Analytical and Numerical Solutions for Nonlinear Equations
Volume 8, Issue 1, pp. 139–151

Research Article

Some Remarks on ϕ-Graded Semi-n-Absorbing Submodules

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Received: 10 June 2024 / Accepted: 15 January 2025

Abstract Let R be a G-graded commutative ring with identity and M be a unitary G-graded R-module. Let $S(M)$ be the set of all graded submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper graded submodule N of M is called *ϕ-graded semi-n-absorbing* submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $r^n m \in N \backslash \phi(N)$, then $r^n \in (N : M)$ or $r^{n-1}m \in N$ ($n \geq 2$). In this work, firstly, we state with deeper results on the structure of generalizations of prime submodules as ϕ-graded prime submodules. Moreover ϕ-graded semi-n-absorbing submodules are studied and some results are obtained.

Keywords ϕ-Graded prime submodule · ϕ-Graded semi-n-absorbing submodule · ψ-Graded semi-n-absorbing ideal · $(n, n-1)$-ϕ-Graded prime submodule

Mathematics Subject Classification (2010) 13C05 · 13C13

1 Introduction

In this work, all rings are commutative with identity and all modules are unitary. Let G be an abelian group with identity e. A ring R is called G-graded ring if there exists a family of subgroups $\{R_g\}_{g\in G}$ of R such that $R = \bigoplus_{g\in G} R_g$ with the property that $R_g R_{g'} \subseteq R_{g+g'}$ for all $g, g' \in G$ where $R_g R_{g'}$ is the set of all elements consisting of all finite sums in the form $r_g r'_{g'}$ with $r_g \in R_g$ and $r'_{g'} \in R_{g'}$. A non-zero element $r \in R_g$ is called a homogeneous element of R

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of degree g. If $r \in R$, then r can be written uniquely as $r = \sum_{g\in G} r_g$, where $r_g \in R_g$ and the sum contains only a finite number of non-zero terms. Each r_g is called the g-component of r in R_g and $h(R) = \cup_{g\in G} R_g$. Moreover, R_e is a subring of R, $1_R \in R_e$ and R_g is an R_e-module for all $g \in G$. An ideal I of R is called graded ideal of R if $I = \bigoplus_{g\in G} I_g$ such that $I_g = I \cap R_g$ for $g \in G$. If I is a graded ideal of R, then $h(I) = \cup_{g\in G} I_g = \cup_{g\in G}(I \cap R_g) = I \cap h(R)$. Furthermore, R/I is a G-graded ring and $R/I = \bigoplus_{g\in G}(R/I)_g$ where $(R/I)_g = (I + R_g)/I$ for $g \in G$. Let R, R' be two G-graded rings. Suppose that $f : R \to R'$ be a ring homomorphism. Then f is said to be graded ring homomorphism of degree σ if $f(R_g) \subseteq R'_{g+\sigma}$ for all $g \in G$.

Now, Let R be a G-graded ring. Then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that $R_g M_{g'} \subseteq M_{g+g'}$ for all $g, g' \in G$. Here $R_g M_{g'}$ denotes the additive subgroup of M consisiting of all finite sums of elements $r_g m_{g'}$ with $r_g \in R_g$ and $m_{g'} \in M_{g'}$. Any element of M_g is said to be a homogeneous element of degree g. Also, we write $h(M) = \cup_{g\in G} M_g$. If $m \in M$, then the element m can be written uniquely as $m = \sum_{g\in G} m_g$ where $m_g \in M_g$ and the sum contains only a finite number of non-zero terms. If $m \neq 0$ is in M_g for some $g \in G$, then m is homogeneous of degree g and we write : $deg\ m = g$(or m is called the g-component). For all $g \in G$, the subgroup M_g of M is an R_e-module. Let $M = \bigoplus_{g\in G} M_g$ is a graded R-module and N a submodule of M. Then N is called a graded submodule of M if $N = \bigoplus_{g\in G} N_g$, where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the g-component of N. Furthermore, M/N becomes a G-graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. If N is a graded submodule of M, then $h(N) = \cup_{g\in G} N_g = \cup_{g\in G}(N \cap M_g) = N \cap h(M)$. Let R be a graded ring and M, M' be graded R-modules. Let $\varphi : M \to M'$ be an R-module homomorphism. Then φ is said to be graded homomorphism of degree σ, $\sigma \in G$, if $\varphi(M_g) \subseteq M'_{g+\sigma}$ for all $g \in G$. For general background and more detail, the reader may consult [14] and [16].

Graded prime submodules, graded primary submodules, graded almost prime submodules, graded weakly prime submodules, graded classical prime submodules, graded almost semiprime submodules and graded weakly semiprime submodules of general graded modules over graded rings have been studied by several authors (See, for example, [4,5,7,15]). In this work, we investigate a generalization of graded semi-n-absorbing submodules and prove some remarks for ϕ-graded semi-n-absorbing submodules. Generalizations of prime submodules and prime ideals play an important role in extended submodules and ideals. D.D. Anderson and Bataineh stated various generalizations of prime ideals (See [3]). Also, Zamani in [17] introduced this concept to ϕ-prime submodule i.e., a proper submodule N of M is a ϕ-prime submodule if $rm \in N\backslash\phi(N)$ whenever $r \in R$ and $m \in M$, then $m \in N$ or $r \in (N : M)$ where $\phi : S(M) \to S(M) \cup \{\emptyset\}$ is a function and $S(M)$ is the set of all submodules of M. D.F. Anderson and Badawi defined an n-absorbing ideal i.e., a proper ideal I of R is called an n-absorbing ideal if $r_1 \dots r_{n+1} \in I$ where $r_1, \dots, r_{n+1} \in R$, then $r_1 \dots r_{i-1} r_{i+1} \dots r_{n+1} \in I$ for some $i \in \{1, \dots, n+1\}$. Also, they called I to be a semi n-absorbing ideal if $r^{n+1} \in I$ where $r \in R$, then $r^n \in I$. A proper submodule N of M is called n-absorbing (or $(n,n+1)$-prime) submodule of M if $r_1 \ldots r_n m \in N$ where $r_1, \ldots, r_n \in R$ and $m \in M$, then $r_1 \ldots r_n \in (N : M)$ or $r_1 \ldots r_{i-1} r_{i+1} \ldots r_n \in N$ for some $i \in \{1, \ldots, n\}$ (See [8]). A proper submodule N of M is said semi-prime if $r^2 m \in N$ where $r \in R$ and $m \in M$, then $rm \in N$. In particular, N is called a semi-n-absorbing submodule of M if whenever $m \in M$ and $r \in R$ with $r^n m \in N$, then $r^n m \in (N : M)$ or $r^{n-1} m \in N$. Some authors extended various generalized prime submodules, prime ideals and obtained some results on generalizations of prime submodules (For example see [1,2,10,12,13]). Now, we will define the ϕ-graded semi-n-absorbing submodules and will give some basic results about ϕ-graded semi-n-absorbing submodules of graded modules. Let R be a G-graded R-module and N a graded submodule of M. The ideal $\{r \in R \mid \; rM \subseteq N\}$ will be denoted by $(N : M)$. The annihilator of M is defined as $(0 : M)$ and is denoted by $Ann(M)$. The graded R-module M is called faithful if $Ann(M) = 0$.

At the end of this section we recall several definitions and remarks. Let M be a graded R-module, $S(M)$ be the set of all graded submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper graded submodule N of M is called ϕ-graded semi-n-absorbing if $r^n m \in N \backslash \phi(N)$ where $r \in h(R)$ and $m \in h(M)$, implies that $r^n \in (N : M)$ or $r^{n-1} m \in N$ $(n \geq 2)$. Let R be a G-graded ring, $\mathcal{I}(R)$ be the set of all graded ideals of R and $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function. We say that a proper graded ideal I of R is ψ-graded semi-n-absorbing ideal if $r^{n+1} \in I \setminus \psi(I)$ for $r \in h(R)$, then $r^n \in I$ $(n \geq 2)$. Let M be a graded R-module and P be a proper graded submodule of M. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ be the set of all graded submodules of M. P is called ϕ-graded prime submodule of M if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P \setminus \phi(P)$, then $r \in (P : M)$ or $m \in P$. Assume that N be a graded submodule of M, N is said ϕ-graded semiprime submodule of M if $N = \cap_{P \in \mathcal{S}} P$ such that

$$\mathcal{S} \subseteq Spec_{\phi}(M) = \{P \in S(M) \mid P \text{ is a } \phi - \text{graded prime submodule}\}.$$

Moreover, suppose that $n \geq 2$ be a positive integer and N be a proper graded submodule of M, N is called $(n-1,n)$-ϕ-graded prime submodule of M if whenever $r_1, \ldots, r_{n-1} \in h(R)$ and $m \in h(M)$ with $r_1 \ldots r_{n-1} m \in N \setminus \phi(N)$, then $r_1 \ldots r_{n-1} \in (N : M)$ or $r_1 \ldots r_{i-1} r_{i+1} \ldots r_{n-1} m \in N$ for some $i \in \{1, \ldots, n-1\}$ (See [10]).

Remark 1 Let R be a G- graded ring and M be a graded R- module. It is clear that, if N is a $(n,n+1)$-ϕ-graded prime submodule of M, then N is a ϕ-graded semi-n-absorbing submodule of M. Also, if N is a ϕ-graded prime submodule of M, then N is a $(n,n-1)$-ϕ-graded prime submodule of M.

Remark 2 Let R be a G- graded ring and M be a graded R- module. If N is a ϕ-graded semi-2-absorbing submodule of M, then N is a ϕ-graded semi-n-absorbing submodule of M. Also, if N is a $(n,n-1)$-ϕ-graded prime submodule of M, then N is a $(n,n+1)$-ϕ-graded prime submodule of M.

Remark 3 *Let R be a G- graded ring, M be a graded R- module and ϕ_1, $\phi_2 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions where $S(M)$ is the set of all graded submodules of M with $\phi_1 \leqslant \phi_2$ (i.e., for every graded submodules of M, $\phi_1(N) \subseteq \phi_2(N)$). If N is a ϕ_1-graded semi-n-absorbing ($(n, n-1)$-ϕ_1-graded prime) submodule of M, then N is a ϕ_2-graded semi-n-absorbing ($(n, n-1)$-ϕ_2-graded prime) submodule of M.*

By the definition of ϕ-graded semi-n-absorbing submodule N, if $\phi(N) = \emptyset$ (resp. $\phi(N) = 0$, $\phi(N) = (N : M)N$, $\phi(N) = (N : M)^{m-1}N$ and $\phi(N) = \cap_{i=1}^{\infty}(N : M)^m N$), then a submodule N is a graded semi-n-absorbing submodule (resp. graded weakly semi-n-absorbing submodule, graded almost semi-n-absorbing submodule, graded m-almost semi-n-absorbing submodule and graded ω-semi-n-absorbing submodule) of M. Now, by Remark 1.6, we have graded semi-n-absorbing submodule $\Rightarrow$ graded weakly semi-n-absorbing submodule $\Rightarrow$ graded ω-semi-n-absorbing submodule $\Rightarrow$ graded m-almost semi-n-absorbing submodule $\Rightarrow$ graded almost semi-n-absorbing submodule
One of the most important questions is "Under what circumstance is the reverse of above condistions held ?"
In the next section, we state some properties of ϕ- graded semi-n-absorbing submodule of M and obtain a number of results concerning ϕ-graded semi-n-absorbing submodules.

2 Properties of ϕ-Graded Semi-n-Absorbing Submodules

The following corollaries be given some results when we utilize the defintion ϕ-graded semi-n-absorbing submodule.

Corollary 1 *Let M be a graded R-module and N be a proper graded submodule of M . Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ be the set of all graded submodules of M. If N is a ϕ-graded prime submodule of M, then N is a ϕ-graded semi-n-absorbing submodule of M.*

Proof Let N be proper graded submodule and $r^n m \in N \backslash \phi(N)$ where $r \in h(R)$ and $m \in h(M)$ $(n \geq 2)$. Since N is a ϕ-prime submodule of M, so $r^n \in (N : M)$ or $m \in N$. Thus $r^{n-1}m \in N$ or $r^n \in (N : M)$, as required.

Corollary 2 *Suppose that N is a ϕ-graded semi-n-absorbing submodule of M, then N is a ϕ-graded semi-$n+1$-absorbing submodule of M.*

Proof It is clear.

Corollary 3 *Let R, S be two G-rings, $\varphi : R \to S$ be a graded ring homomorphism and M be a graded S-module. Suppose that N is a ϕ-graded semi-n-absorbing submodule of graded S-module M, then N is a ϕ-graded semi-n-absorbing submodule of graded R-module M.*

Proof Let $r^n m \in N \setminus \phi(N)$ where $r \in h(R)$ and $m \in h(M)$. We know that $r^n m = \varphi(r)^n m \in N \setminus \phi(N)$ such that $\varphi(r) \in S$. It is clear that N is a ϕ-graded semi-n-absorbing submodule of graded R-module M.

Example 1 Let $R = \mathbb{Z} = R_e$ be a $\mathbb{Z}$-graded ring and $M = \frac{\mathbb{Z}}{p\mathbb{Z}}$ a $\mathbb{Z}$-graded $\mathbb{Z}$-module where $p \in \mathbb{Z}$ is a prime number. Clearly, $N = \{0_M\} = p\mathbb{Z}$ is a proper graded submodule of $\frac{\mathbb{Z}}{p\mathbb{Z}}$ and $(p\mathbb{Z} : \frac{\mathbb{Z}}{p\mathbb{Z}}) = p\mathbb{Z}$. Suppose that $n^p(m + p\mathbb{Z}) \in p\mathbb{Z}$ where $m, n \in \mathbb{Z}$, so $n^p m \in p\mathbb{Z}$ and hence $p \mid n^p m$, therefore $p \mid n^p$ or $p \mid m$. Then $n^p \in p\mathbb{Z} = (p\mathbb{Z} : \frac{\mathbb{Z}}{p\mathbb{Z}})$ or $m \in p\mathbb{Z}$. Accordingly, N is a graded semi-p-absorbing submodule of $M = \frac{\mathbb{Z}}{p\mathbb{Z}}$.
Now, let $R = \mathbb{Z} = R_e$ be as $\mathbb{Z}$-graded ring and $M = \mathbb{Z} \times \mathbb{Z}$ be a $\mathbb{Z}$-graded $\mathbb{Z}$-module with $M_0 = \mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}$. Clearly, $N = 16\mathbb{Z} \times \{0\}$ is a proper graded submodule of $\mathbb{Z} \times \mathbb{Z}$ and $(16\mathbb{Z} \times \{0\} : \mathbb{Z} \times \mathbb{Z}) = 0$. Since, $2^2(4, 0) \in 16\mathbb{Z} \times \{0\}$ but $2(4, 0) \notin 16\mathbb{Z} \times \{0\}$ and $2^2 \notin (16\mathbb{Z} \times \{0\} : \mathbb{Z} \times \mathbb{Z}) = 0$, so $N = 16\mathbb{Z} \times \{0\}$ is not graded semi-2-absorbing submodule of $M = \mathbb{Z} \times \mathbb{Z}$.

Corollary 4 *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ be the set of all graded submodules of graded R-module M and N_i be a proper graded submodule of M for $i \in \Lambda$, with $\phi(\cup_{i\in\Lambda} N_i) \subseteq \phi(\cap_{i\in\Lambda} N_i)$. If N_i is a ϕ- graded semi-n-absorbing submodule of M for each $i \in \Lambda$, then $\cap_{i\in\Lambda} N_i$ is a ϕ-graded semi-n-absorbing submodule of M.*

Proof Let $r^n m \in \cap_{i\in\Lambda} N_i \setminus \phi(\cap_{i\in\Lambda} N_i)$ where $r \in h(R)$ and $m \in h(M)$. So $r^n m \in \cap_{i\in\Lambda} N_i$ and $r^n m \notin \phi(\cap_{i\in\Lambda} N_i)$, since $\phi(\cup_{i\in\Lambda} N_i) \subseteq \phi(\cap_{i\in\Lambda} N_i)$, hence $r^n m \in N_i \setminus \phi(N_i)$ for every $i \in \Lambda$. Because N_i is a ϕ-graded semi- n-absorbing submodule of M, therefore $r^n \in (N_i : M)$ or $r^{n-1} m \in N_i$ for all $i \in \Lambda$. Thus $r^n \in (\cap_{i\in\Lambda} N_i : M)$ or $r^{n-1} m \in \cap_{i\in\Lambda} N_i$, as needed.

Proposition 1 *Let M be a graded R-module , $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and N be a proper graded submodule of M. Assume that $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all graded ideals of R. Then the following conditions hold:*

(1) If $(N : m)$ is a ψ-graded semi-$n-1$-absorbing ideal of R with $\psi(N : m) \subseteq (\phi(N) : m)$ for every $m \in h(M)$, then N is a ϕ-graded semi-n-absorbing submodule of M $(n \geq 2)$.
(2) If N is a ϕ-graded semi-n-absorbing submodule of M with $(\phi(N) : m) \subseteq \psi(N : m)$, then $(N : m)$ is a ψ-graded semi-n-absorbing ideal of R.

Proof (1) Let $r \in h(R)$ and $m \in h(M)$ with $r^n m \in N \setminus \phi(N)$ and $r^n \notin (N : M)$, so $r^n \in (N : m)$ and $r^n \notin (\phi(N) : m)$. Since $\psi(N : m) \subseteq (\phi(N) : m)$, hence $r^n \in (N : m)$ and $r^n \notin \psi(N : m)$. Thus $r^n \in (N : m) \setminus \psi(N : m)$. Since $(N : m)$ is a ψ-graded semi-$n-1$-absorbing ideal of R, hence $r^{n-1} \in (N : m)$. Therefore $r^{n-1} m \in N$, so N is a ϕ-graded semi-n-absorbing submodule of graded R-module M.
(2) Let $r \in h(R)$ and $m \in h(M)$ with $r^{n+1} \in (N : m) \setminus \psi(N : m)$, so $r^{n+1} m \in N$ or $r^{n+1} \notin \psi(N : m)$ and hence $r^n(rm) \in N \setminus \phi(N)$. Since $r \in h(R)$ and $m \in h(M)$, so $r \in R_g$ and $m \in M_{g'}$ for some g, $g' \in G$ and hence $rm \in R_g M_{g'} \subseteq M_{g+g'}$. Then $rm \in h(M)$, therefore $r^n(rm) \in N \setminus \phi(N)$ implies that $r^{n-1}(rm) \in N$ or $r^n \in (N : M)$. So $r^n m \in N$ or $r^n \in (N : M) \subseteq (N : m)$. Consequently, $r^n \in (N : m)$.

Let R be a G-graded ring. A graded R-module M is called graded finitely generated if $M = \sum_{i=1}^{n} Rm_{g_i}$ where $m_{g_i} \in h(M)$ $(1 \leq i \leq n)$.

Proposition 2 *Let M be a graded finitely generated R-module such that $M = \sum_{i=1}^{k} Rm_{g_i}$, N be a proper graded submodule of M and $\psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all graded ideals of G-graded ring R, $m_{g_i} \in h(M)$. Then*

(1) If $(N : m_{g_i})$ is a ψ-graded semi-n-absorbing ideal of R with $\psi(N : m_{g_i}) \subseteq \psi(N : M)$ for every $i = 1, \dots, k$, then $(N : M)$ is a ψ-graded semi-n-absorbing ideal of R.

(2) If $(N : M)$ is a ψ-graded semi-n-absorbing ideal of R, then $(N : m_{g_i})$ is a ψ-graded semi-n-absorbing ideal of R.

Proof (1) Assume that $r^{n+1} \in (N : M) \setminus \psi(N : M)$ and $r^n \notin (N : M)$. Since $(N : \sum_{i=1}^{k} Rm_{g_i}) = \cap_{i=1}^{k}(N : Rm_{g_i}) = \cap_{i=1}^{k}(N : m_{g_i})$, so $r^n \notin (N : m_{g_j})$ for some $j \in \{1, \dots, k\}$ and $m_{g_j} \in h(M)$. Because of $r^{n+1} \notin \psi(N : M)$, hence $r^{n+1} \notin \psi(N : m_{g_i})$ for each $i \in \{1, \dots, k\}$. Thus $r^{n+1} \in (N : m_{g_j}) \setminus \psi(N : m_{g_j})$ for some $j \in \{1, \dots, k\}$ and $m_{g_j} \in h(M)$. Since $(N : m_{g_j})$ is a ψ-graded semi-n-absorbing ideal of graded ring R, so $r^n \in (N : m_{g_j})$ which contradicts with our assumption. Therefore $(N : M)$ is a ψ-graded semi-n-absorbing ideal of R.
(2) Let $r \in h(R)$ with $r^{n+1} \in (N : m_{g_i}) \setminus \psi(N : m_{g_i})$ for each $i \in \{1, \dots, k\}$. We have $r^{n+1} \in \cap_{i=1}^{k}(N : m_{g_i}) = (N : \sum_{i=1}^{k} Rm_{g_i}) = (N : M)$ and since $\psi(\cap_{i=1}^{k}(N : m_{g_i})) \subseteq \cap_{i=1}^{k}\psi(N : m_{g_i}) \subseteq \psi(N : m_{g_i})$ for all i and so $r^{n+1} \notin \psi(N : m_{g_i})$ implies that $r^{n+1} \notin \psi(\cap_{i=1}^{k}(N : m_{g_i})) = \psi(N : M)$. It follows that $r^{n+1} \in (N : M) \setminus \psi(N : M)$. Since $(N : M)$ is a ψ-graded semi-n-absorbing ideal of R, hence $r^n \in (N : M)$, so $r^n \in \cap_{i=1}^{k}(N : m_{g_i})$, therefore $r^n \in (N : m_{g_i})$ for every $i \in \{1, \dots, k\}$. Thus $(N : m_{g_i})$ is a ψ-graded semi-n-absorbing ideal of R.

Suppose that N and L be two graded submodules with $L \subseteq N$, let

$$\phi_L : S(\frac{M}{L}) \to S(\frac{M}{L}) \cup \{\emptyset\},$$

be defined by $\phi_L(\frac{N}{L}) = \frac{\phi(N)+L}{L}$ with $L \subseteq N$ (and $\phi_L(\frac{N}{L}) = \emptyset$ if $\phi(N) = \emptyset$) where $\phi : S(M) \to S(M) \cup \{\emptyset\}$ is a function and $S(\frac{M}{L})$ is the set of all graded submodules of $\frac{M}{L}$. We recall $N = \bigoplus_{g \in G} N_g$ where $N_g = M_g \cap N$ and M_g be an R_e-module, so N_g is an R_e- module and hence

$$(\frac{N}{L})_g = \frac{N_g + L}{L},$$

is a R_e-submodule of R_e-module $\frac{M_g+L}{L}$. Moreover, we define

$$\phi_L(\frac{N}{L}) = \phi_L(\bigoplus_{g \in G} \frac{N_g + L}{L}) = \bigoplus_{g \in G} \frac{\phi(N_g) + L}{L},$$

and $\phi(N) = \phi(\bigoplus_{g \in G} N_g) = \bigoplus_{g \in G} \phi(N_g)$. In connection with this concept, we assert the following theorems.

Theorem 1 *Let M be a graded R-module and $L \subseteq N$ be proper graded submodules of M. Suppose that $\phi_{L_e} : S(\frac{M_g+L}{L}) \to S(\frac{M_g+L}{L}) \cup \{\emptyset\}$ be defined by $\phi_{L_e}(\frac{N_g+L}{L}) = \frac{\phi_e(N_g)+L}{L}$ where $\phi_e : S(M_g) \to S(M_g) \cup \{\emptyset\}$ is a function and $S(\frac{M_g+L}{L})$ is the set of all R_e- submodules of R_e-module $\frac{M_g+L}{L}$, $S(M_g)$ is the set of all R_e-submodules of R_e-module M_g. Then the following statememts hold:*

(1) If N_g is a ϕ_e-semi-n-absorbing submodule of R_e-module M_g, then $\frac{N_g+L}{L}$ is a ϕ_{L_e}-semi-n- absorbing submodule of R_e-module $\frac{M_g+L}{L}$.

(2) If $L \subseteq \phi_e(N_g)$ and $\frac{N_g+L}{L}$ is a ϕ_{L_e}-semi-n-absorbing submodule of $\frac{M_g+L}{L}$, then N_g is a ϕ_e-semi-n-absorbing submodule of R_e-module M_g $(n \geq 2)$.

Proof (1) Let $r_e \in R_e$ and $m_g + l + L \in \frac{M_g+L}{L}$ with

$$r_e^n(m_g + l + L) \in \frac{N_g + L}{L} \setminus \phi_{L_e}(\frac{N_g + L}{L}),$$

so $r_e^n m_g + L \in \frac{N_g+L}{L}$ and $r_e^n m_g + L \notin \phi_{L_e}(\frac{N_g+L}{L}) = \frac{\phi_e(N_g)+L}{L}$. It follows that $r_e^n m_g \in N_g \setminus \phi_e(N_g)$. Since, N_e is a ϕ_e-semi-n-absorbing submodule of R_e-module M_g, that gives $r_e^n \in (N_g : M_g)$ or $r_e^{n-1} m_g \in N_g$. Therefore

$$r_e^n \in (\frac{N_g + L}{L} : \frac{M_g + L}{L}),$$

or $r_e^{n-1}(m_g + l + L) \in \frac{N_g+L}{L}$.
(2) Let $r_e \in R_e$ and $m_g \in M_g$ with $r_e^n m_g \in N_g \setminus \phi_e(N_g)$. Since, $L \subseteq \phi_e(N_g)$, so, $r_e^n m_g \notin \frac{\phi_e(N_g)+L}{L}$. Hence, $r_e^n(m_g + L) \in \frac{N_g+L}{L} \setminus \phi_{L_e}(\frac{N_g+L}{L})$. Since, $\frac{N_g+L}{L}$ is a ϕ_{L_e}-semi-n-absorbing submodule of R_e-module $\frac{M_g+L}{L}$, so $r_e^n \in (\frac{N_g+L}{L} : \frac{M_g+L}{L})$ or $r_e^{n-1}(m_g + l + L) \in \frac{N_g+L}{L}$. It follows that $r_e^n \in (N_g : M_g)$ or $r_e^{n-1} m_g \in N_g$, as required.

Theorem 2 *Let M be a graded R-module and $L \subseteq N$ be proper graded submodules of M. Assume that $\phi_L : S(\frac{M}{L}) \to S(\frac{M}{L}) \cup \{\emptyset\}$ be a function where $\phi : S(M) \to S(M) \cup \{\emptyset\}$ is a function and $S(\frac{M}{L})$ is the set of all graded submodules of $\frac{M}{L}$.*

(1) If N is a ϕ-graded semi-n-absorbing submodule of M, then $\frac{N}{L}$ is a ϕ_L-graded semi-n-absorbing submodule of $\frac{M}{L}$.

(2) If $L \subseteq \phi(N)$ and $\frac{N}{L}$ is a ϕ_L-graded semi-n-absorbing submodule of $\frac{M}{L}$, then N is a ϕ-graded semi-n-absorbing submodule of M.

Proof It is evident.

Now, we introduce function ϕ_{IM}. Let M be a graded R-module and I be a graded ideal of G-graded ring R. Then $I = \sum_{g \in G} I_g$ and $I_{g'} \cap (\sum_{g \in G \setminus \{g'\}} I_g) = \{0\}$. It follow that

$$IM = (\sum_{g \in G} I_g)M = \sum_{g \in G} (I_g M),$$

and

$$I_{g'}M \cap (\sum_{g \in G \setminus \{g'\}} I_g M) = \{0\},$$

and hence, $IM = \bigoplus_{g \in G} I_g M$, so IM is a graded submodule of M. Since, $I \subseteq Ann_R(M/IM)$, so M/IM becomes a G-graded R/I-module with g-component $(M/IM)_g = (M_g + IM)/IM$ for $g \in G$.

Corollary 5 *Let M be a graded R-module, N be a graded submodule of M and I be a graded ideal of G-graded ring R with $IM \subseteq N$. Assume that $\phi_{IM} : S(M/IM) \to S(M/IM) \cup \{\emptyset\}$ be a function where $S(M/IM)$ is the set of all graded submodules of R/I-module M/IM. If N is a ϕ-graded semi-n-absorbing submodule of R-module M, then N/IM is a ϕ_{IM}-graded semi-n-absorbing submodule of R/I-module M/IM.*

Proof By Theorem 2, the proof is clear.

Corollary 6 *Let M be a graded R-module, N be a proper graded submodules of M where N_g is a g-component of N and I be an ideal of R_e. Suppose that*

$$\phi_{IM_e} : S(\frac{M_g + IM}{IM}) \to S(\frac{M_g + IM}{IM}) \cup \{\emptyset\},$$

be defined by

$$\phi_{IM_e}(\frac{N_g + IM}{IM}) = \frac{\phi_e(N_g) + IM}{IM},$$

where $\phi_e : S(M_g) \to S(M_g) \cup \{\emptyset\}$ is a function and $S(\frac{M_g+IM}{IM})$ is the set of all R_e/I- submodules of R_e/I-module $\frac{M_g+IM}{IM}$, $S(M_g)$ is the set of all R_e-submodules of R_e-module M_g. Then the following statememts hold:

(1) If N_g is a ϕ_e-semi-n-absorbing submodule of R_e-module M_g, then $\frac{N_g+IM}{IM}$ is a ϕ_{IM_e}-semi-n- absorbing submodule of R_e/I-module $\frac{M_g+IM}{IM}$.

(2) If $IM \subseteq \phi_e(N_g)$ and $\frac{N_g+IM}{IM}$ is a ϕ_{IM_e}-semi-n-absorbing submodule of R_e/I-module $\frac{M_g+IM}{IM}$, then N_g is a ϕ_e-semi-n-absorbing submodule of R_e-module M_g ($n \geq 2$).

Proof Apply Theorem 1.

We recall a proper graded submodule N of a graded R-module M as a graded weakly semi-n-absorbing submodule of M if whenever $r \in h(R)$, $m \in h(M)$ with $0 \neq r^n m \in N$, then $r^n \in (N : M)$ or $r^{n-1}m \in N$. The following proposition be used for ϕ-semiprime submodules and weakly semiprime submodules (See [9], Proposition 2.15). We state the following proposition for ϕ-graded semi-n-absorbing submodules and graded weakly semi-n-absorbing submodules.

Proposition 3 *Let M be a graded R-module, N be a proper graded submodules of M and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function with*

$$\phi(N_g \cap N_{g'}) = \phi(N_g) \cap \phi(N_{g'}), \quad \phi(\sum_{g\in G} N_g) = \sum_{g\in G} \phi(N_g),$$

and $\phi(0) = 0$ where N_g is a g-component of N. N is a ϕ-graded semi-n-absorbing submodule of graded R-module M if and only if $\frac{N}{\phi(N)}$ is a graded weakly semi-n-absorbing submodule of R-graded module $\frac{M}{\phi(N)}$.

Proof Let N be a proper graded submodules of R-graded module M, hence $N = \Sigma_{g\in G} N_g$ and $N_{g'} \cap (\sum_{g\in G\setminus\{g'\}} N_g) = \{0\}$. In regard to definition ϕ, we have $\phi(N) = \sum_{g\in G} \phi(N_g)$ and $\phi(N_{g'}) \cap (\sum_{g\in G\setminus\{g'\}} \phi(N_g)) = \{0\}$, so $\phi(N) = \bigoplus_{g\in G} \phi(N_g)$ and hence $\phi(N)$ is a garded submodule of M. Then $\frac{M}{\phi(N)}$ becomes a G-graded R-module with g-component $(\frac{M}{\phi(N)})_g = \frac{M_g+\phi(N)}{\phi(N)}$ for $g \in G$. Moreover, $m + \phi(N) \in h(\frac{M}{\phi(N)})$ implies that $m + \phi(N) \in \frac{M_g+\phi(N)}{\phi(N)}$ for some $g \in G$, hence $m \in h(M)$. Now, N be a ϕ-graded semi-n-absorbing submodule of graded R-module M and $\phi(N) \neq r^n(m + \phi(N)) \in \frac{N}{\phi(N)}$ where $r \in h(R)$ and $m + \phi(N) \in \frac{M}{\phi(N)}$, so $r^n m \in N \setminus \phi(N)$ and $m \in h(M)$. Then $r^{n-1}m \in N$ or $r^n \in (N : M)$, therefore $r^{n-1}m + \phi(N) \in \frac{N}{\phi(N)}$ or $r^n \in (\frac{N}{\phi(N)} : \frac{M}{\phi(N)})$ and so $\frac{N}{\phi(N)}$ is a graded weakly semi-n-absorbing submodule of R-graded module $\frac{M}{\phi(N)}$. Now, assume that $\frac{N}{\phi(N)}$ is a graded weakly semi-n-absorbing submodule of R-graded module $\frac{M}{\phi(N)}$. Similarly, we can prove that N is a ϕ-graded semi-n-absorbing submodule of graded R-module M.

Let R and R' be two G-graded rings, so

$$R = \bigoplus_{g\in G} R_g, \quad R' = \bigoplus_{g\in G} R'_g, \quad R_g R_h \subseteq R_{g+h},$$

and $R'_g R'_h \subseteq R'_{g+h}$. Suppose that $(R \times R')_g = R_g \times R'_g$. We define

$$R \times R' = \sum_{g\in G} R_g \times R'_g = \{\sum_{f.s} (r_g, r'_g) \ \mid \ r_g \in R_g, \ r'_g \in R'_g\}.$$

Now, we show that

$$(R_{g'} \times R'_{g'}) \cap (\sum_{g\in G\setminus\{g'\}} R_g \times R'_g) = \{(0, 0)\},$$

and

$$(R \times R')_g (R \times R')_h \subseteq (R \times R')_{g+h}.$$

Let $(r_{g''}, r'_{g''}) \in (R_{g'} \times R'_{g'}) \cap (\sum_{g\in G\setminus\{g'\}} R_g \times R'_g)$. It follows that

$$r_{g''} = \sum_{g'\neq g_i} r_{g_i}, \quad r_{g''} = r_{g'},$$

and

$$r'_{g''} = \sum_{g' \neq g_i} r'_{g_i}, \quad r'_{g''} = r'_{g'},$$

where $r_{g'} \in R_{g'}$, $r'_{g'} \in R'_{g'}$, $r_{g_i} \in R_g$ and $r'_{g_i} \in R'_g$. Then

$$r_{g''} \in R_{g'} \cap (\sum_{g \in G \setminus \{g'\}} R_g) = \{0\},$$

and $r'_{g''} \in R'_{g'} \cap (\sum_{g \in G \setminus \{g'\}} R'_g) = \{0\}$, hence $(r_{g''}, r'_{g''}) = (0, 0)$.
Now, assume that $z \in (R \times R')_g (R \times R')_h$ and hence $z = \sum_{f.s} (r_g, r'_g)(r_h, r'_h)$ where $(r_g, r'_g) \in R_g \times R'_g$ and $(r_h, r'_h) \in R_h \times R'_h$. Therefore

$$z = (\sum_{f.s} r_g r_h, \sum_{f.s} r'_g r'_h).$$

On the other hand, $\sum_{f.s} r_g r_h \in R_g R_h$ and $\sum_{f.s} r'_g r'_h \in R'_g R'_h$. Since $R_g R_h \subseteq R_{g+h}$ and $R'_g R'_h \subseteq R'_{g+h}$, so $z \in R_{g+h} \times R'_{g+h} = (R \times R')_{g+h}$. Accordingly, we have $R \times R' = \bigoplus_{g \in G} R_g \times R'_g$.

Let R and R' be two G-graded rings, M a graded R-module and M' a graded R'-module. We prove that $M \times M'$ is a graded $R \times R'$-module. We write $M \times M' = \sum_{g \in G} M_g \times M'_g$. Since $M_{g'} \cap (\sum_{g \in G \setminus \{g'\}} M_g) = \{0\}$ and $M'_{g'} \cap (\sum_{g \in G \setminus \{g'\}} M'_g) = \{0\}$, so $(M_{g'} \times M'_{g'}) \cap \sum_{g \in G \setminus \{g'\}} (M_g \times M'_g) = \{(0, 0)\}$. Also, we show that $(R \times R')_g (M \times M')_h \subseteq (M \times M')_{g+h}$.
Let $x \in (R \times R')_g (M \times M')_h$ and hence

$$x = \sum_{f.s} (r_g, r'_g)(m_h, m'_h) = (\sum_{f.s} r_g m_h, \sum_{f.s} r'_g m'_h),$$

where $(r_g, r'_g) \in (R \times R')_g$ and $(m_h, m'_h) \in (M \times M')_h$. Since $R_g M_h \subseteq M_{g+h}$ and $R'_g M'_h \subseteq M'_{g+h}$, so $\sum_{f.s} r_g m_h \in M_{g+h}$ and $\sum_{f.s} r'_g m'_h \in M'_{g+h}$, hence $x \in M_{g+h} \times M'_{g+h} = (M \times M')_{g+h}$. Then $M \times M' = \bigoplus_{g \in G} M_g \times M'_g$.

Now, let N be a proper graded submodule of M and N' a proper graded sumodule of M'. We have $N = \bigoplus_{g \in G} N_g$ and $N' = \bigoplus_{g \in G} N'_g$ where $N_g = N \cap M_g$ and $N'_g = N' \cap M'_g$. So,

$$\begin{aligned} N_g \times N'_g &= (N \cap M_g) \times (N' \cap M'_g) \\ &= (N \times N') \cap (M_g \times M'_g) \\ &= (N \times N') \cap (M \times M')_g \\ &= (N \times N')_g, \end{aligned}$$

and we can write

$$N \times N' = \bigoplus_{g \in G} (N \times N')_g = \bigoplus_{g \in G} N_g \times N'_g.$$

Hence, $N \times N'$ is a proper graded submodule of $M \times M'$. Let

$$\phi : S(M \times M') \to S(M \times M') \cup \{\emptyset\}, \quad \phi_1 : S(M) \to S(M) \cup \{\emptyset\},$$

and $\phi_2 : S(M') \to S(M') \cup \{\emptyset\}$ be functions with $\phi(N \times N') = \phi_1(N) \times \phi_2(N')$ where $S(M \times M')$ is a the set of all graded submodules of $M \times M'$. Now, with respect to above results we will assert the next theorems.

Theorem 3 *Let $M \times M'$ be a graded $R \times R'$-module, N a proper graded submodule of M and N' a proper graded submodule of M' with $(R_g \times R'_e) \subseteq (R \times R')_g$ and $M_g \times \{0\} \subseteq (M \times M')_g$ for all $g \in G$. If $N \times N'$ is a ϕ-graded semi-n-absorbing submodule of $M \times M'$, then N is a ϕ_1-graded semi-n-absorbing submodule of M.*

Proof Let N be a proper graded submodule of M, $r \in h(R)$ and $m \in h(M)$ with $r^n m \in N \setminus \phi_1(N)$. We have $(r^n m, 0) \in N \times N'$ and $(r^n m, 0) \notin \phi_1(N) \times \phi_2(N')$. So $(r^n m, 0) = (r, 1)^n (m, 0) \in N \times N' \setminus \phi_1(N) \times \phi_2(N')$ where $r \in h(R)$ and $m \in M$. We have $(r, 1) \in h(R) \times R'_e$, hence $r \in R_g$ for some $g \in G$, therefore $(r, 1) \in R_g \times R'_e$. Since $(R_g \times R'_e) \subseteq (R \times R')_g$, so $(r, 1) \in (R \times R')_g \subseteq h(R \times R')$. Also, $m \in h(M)$ implies that $m \in M_g$ for some $g \in G$ and hence $(m, 0) \in M_g \times \{0\}$. Since $M_g \times \{0\} \subseteq (M_g \times M')_g$, so $(m, 0) \in (M \times M')_g \subseteq h(M \times M')$. Therefore we show that $(m, 0) \in h(M \times M')$ and $(r, 1) \in h(R \times R')$. Since $N \times N'$ is a ϕ-graded semi-n-absorbing submodule of $M \times M'$. Thus $(r, 1)^n \in (N \times N' : M \times M')$ or $(r, 1)^{n-1}(m, 0) \in N \times N'$. It follows that $(r, 1)^n (m'', m') \in N \times N'$, for each $(m'', m') \in h(M \times M') \subseteq h(M) \times h(M')$, or $r^{n-1} m \in N$. Therefore $r^n \in (N : M)$ or $r^{n-1} m \in N$, as required.

Corollary 7 *Let $M \times M'$ be a graded $R \times R'$-module, N a proper graded submodule of M and N' a proper graded submodule of M' with $(R_e \times R'_g) \subseteq (R \times R')_g$ and $\{0\} \times M'_g \subseteq (M \times M')_g$ for all $g \in G$. If $N \times N'$ is a ϕ-graded semi-n-absorbing submodule of $M \times M'$, then N' is a ϕ_2-graded semi-n-absorbing submodule of M'.*

Proof The proof is similar to the proof of Theorem 3.

Theorem 4 *Let $M \times M'$ be a graded $R \times R'$-module and $\phi : S(M \times M') \to S(M \times M') \cup \{\emptyset\}$ be a function with $\phi(N \times N') = \phi_1(N) \times \phi_2(N')$ where $\phi_1 : S(M) \to S(M) \cup \{\emptyset\}$,$\phi_2 : S(M') \to S(M') \cup \{\emptyset\}$ be two functions such that $h(R') = (\phi_2(M') : h(M'))$. If N is a ϕ_1-graded semi-n-absorbing submodule of M, then $N \times M'$ is a ϕ-graded semi-n-absorbing submodule of $M \times M'$.*

Proof Let $(r, r') \in h(R \times R')$ and $(m, m') \in h(M \times M')$ with $(r, r')^n (m, m') \in N \times M' \setminus \phi(N \times M')$. It follows that $r^n m \in N$, $r'^n m' \in M'$ and $(r^n m, r'^n m') \notin \phi_1(N) \times \phi_2(M')$. Since $h(R \times R') \subseteq h(R) \times h(R')$ and $h(M \times M') \subseteq h(M) \times h(M')$, so $(r, r') \in h(R) \times h(R')$, $(m, m') \in h(M) \times h(M')$ and hence $r \in h(R)$, $r' \in h(R')$, $m \in h(M)$ and $m' \in h(M')$. On the other hand, since $h(R') = (\phi_2(M') : h(M'))$, so $r'^n m' \in \phi_2(M')$ and hence $r^n m \notin \phi_1(N)$. Therefore, we show that $r^n m \in N \setminus \phi_1(N)$ where $r \in h(R)$ and $m \in h(M)$. Because N is a ϕ_1-graded semi-n-absorbing submodule of M, hence $r^n \in (N : M)$ or $r^{n-1} m \in N$. Then $(r^n, r'^n) = (r, r')^n \in (N \times M' : M \times M')$ or $(r^{n-1} m, r'^{n-1} m') = (r, r')^{n-1}(m, m') \in N \times M'$, as needed.

Corollary 8 *Let $M \times M'$ be a graded $R \times R'$-module and $\phi : S(M \times M') \to S(M \times M') \cup \{\emptyset\}$ be a function with $\phi(N \times N') = \phi_1(N) \times \phi_2(N')$ where $\phi_1 : S(M) \to S(M) \cup \{\emptyset\}$, $\phi_2 : S(M') \to S(M') \cup \{\emptyset\}$ be two functions such that $h(R) = (\phi_1(M) : h(M))$. If N' is a ϕ_2-graded semi-n-absorbing submodule of M', then $M \times N'$ is a ϕ-graded semi-n-absorbing submodule of $M \times M'$.*

Proof The proof is similar to the proof of Theorem 4.

Let R be a G-ring. A graded R-module M is called a graded multiplication R-module if for every graded submodule N of M, $N = IM$ for some graded ideal of R. One can easily prove that if N is a graded submodule of a graded multiplication module M, then $N = (N : M)M$. Now, suppose that N, K be two graded submodules of graded R-module, $N = IM$ and $K = JM$ for some graded ideals I and J of R, then the product of N and K, denoted by NK, is defined by IJM. We recall that N^m is defined by $N^m = (N : M)^m M$ (See [6] and [11]).

Definition 1 Let M be a graded multipilication R-module and t a positive integer with $t \geqslant 2$. A proper graded submodule N of M is said a t-potent graded semi-n-absorbing submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $r^n m \in N^t$, then $r^{n-1}m \in N$.

Proposition 4 *Let M be a graded multiplication R-module and N be a proper graded submodule of M. If N is a graded m-almost semi-n-absorbing submodule of M for $m \geqslant 2$ and N is a t-potent graded semi-n-absorbing submodule of M such that $t \leqslant m$, then N is a graded semi-n-absorbing submodule of M ($n \geqslant 3$).*

Proof Let $r \in h(R)$ and $x \in h(M)$ with $r^n x \in N$. Since $N^m \subseteq N^t$, so $r^n x \notin N^t$ imples that $r^n x \notin N^m$ and hence $r^n x \in N \setminus N^m$. On the other hand, we have $N \setminus N^m = N \setminus (N : M)^m M = N \setminus (N : M)^{m-1}(N : M)M = N \setminus (N : M)^{m-1}N$. Thus $r^n x \in N \setminus (N : M)^{m-1}N$. Since N is a graded m-almost semi-n-absorbing submodule of M, so $r^{n-1}x \in N$ or $r^n \in (N : M)$. Therefore N is a graded semi-n-absorbing submodule of M. If $r^n x \in N^t$, beacause N is a t-potent graded semi-n-absorbing submodule of M, hence $r^{n-1}x \in N$, as required.

Acknowledgment

We would like to thank the referee for his/her valuable comments and suggestions which improve the quality of this paper.

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