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Research article

On the Index Set of Complete Multipartite Graphs

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Abstract

For an undirected graph G, and an abelian group A, an A-magic labelling is an assignment of non-zero element of A, to the edges of G, such that the sum of the values of all edges incident with each vertex is constant. A constant on magic sum is called an index set of G. Shiu and Low proved that, zero is in the index set of complete multipartite graph. In this paper, for $t \geq 2$ we determine the index set of the complete multipartite graph $K_{n_1,\dots,n_t}$, where $n_i \geq 2$ (for $i = 1,\dots,t$).

Keywords: Index set, Magic, Null set, Zero-sum

Mathematics Subject Classification (2020): 05C15, 05C78

1 Introduction

For an abelian group A, written additively, any mapping $l : E(G) \longrightarrow A \setminus \{0\}$ is called a *labeling*. Given a labeling on the edge set of G, one can introduce a vertex labeling $l^+ : V(G) \longrightarrow A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph G is said to be *A-magic*, if there is a labeling $l : E(G) \longrightarrow A \setminus \{0\}$, such that for each vertex v, the sum of values of all edges incident with v, is a constant; that is, $l^+(v) = c$, for some $c \in A$. In general, a graph G may admit more than one labeling to become A-magic. For example, if $|A| > 2$ and $l : E(G) \longrightarrow A \setminus \{0\}$ is a magic labeling of G with sum c, then $\lambda : E(G) \longrightarrow A \setminus \{0\}$, the inverse labeling of l, defined by $\lambda(uv) = -l(uv)$ will provide another magic labeling of G with sum $-c$. Recently the labeling of graphs have been attracted many researchers to itself, for instance see [1–3, 7, 9]. We use $K_{n_1,\dots,n_k}$ to denote the complete multipartite graph with part sizes $n_1,\dots,n_k$. We denote the complete graph of order n by K_n. Also, if we decompose $E(G)$ into $E(H_1),\dots,E(H_k)$, then we write $G = H_1 \oplus \cdots \oplus H_k$. Within the mathematical literature, various definition of magic graphs have been introduced. The original concept of an A-magic graph is due to Sedlaced [10, 11], who defined it to be a graph with real-valued edge labeling such that distinct edges have distinct non-negative values, and the sum of the values of edges incident with a particular vertex is the same for all vertices. Over the years, there has been a great research interest in graph labeling problems. In fact, many different graph labelings have been introduced in the literature. The interested readers is referred to Wallis' [14] recent monograph on magic graphs. For convenience, a $\mathbb{Z}_h$-magic graph will be referred to as an *h-magic graph*. A constant of a magic sum is called an index of G, an index for short, and we write $\mathrm{I}_A(G) = \{r : G \text{ is } A\text{-magic with index } r\}$. An h-magic graph G is said to have *h-zero-sum magic labeling* if there is a magic labeling of G in $\mathbb{Z}_h$ that induces a vertex labeling with sum 0. The *null* *set* of a graph G, denoted by $N(G)$, is the set of all natural numbers $h \in N$ such G admits a zero-sum magic labeling in $\mathbb{Z}_h$. Salehi in [8] determined the null set of complete graphs.

Theorem 1. *If $n \geq 4$, then*

$$N(K_n) = \begin{cases} \mathbb{N} & n \text{ is odd;} \\ \mathbb{N} \setminus \{2\} & n \text{ is even.} \end{cases}$$

Also, Shiu and M. Low determined in [12], the null set of complete multipartite graphs over an abelian group A.

Theorem 2. *Let A, ($|A| \geq 3$) be an abelian group and $t, n_i \geq 2$, ($i = 1, \dots, t$) be positive integers. Then $K_{n_1,\dots,n_t}$ admits an A-zero-sum magic labeling.*

2 The Index Set of Complete and Complete Bipartite Graphs

In this section we determine the index set of the complete and complete bipartite graphs for an abelian group $\mathbb{Z}_h$, $h \geq 2$. A k-factor of a graph G is a k-regular spanning subgraph of G and a k-factorization is partition the edges of the graph into disjoint k-factors. A graph G is said to be k-factorable if it admits a k-factorization. First, we need the following results.

Theorem 3. [4] *Every complete graph of order $2n$ has 1-factorization.*

Theorem 4. ([10, p.140]) *Every regular graph of even degree has 2-factorization.*

Theorem 5. [13] *Let G be an r-regular graph ($r \geq 2$) which admits a 1-factor, then $\mathrm{I}_h(G) = \mathbb{Z}_h$ for all $h \geq 3$.*

Remark 1. Let G be a graph, $l : E(G) \to \mathbb{Z}_h \setminus \{0\}$ be an edge labeling and $0 \neq c \in \mathrm{I}_h(G)$, then we have

$$2 \sum_{e \in E(G)} l(e) = c|V(G)| \pmod{h}.$$

Also, it is obvious that G admits a 2-magic labeling with sum 1 if and only if the degree of every vertex is odd.

Theorem 6. *Let $h \geq 3$ be positive integers. Then following statements hold:*

(i) $\mathrm{I}_h(K_2) = \mathbb{Z}_h \setminus \{0\}$

(ii) $\mathrm{I}_h(K_3) = \begin{cases} \mathbb{Z}_h \setminus \{0\} & h \text{ is odd;} \\ 2\mathbb{Z}_h & \text{otherwise.} \end{cases}$

(iii) Let $n \geq 4$. Then

$$\mathrm{I}_h(K_n) = \begin{cases} 2\mathbb{Z}_h & \text{if } n \text{ is odd and } h \text{ is even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}$$

Proof. (**i**) It is obvious.

(**ii**) Let $x \in \mathbb{Z}_h \setminus \{0\}$. Note that in any h-magic labeling of an odd cycle, the edges should alternatively be the same value. So, if h is odd, then $0 \notin \mathrm{I}_h(K_3)$. Suppose that $h = 2k+1$. If we assign value $(k+1)x$ to all edges of K_3, then $x \in \mathrm{I}_h(K_3)$. Therefore, $\mathrm{I}_h(K_3) = \mathbb{Z}_h \setminus \{0\}$. Now, assume that $h = 2k$. By Remark 1, if x is odd, then $x \notin \mathrm{I}_h(K_3)$. Let $x = 2t$, for some t. If we assign value t to all edges of K_3, then $x \in \mathrm{I}_h(K_3)$ and if we assign value k to all edges of K_3, then $0 \in \mathrm{I}_h(K_3)$. Thus, $\mathrm{I}_h(K_3) = 2\mathbb{Z}_h$.

(**iii**) By Theorem 1, $0 \in \mathrm{I}_h(K_n)$. Now, we show that K_n has an h-magic labeling with sum 1. Consider two following cases:

Case 1. If $n = 2r+1$, then by Theorem 4, K_n has a 2-factorization. Let $F_1, \dots, F_r$ be its 2-factors. First suppose that $h = 2k+1$. If r is odd, then assign value $k+1$ to the edges of F_1 and value $(-1)^i$ to the edges of F_i, for $i = 2, \dots, r$. If r is even, then assign value k and 1 to the edges of F_1 and F_2, respectively and for $i = 3, \dots, r (r \geq 3)$, assign value $(-1)^i$ to the edges of F_i. Therefore, K_n admits a h-magic labeling with sum 1. So, $\mathrm{I}_h(K_n) = \mathbb{Z}_h$.

Now, assume that $h = 2k$. By Remark 1, if $c \in \mathbb{Z}_h \setminus \{0\}$ is odd, then $c \notin \mathrm{I}_h(K_n)$. We prove that $2 \in \mathrm{I}_h(K_n)$. If r is odd, then assign value 1 to the edges of F_1 and for $i = 2, \dots, r$, assign value $(-1)^i$ to the edges of F_i. If r is even, then assign value -1 and 2 to the edges of F_1 and F_2, respectively and for $i = 3, \dots, r$, assign $(-1)^i$ to the edges of F_i. Therefore, if k is odd, then by Remark 1, $\mathrm{I}_h(K_n) = 2\mathbb{Z}_h \setminus \{k\}$. Now, if $k = 2t$, then assign value $t-1$ and 1 to the edges of F_1 and F_2, respectively and value $(-1)^i$ to the edges of F_i. So, $k \in \mathrm{I}_h(K_n)$, as desired.

Case 2. If n is even, then by Theorem 3, K_n has 1-factorization. Let $E_1, \dots, E_{n-1}$ be a 1-factorization for K_n. We assign value 1 to the edges of E_1 and value $(-1)^i$ to the edges of E_i $(2 \le i \le n-1)$. So, $1 \in \mathrm{I}_h(K_n)$ and the proof is complete. □

Theorem 7. *Let $h \ge 3$, $m, n \ge 2$ be positive integers. Then $\mathrm{I}_h(K_{m,n}) = \mathbb{Z}_h$ if and only if $m = n \pmod h$.*

Proof. Let $\mathrm{I}_h(K_{m,n}) = \mathbb{Z}_h$. Since $1 \in \mathrm{I}_h(K_{m,n})$, there is an edge labeling $l : E(K_{m,n}) \to \mathbb{Z}_h \setminus \{0\}$ such that for every $v \in V(G)$, $l^+(v) = 1 \pmod h$. By double counting we have, $m = \sum_{e \in E(K_{m,n})} l(e) = n \pmod h$. Thus $m = n \pmod h$.

Conversely, by Theorem 2, $K_{m,n}$ admits an h-zero-sum magic labeling. Let $\{1, 2, \dots, m\}$, $\{1', 2', \dots, n'\}$ be the vertex sets of two parts of $K_{m,n}$ and with no loss of generality assume that $m \ge n$. We partition all edges of $K_{m,n}$ as follows:

$$K_{m,n} = K_{n,n} \oplus K_{m-n,n}.$$

By Theorem 5, $\mathrm{I}_h(K_{n,n}) = \mathbb{Z}_h$. We claim that $K_{m-n,n}$ has a magic labeling in $\mathbb{Z}_h$ such that for every vertex v of part of size n, the sum of values of all edges incident with v is 0 and for every vertex w of part of size $m-n$, the sum of values of all edges incident with w is 1. Consider two cases:

Case 1. Let n be odd. Assign value $(-1)^{i'+1}$ to all edges incident with i' $(1' \le i' \le n')$.

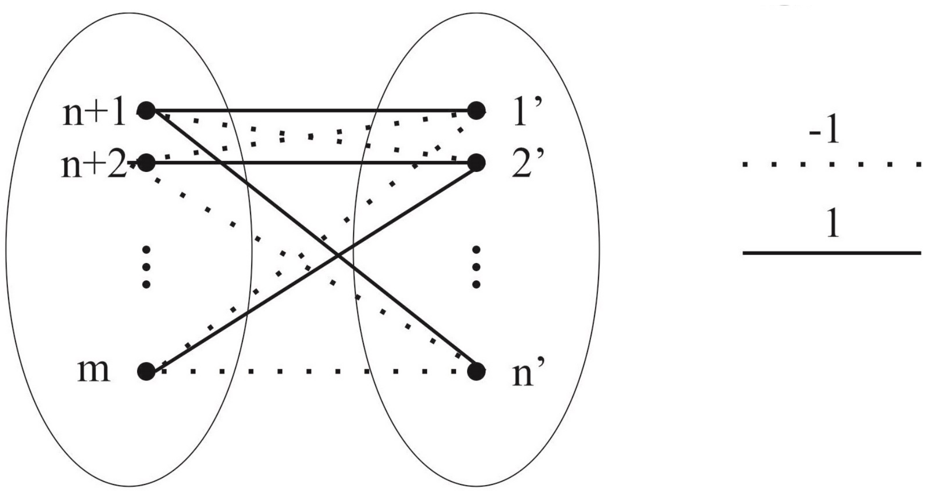

Figure 1. $K_{m-n,n}$

Case 2. Let n be even. Consider the following edge labeling of $K_{m-n,n}$ for all edges incident with i' $(1' \le i' \le n')$.

$$2, \dots, 2, \quad -1, \dots, -1, \quad 1, \dots, 1, \quad -1, \dots, -1, \quad \dots \quad 1, \dots, 1, \quad -1, \dots, -1.$$

Hence $1 \in \mathrm{I}_h(K_{m,n})$ (See Figure 2, (a)). If h is odd, then $\mathrm{I}_h(K_{m,n}) = \mathbb{Z}_h$. If $h = 2k$, then the above edge labeling shows that $\mathbb{Z}_h \setminus \{k\} \subseteq \mathrm{I}_h(K_{m,n})$. The following edge labeling of $K_{m-n,n}$ for all edges incident with i' $(1' \le i' \le n')$ shows that $k \in \mathrm{I}_h(K_{m,n})$ and so $\mathrm{I}_h(K_{m,n}) = \mathbb{Z}_h$ (See Figure 2, (b)).

$$k-1, \dots, k-1, \quad 1, \dots, 1, \quad 1, \dots, 1, \quad -1, \dots, -1, \quad \dots \quad 1, \dots, 1, \quad -1, \dots, -1.$$

The proof is complete. □

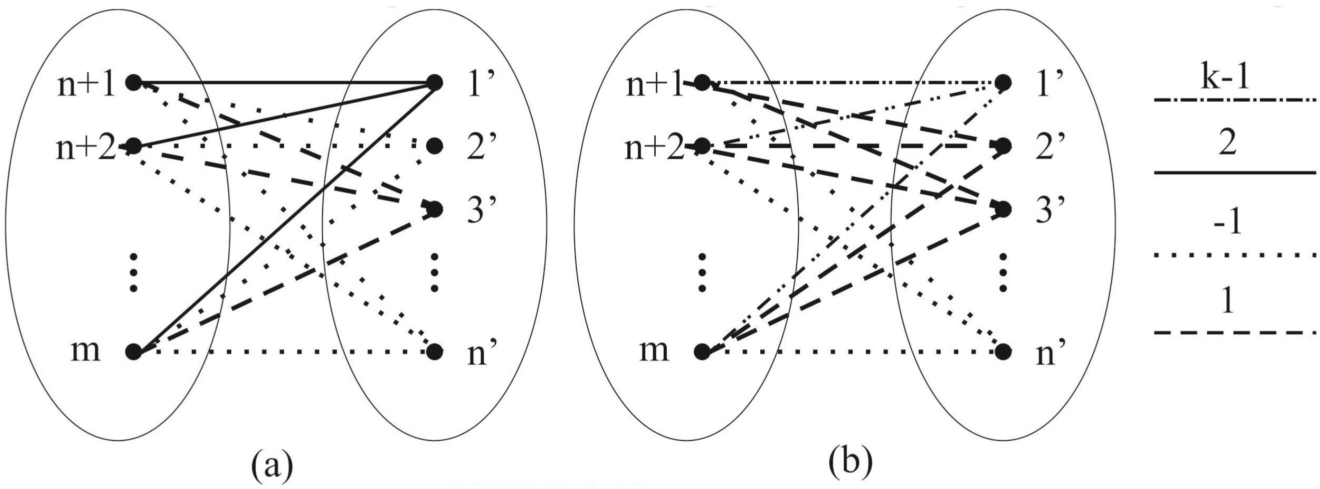

Figure 2. $K_{m-n,n}$

3 The Index Set of the Complete Multipartite Graphs

In this section we would like to determine the index set of the complete multipartite graphs over an abelian group $\mathbb{Z}_h$, $h \geq 2$. From now on, we denote the vertex set of part of size n_i by X_i in $K_{n_1,\ldots,n_t}$, for $i = 1,\ldots,t$.

Lemma 1. *Let $h \geq 3$, $m,n,t \geq 2$ be positive integers and $m = n = t \pmod h$. Then the following holds:*

$$\mathrm{I}_h(K_{m,n,t}) = \begin{cases} 2\mathbb{Z}_h & \text{if } m+n+t \text{ is odd and } h \text{ is even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2, $K_{m,n,t}$ has an h-zero-sum magic labeling. We partition all edges of $K_{m,n,t}$ as follows:

$$K_{m,n,t} = K_{m,n} \oplus K_{n,t} \oplus K_{m,t}.$$

Let $x \in \mathbb{Z}_h \setminus \{0\}$. First suppose that $h = 2k+1$. By Theorem 7, $x(k+1) \in \mathrm{I}_h(K_{m,n})$, $x(k+1) \in \mathrm{I}_h(K_{n,t})$ and $x(k+1) \in \mathrm{I}_h(K_{m,t})$. So, $x \in \mathrm{I}_h(K_{m,n,t})$, as desired. Now, assume that h is even and $x = 2t$. By Theorem 7, $t \in \mathrm{I}_h(K_{m,n})$, $t \in \mathrm{I}_h(K_{n,t})$ and $t \in \mathrm{I}_h(K_{m,t})$. So, $x \in \mathrm{I}_h(K_{m,n,t})$. If $m+n+t$ and $c \in \mathbb{Z}_h \setminus \{0\}$ are odd, then by Remark 1, $c \notin \mathrm{I}_h(K_{m,n,t})$. Thus $\mathrm{I}_h(K_{m,n,t}) = 2\mathbb{Z}_h$. Now, suppose that $m+n+t$ is even. If $m = n = t$, then by Dirac Theorem [8, p.288], $K_{n,n,n}$ has a hamilton cycle and so it contains a 1-factor. Thus by Theorem 5, $\mathrm{I}_h(K_{n,n,n}) = \mathbb{Z}_h$. Let X_1, X_2 and X_3 be three parts of $K_{m,n,t}$ such that $|X_1| = m$, $|X_2| = n$ and $|X_3| = t$. Let $Y_2 \subseteq X_2$ and $Y_3 \subseteq X_3$ be two sets such that $|Y_2| = |Y_3| = m$. Also, let $W_2 = X_2 \setminus Y_2$, $W_3 = X_3 \setminus Y_3$. We consider two cases:

Case 1. Let $m < n \leq t$. We partition all edges of $K_{m,n,t}$ as follows:

$$K_{m,n,t} = K_{X_1,Y_2,Y_3} \oplus K_{W_2,W_3} \oplus K_{W_2,X_1\cup Y_3} \oplus K_{W_3,X_1\cup Y_2}.$$

By Theorem 2, both $K_{W_2,X_1\cup Y_3}$ and $K_{W_3,X_1\cup Y_2}$ have an h-zero-sum magic labeling. Also, as we did before, $\mathrm{I}_h(K_{X_1,Y_2,Y_3}) = \mathbb{Z}_h$ and by Theorem 7, $\mathrm{I}_h(K_{W_2,W_3}) = \mathbb{Z}_h$. Therefore, $\mathrm{I}_h(K_{m,n,t}) = \mathbb{Z}_h$.

Case 2. Let $m = n < t$. We partition all edges of $K_{m,n,t}$ as follows:

$$K_{m,n,t} = K_{X_1,X_2,Y_3} \oplus K_{X_1,W_3} \oplus K_{X_2,W_3}.$$

By Theorem 2, K_{X_1,W_3} has an h-zero-sum magic labeling and as we did before, $\mathrm{I}_h(K_{X_1,X_2,Y_3}) = \mathbb{Z}_h$. Now, let $x \in \mathbb{Z}_h \setminus \{0\}$. We would like to assign some values to the edges of K_{X_2,W_3} such that every vertex of part of size X_2 has vertex labeling 0 and every vertex of part of size W_3 has vertex labeling x. If $x \neq 1$, then the following edge labeling of K_{X_2,W_3} shows that $x \in \mathrm{I}_h(K_{X_2,W_3})$.

$$x-1,\dots,x-1, \quad 1,\dots,1, \quad 1,\dots,1 \quad -1,\dots,-1 \quad \dots \quad 1,\dots,1, \quad -1,\dots,-1,$$

where the size of each block is $|W_3|$. Also, to obtain the index 1 consider the following edge labeling of K_{X_2,W_3}.

$$2,\dots,2, \quad -1,\dots,-1, \quad 1,\dots,1 \quad -1,\dots,-1 \quad \dots \quad 1,\dots,1, \quad -1,\dots,-1,$$

where the size of each block is $|W_3|$. So, $\mathrm{I}_h(K_{m,n,t}) = \mathbb{Z}_h$ and the proof is complete. □

Theorem 8. *Let $h,t \geq 3$ and $n_i \geq 2$ $(i = 1,\dots,t)$* be positive integers and $n_1 = \cdots = n_t \pmod h$. Then the following holds:

$$\mathrm{I}_h(K_{n_1,\dots,n_t}) = \begin{cases} 2\mathbb{Z}_h & \text{if } n_1 + \cdots + n_t \text{ is odd and } h \text{ is even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2, $K_{n_1,\dots,n_t}$ has an h-zero-sum magic labeling. Note that by Lemma 1 we can assume that $t \geq 4$. Consider two cases:

Case 1. Suppose that t is even. We partition all edges of $K_{n_1,\dots,n_t}$ as follows:

$$K_{n_1,\dots,n_t} = K_{n_1,n_2} \oplus K_{n_3,n_4} \oplus \dots \oplus K_{n_{t-1},n_t} \oplus H,$$

where H is a complete $\frac{t}{2}$ -partite graph with parts $X_{2i-1} \cup X_{2i}$, $(1 \leq i \leq \frac{t}{2})$. By Theorems 2 and 7, H has an h-zero-sum magic labeling and

$$\mathrm{I}_h(K_{n_{2i-1},n_{2i}}) = \mathbb{Z}_h, \qquad (1 \leq i \leq \frac{t}{2}),$$

as desired.

Case 2. If t is odd, then we partition all edges of $K_{n_1,\dots,n_t}$ as follows:

$$K_{n_1,\dots,n_t} = K_{n_1,n_2,n_3} \oplus K_{n_4,\dots,n_t} \oplus H',$$

where H' is a complete bipartite graph with parts $X_1 \cup X_2 \cup X_3$ and $X_4 \cup \dots \cup X_t$. By Theorem 2, H' has an h-zero-sum magic labeling. Now, if h is odd or both h and $n_1 + \cdots + n_t$ are even, then by Lemma 1 and Case 1, $\mathrm{I}_h(K_{n_1,n_2,n_3}) = \mathbb{Z}_h$ and $\mathrm{I}_h(K_{n_4,\dots,n_t}) = \mathbb{Z}_h$, respectively, as desired. Now, assume that $|V(G)| = n_1 + \cdots + n_t$ is odd and h is even, then by Remark 1, if $c \in \mathbb{Z}_h \setminus \{0\}$ is odd, then $c \notin \mathrm{I}_h(K_{n_1,\dots,n_t})$. So, by Lemma 1 and by Case 1, $2\mathbb{Z}_h \subseteq \mathrm{I}_h(K_{n_1,n_2,n_3})$ and $2\mathbb{Z}_h \subseteq \mathrm{I}_h(K_{n_4,\dots,n_t})$, respectively. So, $\mathrm{I}_h(K_{n_1,\dots,n_t}) = 2\mathbb{Z}_h$ and the proof is complete. □

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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