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On the Index Set of Complete Multipartite Graphs

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Abstract For an undirected graph G, and an abelian group A, an A-magic labelling is an assignment of non-zero element of A, to the edges of G, such that the sum of the values of all edges incident with each vertex is constant. A constant on magic sum is called an index set of G. Shiu and Low proved that, zero is in the index set of complete multipartite graph. In this paper, for $t \geq 2$ we determine the index set of the complete multipartite graph $K_{n_1,\ldots,n_t}$, where $n_i \geq 2$ (for $i = 1, \ldots, t$).

Keywords Index set · Magic · Null set · Zero-sum

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1 Introduction

For an abelian group A, written additively, any mapping $l : E(G) \longrightarrow A \setminus \{0\}$ is called a *labeling*. Given a labeling on the edge set of G, one can introduce a vertex labeling $l^+ : V(G) \longrightarrow A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph G is said to be A-*magic*, if there is a labeling $l : E(G) \longrightarrow A \setminus \{0\}$, such that for each vertex v, the sum of values of all edges incident with v, is a constant; that is, $l^+(v) = c$, for some $c \in A$. In general, a graph G may admit more than one labeling to become A-magic. For example, if $|A| > 2$ and $l : E(G) \longrightarrow A \setminus \{0\}$ is a magic labeling of G with sum c, then λ :

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$E(G) \longrightarrow A \setminus \{0\}$, the inverse labeling of l, defined by $\lambda(uv) = -l(uv)$ will provide another magic labeling of G with sum $-c$. Recently the labeling of graphs have been attracted many researchers to itself, for instance see [1–3, 7,9]. We use $K_{n_1,\ldots,n_k}$ to denote the complete multipartite graph with part sizes $n_1, \ldots, n_k$. We denote the complete graph of order n by K_n. Also, if we decompose $E(G)$ into $E(H_1), \ldots, E(H_k)$, then we write $G = H_1 \oplus \cdots \oplus H_k$. Within the mathematical literature, various definition of magic graphs have been introduced. The original concept of an A-magic graph is due to Sedlaced [10,11], who defined it to be a graph with real-valued edge labeling such that distinct edges have distinct non-negative values, and the sum of the values of edges incident with a particular vertex is the same for all vertices. Over the years, there has been a great research interest in graph labeling problems. In fact, many different graph labelings have been introduced in the literature. The interested readers is referred to Wallis' [14] recent monograph on magic graphs. For convenience, a $\mathbb{Z}_h$-magic graph will be referred to as an *h-magic graph*. A constant of a magic sum is called an index of G, an index for short, and we write $\mathrm{I}_A(G) = \{r : \ G \text{ is } A\text{-magic with index } r\}$. An h-magic graph G is said to have *h-zero-sum magic labeling* if there is a magic labeling of G in $\mathbb{Z}_h$ that induces a vertex labeling with sum 0. The *null set* of a graph G, denoted by $N(G)$, is the set of all natural numbers $h \in N$ such G admits a zero-sum magic labeling in $\mathbb{Z}_h$. Salehi in [8] determined the null set of complete graphs.

Theorem 1 *If $n \geq 4$, then*

$$N(K_n) = \begin{cases} \mathbb{N} & n \text{ is odd;} \\ \mathbb{N} \setminus \{2\} & n \text{ is even.} \end{cases}$$

Also, Shiu and M. Low determined in [12], the null set of complete multipartite graphs over an abelian group A.

Theorem 2 *Let A, ($|A| \geq 3$) be an abelian group and $t, n_i \geq 2$, ($i = 1, \ldots, t$) be positive integers. Then $K_{n_1,\ldots,n_t}$ admits an A-zero-sum magic labeling.*

2 The Index set of complete and complete bipartite graphs

In this section we determine the index set of the complete and complete bipartite graphs for an abelian group $\mathbb{Z}_h$, $h \geq 2$. A k-factor of a graph G is a k-regular spanning subgraph of G and a k-factorization is partition the edges of the graph into disjoint k-factors. A graph G is said to be k-factorable if it admits a k-factorization. First, we need the following results.

Theorem 3 [4] *Every complete graph of order $2n$ has 1-factorization.*

Theorem 4 ([10, p.140]) *Every regular graph of even degree has 2-factorization.*

Theorem 5 [13] *Let G be an r-regular graph ($r \geq 2$) which admits a 1-factor, then $\mathrm{I}_h(G) = \mathbb{Z}_h$ for all $h \geq 3$.*

Remark 1 Let G be a graph, $l : E(G) \to \mathbb{Z}_h \setminus \{0\}$ be an edge labeling and $0 \neq c \in \mathrm{I}_h(G)$, then we have

$$2 \sum_{e \in E(G)} l(e) = c|V(G)| \pmod{h}.$$

Also, it is obvious that G admits a 2-magic labeling with sum 1 if and only if the degree of every vertex is odd.

Theorem 6 *Let $h \geq 3$ be positive integers. Then following statements hold:*

(i) $\mathrm{I}_h(K_2) = \mathbb{Z}_h \setminus \{0\}$

(ii) $\mathrm{I}_h(K_3) = \begin{cases} \mathbb{Z}_h \setminus \{0\} & h \text{ is odd;} \\ 2\mathbb{Z}_h & \text{otherwise.} \end{cases}$

(iii) Let $n \geq 4$. Then

$$\mathrm{I}_h(K_n) = \begin{cases} 2\mathbb{Z}_h & \text{if } n \text{ is odd and } h \text{ is even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}$$

Proof (**i**) It is obvious.

(**ii**) Let $x \in \mathbb{Z}_h \setminus \{0\}$. Note that in any h-magic labeling of an odd cycle, the edges should alternatively be the same value. So, if h is odd, then $0 \notin \mathrm{I}_h(K_3)$. Suppose that $h = 2k + 1$. If we assign value $(k + 1)x$ to all edges of K_3, then $x \in \mathrm{I}_h(K_3)$. Therefore, $\mathrm{I}_h(K_3) = \mathbb{Z}_h \setminus \{0\}$. Now, assume that $h = 2k$. By Remark 1, if x is odd, then $x \notin \mathrm{I}_h(K_3)$. Let $x = 2t$, for some t. If we assign value t to all edges of K_3, then $x \in \mathrm{I}_h(K_3)$ and if we assign value k to all edges of K_3, then $0 \in \mathrm{I}_h(K_3)$. Thus, $\mathrm{I}_h(K_3) = 2\mathbb{Z}_h$.

(**iii**) By Theorem 1, $0 \in \mathrm{I}_h(K_n)$. Now, we show that K_n has an h-magic labeling with sum 1. Consider two following cases:

Case 1. If $n = 2r + 1$, then by Theorem 4, K_n has a 2-factorization. Let $F_1, \dots, F_r$ be its 2-factors. First suppose that $h = 2k + 1$. If r is odd, then assign value $k + 1$ to the edges of F_1 and value $(-1)^i$ to the edges of F_i, for $i = 2, \dots, r$. If r is even, then assign value k and 1 to the edges of F_1 and F_2, respectively and for $i = 3, \dots, r (r \geq 3)$, assign value $(-1)^i$ to the edges of F_i. Therefore, K_n admits a h-magic labeling with sum 1. So, $\mathrm{I}_h(K_n) = \mathbb{Z}_h$.

Now, assume that $h = 2k$. By Remark 1, if $c \in \mathbb{Z}_h \setminus \{0\}$ is odd, then $c \notin \mathrm{I}_h(K_n)$. We prove that $2 \in \mathrm{I}_h(K_n)$. If r is odd, then assign value 1 to the edges of F_1 and for $i = 2, \dots, r$, assign value $(-1)^i$ to the edges of F_i. If r is even, then assign value -1 and 2 to the edges of F_1 and F_2, respectively and for $i = 3, \dots, r$, assign $(-1)^i$ to the edges of F_i. Therefore, if k is odd, then by Remark 1, $\mathrm{I}_h(K_n) = 2\mathbb{Z}_h \setminus \{k\}$. Now, if $k = 2t$, then assign value $t - 1$ and 1 to the edges of F_1 and F_2, respectively and value $(-1)^i$ to the edges of F_i. So, $k \in \mathrm{I}_h(K_n)$, as desired.

Case 2. If n is even, then by Theorem 3, K_n has 1-factorization. Let $E_1, \ldots, E_{n-1}$ be a 1-factorization for K_n. We assign value 1 to the edges of E_1 and value $(-1)^i$ to the edges of E_i $(2 \leq i \leq n-1)$. So, $1 \in \mathrm{I}_h(K_n)$ and the proof is complete.

Theorem 7 *Let $h \geq 3$, $m, n \geq 2$ be positive integers. Then $\mathrm{I}_h(K_{m,n}) = \mathbb{Z}_h$ if and only if $m = n \pmod h$.*

Proof Let $\mathrm{I}_h(K_{m,n}) = \mathbb{Z}_h$. Since $1 \in \mathrm{I}_h(K_{m,n})$, there is an edge labeling $l : E(K_{m,n}) \to \mathbb{Z}_h \setminus \{0\}$ such that for every $v \in V(G)$, $l^+(v) = 1 \pmod h$. By double counting we have, $m = \sum_{e \in E(K_{m,n})} l(e) = n \pmod h$. Thus $m = n$ $(\mathrm{mod}\, h)$.

Conversely, by Theorem 2, $K_{m,n}$ admits an h-zero-sum magic labeling. Let $\{1, 2, \ldots, m\}$, $\{1^{'}, 2^{'}, \ldots, n^{'}\}$ be the vertex sets of two parts of $K_{m,n}$ and with no loss of generality assume that $m \geq n$. We partition all edges of $K_{m,n}$ as follows:

$$K_{m,n} = K_{n,n} \oplus K_{m-n,n}.$$

By Theorem 5, $\mathrm{I}_h(K_{n,n}) = \mathbb{Z}_h$. We claim that $K_{m-n,n}$ has a magic labeling in $\mathbb{Z}_h$ such that for every vertex v of part of size n, the sum of values of all edges incident with v is 0 and for every vertex w of part of size $m-n$, the sum of values of all edges incident with w is 1. Consider two cases:

Case 1. Let n be odd. Assign value $(-1)^{i'+1}$ to all edges incident with i' $(1' \leq i' \leq n')$.

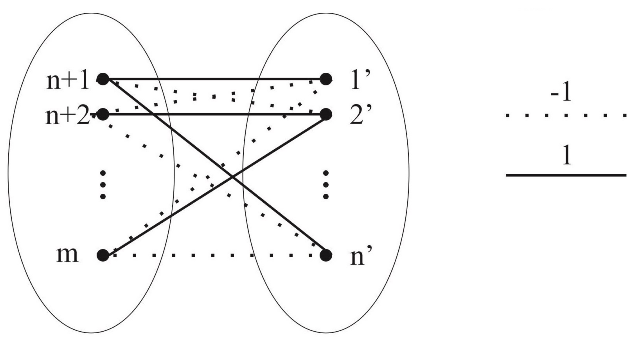

Fig. 1 $K_{m-n,n}$

Case 2. Let n be even. Consider the following edge labeling of $K_{m-n,n}$ for all edges incident with i' $(1' \leq i' \leq n')$.

$$2, \ldots, 2, \quad -1, \ldots, -1, \quad 1, \ldots, 1, \quad -1, \ldots, -1, \quad \ldots \quad 1, \ldots, 1, \quad -1, \ldots, -1.$$

Hence $1 \in \mathrm{I}_h(K_{m,n})$ (See Figure 2, (a)). If h is odd, then $\mathrm{I}_h(K_{m,n}) = \mathbb{Z}_h$. If $h = 2k$, then the above edge labeling shows that $\mathbb{Z}_h \setminus \{k\} \subseteq \mathrm{I}_h(K_{m,n})$. The following edge labeling of $K_{m-n,n}$ for all edges incident with i' $(1' \leq i' \leq n')$ shows that $k \in \mathrm{I}_h(K_{m,n})$ and so $\mathrm{I}_h(K_{m,n}) = \mathbb{Z}_h$ (See Figure 2, (b)).

$$k-1, \ldots, k-1, \; 1, \ldots, 1, \; 1, \ldots, 1, \; -1, \ldots, -1, \; \ldots \; 1, \ldots, 1, \; -1, \ldots, -1.$$

The proof is complete.

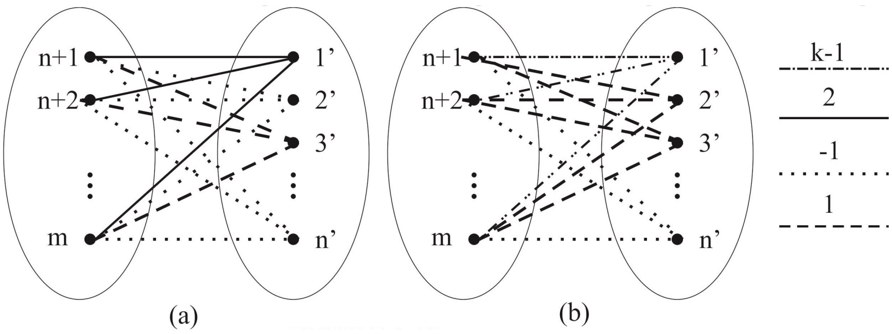

Fig. 2 $K_{m-n,n}$

3 The Index set of the complete multipartite graphs

In this section we would like to determine the index set of the complete multipartite graphs over an abelian group $\mathbb{Z}_h$, $h \geq 2$.
From now on, we denote the vertex set of part of size n_i by X_i in $K_{n_1,\ldots,n_t}$, for $i = 1, \ldots, t$.

Lemma 1 *Let $h \geq 3$, $m, n, t \geq 2$ be positive integers and $m = n = t \pmod{h}$. Then the following holds:*

$$\mathrm{I}_h(K_{m,n,t}) = \begin{cases} 2\mathbb{Z}_h & \textit{if } m+n+t \textit{ is odd and } h \textit{ is even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}$$

Proof By Theorem 2, $K_{m,n,t}$ has an h-zero-sum magic labeling. We partition all edges of $K_{m,n,t}$ as follows:

$$K_{m,n,t} = K_{m,n} \oplus K_{n,t} \oplus K_{m,t}.$$

Let $x \in \mathbb{Z}_h \setminus \{0\}$. First suppose that $h = 2k+1$. By Theorem 7, $x(k+1) \in \mathrm{I}_h(K_{m,n})$, $x(k+1) \in \mathrm{I}_h(K_{n,t})$ and $x(k+1) \in \mathrm{I}_h(K_{m,t})$. So, $x \in \mathrm{I}_h(K_{m,n,t})$, as desired. Now, assume that h is even and $x = 2t$. By Theorem 7, $t \in \mathrm{I}_h(K_{m,n})$, $t \in \mathrm{I}_h(K_{n,t})$ and $t \in \mathrm{I}_h(K_{m,t})$. So, $x \in \mathrm{I}_h(K_{m,n,t})$. If $m+n+t$ and $c \in \mathbb{Z}_h \setminus \{0\}$

are odd, then by Remark 1, $c \notin \mathrm{I}_h(K_{m,n,t})$. Thus $\mathrm{I}_h(K_{m,n,t}) = 2\mathbb{Z}_h$. Now, suppose that $m+n+t$ is even. If $m = n = t$, then by Dirac Theorem [8, p.288], $K_{n,n,n}$ has a hamilton cycle and so it contains a 1-factor. Thus by Theorem 5, $\mathrm{I}_h(K_{n,n,n}) = \mathbb{Z}_h$. Let X_1, X_2 and X_3 be three parts of $K_{m,n,t}$ such that $|X_1| = m$, $|X_2| = n$ and $|X_3| = t$. Let $Y_2 \subseteq X_2$ and $Y_3 \subseteq X_3$ be two sets such that $|Y_2| = |Y_3| = m$. Also, let $W_2 = X_2 \setminus Y_2$, $W_3 = X_3 \setminus Y_3$. We consider two cases:

Case 1. Let $m < n \leq t$. We partition all edges of $K_{m,n,t}$ as follows:

$$K_{m,n,t} = K_{X_1,Y_2,Y_3} \oplus K_{W_2,W_3} \oplus K_{W_2,X_1\cup Y_3} \oplus K_{W_3,X_1\cup Y_2}.$$

By Theorem 2, both $K_{W_2,X_1\cup Y_3}$ and $K_{W_3,X_1\cup Y_2}$ have an h-zero-sum magic labeling. Also, as we did before, $\mathrm{I}_h(K_{X_1,Y_2,Y_3}) = \mathbb{Z}_h$ and by Theorem 7, $\mathrm{I}_h(K_{W_2,W_3}) = \mathbb{Z}_h$. Therefore, $\mathrm{I}_h(K_{m,n,t}) = \mathbb{Z}_h$.

Case 2. Let $m = n < t$. We partition all edges of $K_{m,n,t}$ as follows:

$$K_{m,n,t} = K_{X_1,X_2,Y_3} \oplus K_{X_1,W_3} \oplus K_{X_2,W_3}.$$

By Theorem 2, K_{X_1,W_3} has an h-zero-sum magic labeling and as we did before, $\mathrm{I}_h(K_{X_1,X_2,Y_3}) = \mathbb{Z}_h$. Now, let $x \in \mathbb{Z}_h \setminus \{0\}$. We would like to assign some values to the edges of K_{X_2,W_3} such that every vertex of part of size X_2 has vertex labeling 0 and every vertex of part of size W_3 has vertex labeling x. If $x \neq 1$, then the following edge labeling of K_{X_2,W_3} shows that $x \in \mathrm{I}_h(K_{X_2,W_3})$.

$$x-1,\ldots,x-1,\ \ 1,\ldots,1,\ \ 1,\ldots,1\ \ -1,\ldots,-1\ \ \ \ldots\ \ \ 1,\ldots,1,\ \ -1,\ldots,-1,$$

where the size of each block is $|W_3|$. Also, to obtain the index 1 consider the following edge labeling of K_{X_2,W_3}.

$$2,\ldots,2,\ \ -1,\ldots,-1,\ \ 1,\ldots,1\ \ -1,\ldots,-1\ \ \ \ldots\ \ \ 1,\ldots,1,\ \ -1,\ldots,-1,$$

where the size of each block is $|W_3|$. So, $\mathrm{I}_h(K_{m,n,t}) = \mathbb{Z}_h$ and the proof is complete.

Theorem 8 *Let* $h, t \geq 3$ *and* $n_i \geq 2$ $(i = 1,\ldots,t)$ be positive integers and $n_1 = \cdots = n_t \pmod h$. Then the following holds:

$$\mathrm{I}_h(K_{n_1,\ldots,n_t}) = \begin{cases} 2\mathbb{Z}_h & \text{if } n_1+\cdots+n_t \text{ is odd and } h \text{ is even;} \\ \mathbb{Z}_h & \text{otherwise.} \end{cases}$$

Proof By Theorem 2, $K_{n_1,\ldots,n_t}$ has an h-zero-sum magic labeling. Note that by Lemma 1 we can assume that $t \geq 4$. Consider two cases:

Case 1. Suppose that t is even. We partition all edges of $K_{n_1,\ldots,n_t}$ as follows:

$$K_{n_1,\ldots,n_t} = K_{n_1,n_2} \oplus K_{n_3,n_4} \oplus \ldots \oplus K_{n_{t-1},n_t} \oplus H,$$

where H is a complete $\frac{t}{2}$ -partite graph with parts $X_{2i-1} \cup X_{2i}$, $(1 \leq i \leq \frac{t}{2})$. By Theorems 2 and 7, H has an h-zero-sum magic labeling and

$$\mathrm{I}_h(K_{n_{2i-1},n_{2i}}) = \mathbb{Z}_h, \qquad (1 \leq i \leq \frac{t}{2}),$$

as desired.

Case 2. If t is odd, then we partition all edges of $K_{n_1,\ldots,n_t}$ as follows:

$$K_{n_1,\ldots,n_t} = K_{n_1,n_2,n_3} \oplus K_{n_4,\ldots,n_t} \oplus H',$$

where H' is a complete bipartite graph with parts $X_1 \cup X_2 \cup X_3$ and $X_4 \cup \ldots \cup X_t$. By Theorem 2, H' has an h-zero-sum magic labeling. Now, if h is odd or both h and $n_1 + \cdots + n_t$ are even, then by Lemma 1 and Case 1, $\mathrm{I}_h(K_{n_1,n_2,n_3}) = \mathbb{Z}_h$ and $\mathrm{I}_h(K_{n_4,\ldots,n_t}) = \mathbb{Z}_h$, respectively, as desired. Now, assume that $|V(G)| = n_1 + \cdots + n_t$ is odd and h is even, then by Remark 1, if $c \in \mathbb{Z}_h \setminus \{0\}$ is odd, then $c \notin \mathrm{I}_h(K_{n_1,\ldots,n_t})$. So, by Lemma 1 and by Case 1, $2\mathbb{Z}_h \subseteq \mathrm{I}_h(K_{n_1,n_2,n_3})$ and $2\mathbb{Z}_h \subseteq \mathrm{I}_h(K_{n_4,\ldots,n_t})$, respectively. So, $\mathrm{I}_h(K_{n_1,\ldots,n_t}) = 2\mathbb{Z}_h$ and the proof is complete.

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