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Research Article

Metric Dimension Of $C_n(1, 2, 3)$ For $n \equiv 0 \pmod 6$

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Abstract The *metric dimension* of a connected graph G is the minimum number of vertices in a subset B of G such that all other vertices are uniquely determined by their distances to the vertices in B. In this case, B is called a *metric basis* for G and written $dim(G) = \|B\|$. We have solved an open problem which shows dimension of circulant graph, $dim(C_n(1, 2, 3)) = 4, n \equiv 0 \pmod 6$. To prove this result, we employ a combination of combinatorial techniques, including distance-based analysis and structural properties of circulant graphs, to carefully analyze the relationship between the graphs structure and its metric dimension. The solution not only answers a previously unresolved question in graph theory but also provides valuable insights into the metric dimensions of more general classes of graphs, particularly in network theory, where understanding the metric dimension is essential for applications in sensor networks, graph-based data storage, and network routing. This work lays the groundwork for future research on the metric dimensions of other families of graphs and has potential applications in optimizing communication and sensor placement in large-scale networks.

Keywords Metric dimension · resolving set · metric basis · circulant graph.

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1 Introduction

In 1975, when Slater [6] was working on tracking submarines and determining the route of planes and ships by sea telecommunications, he discovered the importance of the explorers' collection. He used the terms of the locating set

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and the locating number. The next year, in 1976, Harary F. and Melter [1], independently of Slater, defined these concepts and used modern terminology.

As one of the applications of the metric dimension in data science and machine learning, the principles of metric dimensions offer powerful tools for feature selection, dimensionality reduction, clustering, graph-based analysis, and model optimization. They improve efficiency, interpret ability, and scalability, especially in high-dimensional or complex data scenarios[3]. Graphs with a metric dimension of two were characterized in [4] and graphs with a unique metric dimension have been investigated [5].
Let $G = (V, E)$ be a connected simple graph. For two vertex u and v of G, the distance $d(x, y)$ of x and y is the length of a minimum path connecting x to y. For a subsets $R = \{r_1, \ldots, r_k\}$ of V and a vertex v, the representation of v with respect to R is the k-tuple $\langle v|R\rangle = (d(v, v_1), \ldots, d(v, v_k))$. The subset R is called a resolving set for G if any vertex has a unique representation with respect to R. A resolving set B of V is called a metric basis for G if it has the minimum possible number of elements for a resolving set. The metric dimension G, denoted by $\dim(G)$ is then equal to this minimum number. See [1,6]

Let n, m and $a_1, a_2, \ldots, a_m$ be positive integers, $1 \leq a_i \leq \lfloor \frac{n}{2} \rfloor$ and $a_i \neq a_j$ for all $1 \leq i < j \leq m$. An undirected graph with the set of vertices $V = \{v_1, \ldots, v_n\}$ and the set of edges $E = \{v_i v_{i+a_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$, the indices being taken modulo n, is called a circulant graph and is denoted by $C_n(a_1, \ldots, a_m)$. For $n = 12$, $m = 3$ see Fig. 1.

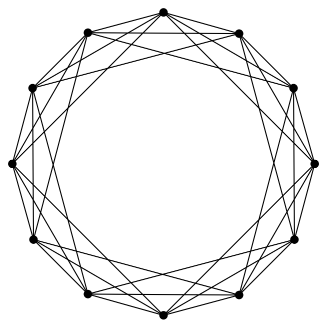

Fig. 1 The circulant graph $C_{12}(1, 2, 3)$.

In [2] M. Imran et al. proved the following theorem.

Theorem 1 *[2] For circulant graphs* $Cn(1, 2, 3)$*, we have* $dim(C_n(1, 2, 3)) = 4$ *when* $n \equiv 2, 3, 4, 5 \pmod 6$ *and* $n \geq 13$*.*

And raised the following problem.

Open Problem 1 [2] Find the exact value of the metric dimension of

$$C_n(1, 2, 3), n \equiv 0, 1 \pmod 6$$

.

2 Obtain the metric dimension of $C_n(1,2,3), n \equiv 0 \pmod 6$

Lemma 1 *$dim(C_n(1,2,3)) \geq 4$, $n \equiv 0 \pmod 6, n \geq 12$.*

Proof Let the vertices of $C_n(1,2,3)$ are named as $\{1,2,\ldots,n\}$ respectively. Without loss of generality, suppose that $B = \{4,x,y\}$ with $5 \leq x \leq y \leq n$ is the basis. Note that $d(t,4) = 1$ for $t \in \{1,2,3,5,6,7\}$. Table 1 shows how in each case at least two vertices will not be resolved by B. This is a contradiction; therefore $dim(C_n(1,2,3)) > 3$.

Table 1 Vertices that not be resolved by $\{4,x,y\}$

x,y	5	6	7	$3k$, $\frac{n}{2}+3k-1$ $(3 \leq k \leq \frac{n}{6})$	$3k+1$, $\frac{n}{2}+3k-3$ $(3 \leq k \leq \frac{n}{6})$	$3k+2$, $\frac{n}{2}+3k+1$ $(2 \leq k \leq \frac{n}{6}-1)$	$\frac{n}{2}+2$	$\frac{n}{2}+3$	$\frac{n}{2}+4$	$\frac{n}{2}+5$	$\frac{n}{2}+6$
5	2,3	3,7	2,3	6,7	2,3	2,3	2,3	2,3	2,3	6,7	2,3
6		1,2	1,2	1,2	1,2	5,7	1,2	1,2	3,5	1,2	1,2
7			1,2	1,2	1,2	2,3	1,2	1,2	2,3	1,2	1,2
$3k$, $\frac{n}{2}+3k-1$ $(3 \leq k \leq \frac{n}{6})$				1,2	1,2	6,7	1,2	1,2	3,5	1,2	1,2
$3k+1$, $\frac{n}{2}+3k-3$ $(3 \leq k \leq \frac{n}{6})$					1,2	5,6	1,2	1,2	5,6	1,2	1,2
$3k+2$, $\frac{n}{2}+3k+1$ $(2 \leq k \leq \frac{n}{6}-1)$						2,3	2,3	2,3	2,3	1,2	1,2
$\frac{n}{2}+2$							1,2	1,2	2,3	1,2	1,2
$\frac{n}{2}+3$								1,2	2,3	1,2	1,2
$\frac{n}{2}+4$									2,3	3,5	2,3
$\frac{n}{2}+5$										1,2	1,2
$\frac{n}{2}+6$											1,2

Lemma 2 *$dim(C_n(1,2,3)) \leq 4$, $n \equiv 0 \pmod 6, n \geq 12$.*

Proof Let $n \equiv 0 \pmod 6, n \geq 12$ and the vertices $C_n(1,2,3)$ with $1,2,\ldots,n$ are labeled. We show that $W = \{1,3,5,7\}$ is a resolving set for $C_n(1,2,3)$.

Let $x \in \{1,2,\ldots,n\}$ be a vertex of $C_n(1,2,3)$. Then

$$\langle x|W\rangle = F_n(x) = (f_1(x), f_2(x), f_3(x), f_4(x)),$$

where

$$f_1(x) = \begin{cases} \lceil \frac{x-1}{3} \rceil, & 1 \leq x \leq \frac{n}{2}, \\ \frac{n}{3} + 1 - \lceil \frac{x}{3} \rceil, & \frac{n}{2} + 1 \leq x \leq n. \end{cases}$$

$$f_2(x) = \begin{cases} 2 - \lceil \frac{x}{3} \rceil, & 1 \leq x \leq 2, \\ \lceil \frac{x}{3} \rceil - 1, & 3 \leq x \leq \frac{n}{2} + 2, \\ \frac{n}{3} + 1 - \lceil \frac{x-2}{3} \rceil, & \frac{n}{2} + 3 \leq x \leq n. \end{cases}$$

$$f_3(x) = \begin{cases} 2 - \lceil \frac{x-1}{3} \rceil, & 1 \leq x \leq 4, \\ \lceil \frac{x-2}{3} \rceil - 1, & 5 \leq x \leq \frac{n}{2} + 4, \\ \frac{n}{3} + 2 - \lceil \frac{x-1}{3} \rceil, & \frac{n}{2} + 5 \leq x \leq n. \end{cases}$$

$$f_4(x) = \begin{cases} 3 - \lceil \frac{x}{3} \rceil, & 1 \leq x \leq 6, \\ \lceil \frac{x-1}{3} \rceil - 2, & 7 \leq x \leq \frac{n}{2} + 6, \\ \frac{n}{3} + 3 - \lceil \frac{x}{3} \rceil, & \frac{n}{2} + 7 \leq x \leq n. \end{cases}$$

To prove that $F_n(x)$ is an injective function, you can partition the vertex set $\{1,2,\ldots,n\}$ into A, B, C, D as follows:

$$A = \{1,2,\ldots,6\}, \qquad B = \{7,8,\ldots,\frac{n}{2}+1\},$$
$$C = \{\frac{n}{2}+2, \frac{n}{2}+3, \ldots, \frac{n}{2}+6\}, \qquad D = \{\frac{n}{2}+7, \frac{n}{2}+8, \ldots, n\}.$$

We see that if $x \in A$,

$$F_n(1) = (0,1,2,2), \qquad F_n(2) = (1,1,1,2),$$
$$F_n(3) = (1,0,1,2), \qquad F_n(4) = (1,1,1,1),$$
$$F_n(5) = (2,1,0,1), \qquad F_n(6) = (2,1,1,1).$$

If $x \in B$,

$$F_n(x) = \begin{cases} (k+1, k, k, k-1), & k = \frac{x-3}{3},\ x \overset{3}{\equiv} 0, \\ (k, k, k-1, k-2), & k = \frac{x-1}{3},\ x \overset{3}{\equiv} 1, \\ (k+1, k, k-1, k-1), & k = \frac{x-2}{3},\ x \overset{3}{\equiv} 2. \end{cases}$$

If $x \in C$, put $k = \frac{n}{6}$,

$$\begin{aligned}
F_n(\frac{n}{2}+2) &= (k,k,k-1,k-1),\\
F_n(\frac{n}{2}+3) &= (k,k,k,k-1),\\
F_n(\frac{n}{2}+4) &= (k-1,k,k,k-1),\\
F_n(\frac{n}{2}+5) &= (k-1,k,k,k),\\
F_n(\frac{n}{2}+6) &= (k-1,k-1,k,k).
\end{aligned}$$

If $x \in D$,

$$F_n(x) = \begin{cases} (k,k,k+1,k+2), & k = \frac{n-x+3}{3},\ x \overset{3}{\equiv} 0,\\ (k,k+1,k+2,k+2), & k = \frac{n-x+1}{3},\ x \overset{3}{\equiv} 1,\\ (k,k+1,k+1,k+2), & k = \frac{n-x+2}{3},\ x \overset{3}{\equiv} 2. \end{cases}$$

In all the above expression with $k \in \mathbb{N}$, clearly, there is no repetition.

According to Lemmas 4.2 and 4.3 we conclude the following theorem.

Theorem 2 *$dim(C_n(1,2,3)) = 4$ for $n \equiv 0 \pmod 6$, $n \geq 12$*

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