots, \Psi_m, \ldots, \Psi_n$ be regular uncertainty distributions of these variables, respectively. if the function $f(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n)$ is strictly increasing function with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing function with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain process $\eta = f(\eta_1, \ldots, \eta_m, \ldots, \eta_n)$ has an inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f(\Psi_1^{-1}(\alpha), \ldots, \Psi_m^{-1}(\alpha), \ldots, \Psi_{m+1}^{-1}(1-\alpha), \ldots, \Psi_n^{-1}(1-\alpha))$$

where $\Psi^{-1}(\alpha) = X_t^\alpha(\alpha - path\ of\ X_t)$

3 Uncertain stock model for barrier option pricing

Assum that the stock price S_t follows,

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dC_t \\ dp_t = r p_t dt \end{cases} \tag{6}$$

where P_t is the bond price, positive constants r, μ, σ are the risk-less interest rate, log-drift and log-diffusion respectively, and C_t represents a Liu process.

Theorem 3 *Assume that the stock price follows*

$$dS_t = \mu S_t dt + \sigma S_t dC_t \tag{7}$$

where S_t is the stock price at the moment t. Then we obtain an α-path for S_t as

$$S_t^\alpha = S_0 exp(\mu t + \frac{\sqrt{3}\sigma t}{\pi} ln \frac{\alpha}{1-\alpha}) \tag{8}$$

Proof . According to Definition[4], we have

$$dS_t^\alpha = \mu S_t^\alpha dt + \sigma |S_t^\alpha| \Phi^{-1}(\alpha) dt$$

so

$$\frac{dS_t^\alpha}{S_t^\alpha} = \mu dt + \frac{\sqrt{3}\sigma}{\pi} ln \frac{\alpha}{1-\alpha} dt$$

and

$$dln S_t^\alpha = \mu dt + \frac{\sqrt{3}\sigma}{\pi} ln \frac{\alpha}{1-\alpha} dt$$

By solving the above differential equation we have

$$S_t^\alpha = S_0 exp(\mu t + \frac{\sqrt{3}\sigma t}{\pi} ln \frac{\alpha}{1-\alpha}) \tag{9}$$

4 double barrier options

In double barrier options, one barrier is established above the current stock price, and the other is below it. Since the payoff of the option depends on the behavior of the stock price process due to these two obstacles, thus classifying it as a path-dependent option. Throughout the duration of double barrier options, the payoff can be categorized into three types, depending on whether the upper barrier, the lower barrier, or neither is breached. When the payoff of an option diminishes upon reaching the barrier, that barrier is deemed worthless; conversely, if the payoff rises, the barrier is considered valuable. One of the aspect of the barrier is its potential use for the throughout the life of the option or for a segment of the life of the option. In this section, we have presented the formula for pricing double barrier options, which asset price follows Eq [6].

4.1 European knock-in options

One variant of barrier options is the knock-in option, which is activated only when the underlying asset exceeds a designated price level. This condition restricts traders to buying or selling this option solely at the moment the asset's price attains the predetermined level. If the knock-in price is breached at any time before the option's maturity date, the payoff of the option is converted into a vanilla option, and the knock-in barrier option expires without value. In this section, we provide the formula for pricing European knock-in options, where the asset price follows Eq [6].

4.1.1 Pricing formula for double knock-in call option

Consider a double barrier option which the lower barrier level is B_L, the upper barrier level is B_U, the exercise price is K, and the expiration time is T. If before the maturity T, the underlying asset price S_t hits the lower or upper barrier level and exceeds them, then this call option will become into existence, and its payoff will be $max(S_t - K, 0)$ on the maturity date. Now we assign $\eta^+ = max(\eta, 0)$ and apply an indicator function

$$I_B(\eta) = \begin{cases} 1, & S_t < B_L \ or \ S_t > B_U \\ 0, & B_L < \eta < B_U \end{cases}$$

Hence, the payoff on the maturity time is written as;

$$payoff = (S_T - K)^+(I_B(S_t)) \tag{10}$$

By taking into account the discount rate on the initial date, the discounted expectation of payoff is

$$B_{dki} = e^{-rT}(S_T - K)^+(I_B(S_t)) \tag{11}$$

and a price of this kind of double barrier options is

$$f^c_{dki} = E[B_{dki}] = E[e^{-rT}(S_T - K)^+(I_B(S_t))] \tag{12}$$

Theorem 4 *Consider a double knock-in call option for stock pricing model that underlying uncertain Eq. [6] has a lower barrier level B_L, upper barrier level B_U, exercise price K, and the expiration time T. Then the price of this option is defined by*

$$f^c_{dki} = e^{-rT}[\int_0^{\theta_0} (S^\alpha_T - K)^+ d\alpha + \int_{\theta_1}^1 (S^\alpha_T - K)^+ d\alpha] \tag{13}$$

where

$$\theta_0 = \frac{exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t)]}{1 + exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t)]} \tag{14}$$

and

$$\theta_1 = \frac{exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t)]}{1 + exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t)]} \tag{15}$$

and

$$S^\alpha_t = S_0 exp(\mu t + \frac{\sqrt{3}\sigma t}{\pi} ln \frac{\alpha}{1-\alpha}) \tag{16}$$

Proof . Note that

$$I_B(S^\alpha_t) = 1 \tag{17}$$

if and only if

$$S^\alpha_t < B_L \quad or \quad S^\alpha_t > B_U \tag{18}$$

In addition

$$\begin{aligned} &S_0 exp(\mu t + \frac{\sqrt{3}\sigma t}{\pi} ln \frac{\alpha}{1-\alpha}) < B_L \\ \Rightarrow &\mu t + \frac{\sqrt{3}\sigma t}{\pi} ln \frac{\alpha}{1-\alpha} < ln(\frac{B_L}{S_0}) \\ \Rightarrow &\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t) > ln \frac{\alpha}{1-\alpha} \end{aligned}$$

By taking

$$M = \frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t) \tag{19}$$

we have

$$e^M > \frac{\alpha}{1-\alpha} \tag{20}$$

then

$$\alpha < \frac{e^M}{1+e^M} = \theta_0 \tag{21}$$

On the other hand

$$S_0exp(\mu t + \frac{\sqrt{3}\sigma t}{\pi}ln\frac{\alpha}{1-\alpha}) > B_U$$
$$\Rightarrow \mu t + \frac{\sqrt{3}\sigma t}{\pi}ln\frac{\alpha}{1-\alpha} > ln(\frac{B_U}{S_0})$$
$$\Rightarrow \frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t) < ln\frac{\alpha}{1-\alpha}$$

By taking

$$N = \frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t) \tag{22}$$

we have

$$e^N < \frac{\alpha}{1-\alpha} \tag{23}$$

then

$$\alpha > \frac{e^N}{1+e^N} = \theta_1 \tag{24}$$

Example 1 Assume the initial stock price $S_0 = 8$, risk-less interest rate $r = 0.03$, lower barrier level $B_L = 6$, upper barrier level $B_U = 12$, strike price $K = 20$, time to maturity $T = 15$, log-diffusion $\sigma = 0.05$ and log-drift $\mu = 0.04$. Then the price of double knock-in call option is 3.4394.

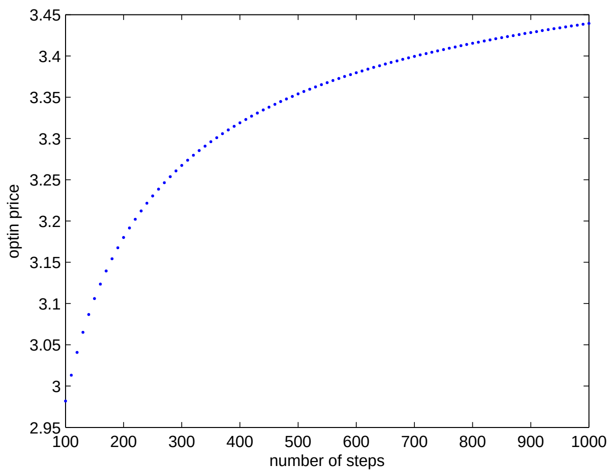

Fig. 1 The barrier option price f^c_{dki} with respect to different step N in Example 1.

4.1.2 Pricing formula for double knock-in put option

Consider a double barrier option which the lower barrier level is B_L, the upper barrier level is B_U, the exercise price is K, and the expiration time is T. If before the maturity T, the underlying asset price S_t hits the lower or upper barrier level and exceeds them, then this call option will become into existence, and its payoff will be $max(K - S_t, 0)$ on the maturity date. Now we assign $\eta^+ = max(\eta, 0)$ and apply an indicator function

$$I_B(\eta) = \begin{cases} 1, & S_t < B_L \ or \ S_t > B_U \\ 0, & B_L < \eta < B_U \end{cases}$$

Hence, the payoff on the maturity time is written as;

$$payoff = (K - S_T)^+(I_B(S_t)) \tag{25}$$

By taking into account the discount rate on the initial date, the discounted expectation of payoff is

$$B_{dki} = e^{-rT}(K - S_T)^+(I_B(S_t)) \tag{26}$$

and a price of this kind of double barrier options is

$$f^p_{dki} = E[B_{dki}] = E[e^{-rT}(K - S_T)^+(I_B(S_t))] \tag{27}$$

Theorem 5 *Consider a double knock-in put option for stock pricing model that underlying uncertain Eq. [6] has a lower barrier level B_L, upper barrier level B_U, exercise price K, and the expiration time T. Then the price of this option is defined by*

$$f^p_{dki} = e^{-rT}[\int_0^{\theta_0}(K - S^\alpha_T)^+ d\alpha + \int_{\theta_1}^1 (K - S^\alpha_T)^+ d\alpha] \tag{28}$$

where

$$\theta_0 = \frac{exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t)]}{1 + exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t)]} \tag{29}$$

and

$$\theta_1 = \frac{exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t)]}{1 + exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t)]} \tag{30}$$

and

$$S^\alpha_t = S_0 exp(\mu t + \frac{\sqrt{3}\sigma t}{\pi} ln\frac{\alpha}{1 - \alpha}) \tag{31}$$

Proof . Similar to theorem 4 it will be proved.

Example 2 Assume the initial stock price $S_0 = 10$, risk-less interest rate $r = 0.03$, lower barrier level $B_L = 8$, upper barrier level $B_U = 12$, strike price $K = 15$, time to maturity $T = 15$, log-diffusion $\sigma = 0.05$ and log-drift $\mu = 0.04$.Then the price of double knock-in put option is 1.2967.

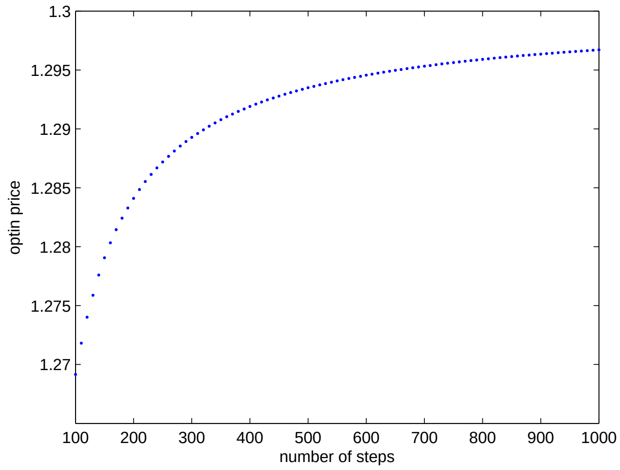

Fig. 2 The barrier option price f^p_{dki} with respect to different step N in Example 2.

4.2 European knock-out options

A knock-out option is a type of barrier option that if the price of the underlying asset does not exceed a predetermined barrier level throughout the option's duration, the option will yield a payoff. However, should the asset's price surpass this barrier level at any time before the maturity date T, the payoff is rendered null. In this section, we have presented the formula for pricing European knock-out option, which asset price follows Eq [6].

4.2.1 Pricing formula for double knock-out call option

Consider a double barrier option which the lower barrier level is B_L, the upper barrier level is B_U, the exercise price is K, and the expiration time is T. If before the maturity T, the spot price S_t always be between the lower barrier level B_L and upper barrier level B_U, then this call option will become into existence, and its payoff will be $max(S_t - K, 0)$ on the maturity date. Now we assign $\eta^+ = max(\eta, 0)$ and apply an indicator function

$$I_B(\eta) = \begin{cases} 1, & B_L < \eta < B_U \\ 0, & \eta < B_L \ or \ \eta > B_U \end{cases}$$

Hence, the payoff on the maturity time is written as;

$$payoff = (S_T - K)^+ (I_B(S_t)) \tag{32}$$

By taking into account the discount rate on the initial date, the discounted expectation of payoff is

$$B_{dko} = e^{-rT}(S_T - K)^+(I_B(S_t)) \tag{33}$$

and a price of this kind of double barrier options is

$$f^c_{dko} = E[B_{dko}] = E[e^{-rT}(S_T - K)^+(I_B(S_t))] \tag{34}$$

Theorem 6 *Consider a double knock-out call option for stock pricing model that underlying uncertain Eq. [6] has a lower barrier level B_L, upper barrier level B_U, exercise price K, and the expiration time T. Then the price of this option is defined by*

$$f^c_{dko} = e^{-rT}\int_{\theta_0}^{\theta_1}(S^\alpha_T - K)^+ d\alpha \tag{35}$$

where

$$\theta_0 = \frac{exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t)]}{1 + exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t)]} \tag{36}$$

and

$$\theta_1 = \frac{exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t)]}{1 + exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t)]} \tag{37}$$

and

$$S^\alpha_t = S_0 exp(\mu t + \frac{\sqrt{3}\sigma t}{\pi} ln\frac{\alpha}{1-\alpha}) \tag{38}$$

Proof . Note that

$$I_B(S^\alpha_t) = 1 \tag{39}$$

if and only if

$$B_L < S^\alpha_t < B_U \tag{40}$$

In addition

$$\begin{aligned} &B_L < S^\alpha_t = S_0 exp(\mu t + \frac{\sqrt{3}\sigma t}{\pi} ln\frac{\alpha}{1-\alpha}) < B_U \\ \Rightarrow &ln(\frac{B_L}{S_0}) < \mu t + \frac{\sqrt{3}\sigma t}{\pi} ln\frac{\alpha}{1-\alpha} < ln(\frac{B_U}{S_0}) \\ \Rightarrow &\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t) < ln\frac{\alpha}{1-\alpha} < \frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t) \end{aligned}$$

By taking

$$M = \frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t) \tag{41}$$

and

$$N = \frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t) \tag{42}$$

we have

$$e^M < \frac{\alpha}{1-\alpha} < e^N \tag{43}$$

then

$$\theta_0 = \frac{e^M}{1+e^M} < \alpha < \frac{e^N}{1+e^N} = \theta_1 \tag{44}$$

Example 3 Assume the initial stock price $S_0 = 10$, risk-less interest rate $r = 0.03$, lower barrier level $B_L = 8$, upper barrier level $B_U = 25$, strike price $K = 25$, time to maturity $T = 15$, log-diffusion $\sigma = 0.05$ and log-drift $\mu = 0.04$. Then the price of knock-out call option is 4.3286.

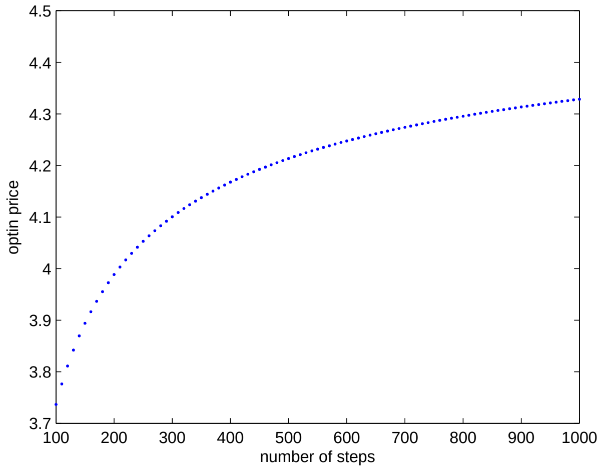

Fig. 3 The double barrier option price f^c_{dko} with respect to different step N in Example 3.

4.2.2 Pricing formula for double knock-out put option

Consider a double barrier option which the lower barrier level is B_L, the upper barrier level is B_U, the exercise price is K, and the expiration time is T. If before the maturity T, the spot price S_t always be between the lower barrier level B_L and upper barrier level B_U, then this call option will become into existence, and its payoff will be $max(K - S_t, 0)$ on the maturity date. Now we assign $\eta^+ = max(\eta, 0)$ and apply an indicator function

$$I_B(\eta) = \begin{cases} 1, & B_L < \eta < B_U \\ 0, & \eta < B_L \text{ } or \text{ } \eta > B_U \end{cases}$$

Hence, the payoff on the maturity time is written as;

$$payoff = (K - S_T)^+ (I_B(S_t)) \tag{45}$$

By taking into account the discount rate on the initial date, the discounted expectation of payoff is

$$B_{dko} = e^{-rT}(K - S_T)^+ (I_B(S_t)) \tag{46}$$

and a price of this kind of double barrier options is

$$f^p_{dko} = E[B_{dko}] = E[e^{-rT}(K - S_T)^+ (I_B(S_t))] \tag{47}$$

Theorem 7 *Consider a double knock-out put option for stock pricing model that underlying uncertain Eq. [6] has a lower barrier level B_L, upper barrier level B_U, exercise price K, and the expiration time T. Then the price of this option is defined by*

$$f^p_{dko} = e^{-rT} \int_{\theta_0}^{\theta_1} (K - S^\alpha_T)^+ d\alpha \tag{48}$$

where

$$\theta_0 = \frac{exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t)]}{1 + exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_L}{S_0}) - \mu t)]} \tag{49}$$

and

$$\theta_1 = \frac{exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t)]}{1 + exp[\frac{\pi}{\sqrt{3}\sigma t}(ln(\frac{B_U}{S_0}) - \mu t)]} \tag{50}$$

and

$$S^\alpha_t = S_0 exp(\mu t + \frac{\sqrt{3}\sigma t}{\pi} ln \frac{\alpha}{1 - \alpha}) \tag{51}$$

Proof . Similar to theorem 6 it will be proved.

Example 4 Assume the initial stock price $S_0 = 10$, risk-less interest rate $r = 0.03$, lower barrier level $B_L = 8$, upper barrier level $B_U = 25$, strike price $K = 20$, time to maturity $T = 15$, log-diffusion $\sigma = 0.05$ and log-drift $\mu = 0.04$. Then the price of double knock-out put option is 2.8072.

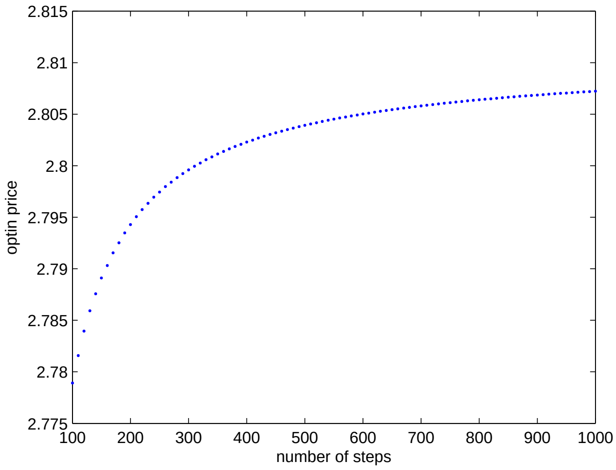

Fig. 4 The barrier option price f^p_{dko} with respect to different step N in Example 4.

5 Conclusion

Since the probability space and randomness are insufficient for effectively simulating investor decisions, prompting many researchers to recommend Liu's uncertain space for these applications. In this study, we introduce an uncertain process to validate the double barrier option pricing formula. The pricing formulas for both knocked-in and knocked-out options are derived through α-paths of uncertain differential equations (UDEs) within this uncertain environment. Furthermore, several numerical examples are provided to demonstrate the pricing of double barrier options using the proposed model. Further research may use other types of exotic options on this uncertain stock pricing model with similar conditions and may consider multi-asset options in the uncertain environment and derive formulas for option pricing.

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