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Prey-Predator System; Having Stable Periodic Orbit

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Abstract The study of differential equations is useful in to analyze the possible past or future with help of present information. In this paper, the behavior of solutions has been analyzed around the equilibrium points for Gause model. Finally, some results are worked out to exist the stable periodic orbit for mentioned predator-prey system.

Keywords Prey-predator · Dynamics · Periodic orbit · Monotonic · Unstable

Mathematics Subject Classification (2010) 35B10 · 92D40 · 83C10

1 Introduction

The extensive work of Kolmogorov has been done on the existence and uniqueness of limit cycles in a predator-prey system modeled by autonomous differential equations. One of the popular version and well-known of this systems is Gause-type model which can be written as follows:

$$\begin{cases} \frac{dx}{dt} = xg(x) - yp(x) \\ \frac{dy}{dt} = y[-d + cp(x)]. \end{cases} \tag{1}$$

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Here, the parameters c and d are positive real numbers, the function $g(x)$ is the individual growth rate of the prey species in the absence of the predators, $p(x)$ represents the functional response of predators to the growth of prey.

Now, we review some basic concepts which are employed in this work.

Remark 1 Compact limit set is limit of a closed bounded set when mapped through time.

Remark 2 A nonempty compact limit set of a C^1planar dynamical system is a closed orbit if the mentioned limit cycle contains no equilibrium point.

Remark 3 C^1 has a continuous first order derivative.

2 Main Results

Consider the Gause Model (1). Having payed attention to the this model, one can see

$$\tfrac{dg(x)}{dx} \leq 0, \qquad g(0) > 0;$$

and

$$\tfrac{dp(x)}{dx} > 0, \qquad p(0) = 0.$$

A prototype of $g(x)$ is the logistic growth pattern, while $p(x)$ is usually assumed to be monotonically increasing. For the background on this model and its generalizations, see Freedman [1], Kuang and Freedman [3], Huang and Merrill [2], and the references therein.

Now, let us assume that:

$$\tfrac{dz(t)}{dt} = f(z) = [f_1(z_1, ..., z_n), ..., f_n(z_1, ..., z_n)]$$

$$\tfrac{dF(z,t)}{dt} = f(z), \qquad F^{(n)}(z) = \tfrac{d^n F(z)}{dz^n}, \qquad n = 0, ..., \infty.$$

Note that in the above model we have

$$z_1 = x, \qquad z_2 = y;$$

and

$$f_1(z) = xg(x) - yp(x), \qquad f_2(z) = y[-d + cp(x)].$$

If we assume that

$$z \epsilon B(z_0, \varepsilon) = \{z \quad | \quad |z - z_0| < \varepsilon\},$$

then the Taylor series is as follows:

$$F(z) = \sum \frac{F^{(k)}(z_0)(z-_{z0})^k}{k!} = F(z_0) + \sum \frac{f^{(k)}(z_0)(z-z_0)^{k+1}}{(k+1)!} \tag{2}$$

where

$$f^{(k)}(z_0) = \begin{pmatrix} \frac{\partial^k f_1(z_1,..,z_n)}{\partial z_1^k} & \dots & \frac{\partial^k f_1(z_1,..,z_n)}{\partial z_n^k} \\ \vdots & \vdots & \vdots \\ \frac{\partial^k f_n(z_1,..,z_n)}{\partial z_1^k} & \dots & \frac{\partial^k f_n(z_1,..,z_n)}{\partial z_n^k} \end{pmatrix}. \tag{3}$$

Here, $f^{(k)}(z_0)$ describes the k^{th} derivative of $f(z_0)$.

Indeed, $f^{(1)}(z_0)$ explains the Jacobian matrix which is evaluated at the point z_0.

Theorem 1 *For system (1) the following statements hold:*

i) *It has three equilibrium points which are origin,* $(k, 0)$ *and* (x^*, y^*).

ii) *Origin and the point* $(k, 0)$ *are unstable points. The stability of point* (x^*, y^*) *depends to the sign of the following term*

$$B = g(x^*) + x^* g^{'}(x^*) - y^* p^{'}(x^*),$$

if $B < 0$ *then the said point is unstable and if* $B > 0$ *then the said point is stable point.*

iii) *There is stable periodic orbit around the unstable fixed point.*

Proof ***i***) To find the equilibrium points of system (1), we should set

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0, \end{cases}$$

and so, we see that system (1) has three fixed points $(0, 0), (k, 0), (x^*, y^*)$.

ii) First,we calculate the Jacobian matrix for origin, and so we have

$$F(0) = 0, f(0) = 0.$$

Eigenvalue λ is solution of the following characteristic equation

$$|f^{(1)}(z_0) - \lambda I| = 0$$

which is evaluated at fixed point z_0.

Here, I is identity matrix.

For calculating the eigenvalues of Jacobian matrix of system (1), we have

$$f^{(1)}(z_0) = \begin{pmatrix} g(x_0) + x_0 g^{'}(x_0) - y_0 p'(x_0) & p(x_0) \\ c y_0 p^{'}(x_0) & -d + cp(x_0) \end{pmatrix}.$$

Thus,

$$|f^{(1)}(z) - \lambda I| = 0.$$

Hence,

$$\left| \begin{matrix} g(x_0) + x_0 g^{'}(x_0) - y_0 p'(x_0) - \lambda & p(x_0) \\ c y_0 p^{'}(x_0) & -d + cp(x_0) - \lambda \end{matrix} \right| = 0.$$

And so, the following polynomial equation of second order will be obtained:

$$\Big(g(x_0) + x_0 g^{'}(x_0) - y_0 p'(x_0) - \lambda)(-d + cp(x_0) - \lambda) - c y_0 p^{'}(x_0) p(x_0) = 0 \quad (4)$$

Therefore, by calculating its solution, we see that
the point z_0 is stable if

$$max(Re(\lambda_k), fork = 1, ..., n) < 0.$$

And so, the mentioned point is unstable, if $max(Re(\lambda_k), k = 1, ..., n) > 0$.

Now, let us consider the point $(0, 0)$, so that $p(0) = 0$.
Now, set it into the equation (1).
Note that

$$x_0 = y_0 = p(x_0) = 0,$$

and

$$(g(0) - \lambda)(-d - \lambda) = 0.$$

Thus, the eigenvalue may be obtained as follows:

$$\lambda = g(0), -d.$$

$\lambda_1 = g(0) > 0$ which corresponds to x-axis,
$\lambda_2 = -d < 0$ which corresponds to y-axis.

As regarding the multiplication of the above eigenvalues is negative we see that origin is unstable point for system (1).

Now, consider the point $(k, 0)$ so that $g(k) = 0$. After setting it into the equation (1) and assuming

$$x_0 = k \quad , \quad y_0 = g(k) = 0$$

we have:

$$(kg^{'}(k) - \lambda)(-d + cp(k) - \lambda) = 0$$

Hence, the related eigenvalues can be obtained as follows:

$$\lambda = kg^{'}(k), -d + cp(k).$$

Note that

$$-d + cp(x^*) = 0 \quad , \quad x^* < k \quad , \quad p(x) > 0.$$

Thus,

$$p(x^*) < p(k).$$

And so,

$$-d + cp(k) > 0,$$

which implies that its first eigenvalue is negative by following value:

$$\lambda_1 = kg(k) < 0$$

And also,

$$\lambda_2 = -d + cp(k) > 0.$$

Therefor, regarding that the multiplication of the last eigenvalues is negative the point $(k, 0)$ is unstable for system (1). Indeed, it is saddle point

Now, let us consider the point (x^*, y^*), then

$$0 < x^* < k$$

and

$$y^* > 0.$$

Setting it into the equation (1), one can see that

$$-d + cp(x_0) = 0.$$

And so,

$$x_0 = x^*, \quad y_0 = y^*$$

The following equation becomes:

$$(g(x^*) + x^*g^{'}(x^*) - y^*p^{'}(x^*) - \lambda)(-\lambda) - cy^*p^{'}(x^*)p(x^*) = 0,$$

which is equivalent by the following equation:

$$\lambda^2 - [g(x^*) + x^*g^{'}(x^*) - y^*p^{'}(x^*)]\lambda - cy^*p^{'}(x^*)p(x^*) = 0.$$

Now assume that

$$B = g(x^*) + x^*g^{'}(x^*) - y^*p^{'}(x^*),$$

and

$$C = -cy^*p^{'}(x^*)p(x^*).$$

Hence,

$$\lambda = \frac{B \pm \sqrt{B^2 + 4C}}{2}.$$

First, we assume that the sign of B is positive.
If $B > 0$, then the first eigenvalue is as follows:

$$\lambda_1 = \frac{B - \sqrt{B^2 + 4C}}{2} < 0,$$

and for the second eigenvalue we have

$$\lambda_2 = \frac{B + \sqrt{B^2 + 4C}}{2} > 0,$$

and so, in this case we have the point (x^*, y^*) is unstable for system (1).

For the second case consider the sign of B is negative.
If $B < 0$, then

$$\lambda_1 = \frac{B - \sqrt{B^2 + 4C}}{2} < 0$$

and

$$\lambda_2 = \frac{B + \sqrt{B^2 + 4C}}{2} < 0.$$

Therefore, the point (x^*, y^*) is stable for system (1).

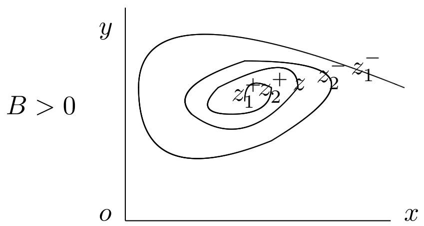

Fig1: Periodic Orbits

$$\begin{array}{ll} F(z^+, t_1) = z_1^+ = (x_1, y_1) & F(z^-, t_1) = z_1^- = (x_1, y_1) \\ F(z^+, t_2) = z_2^+ = (x_2, y_2) & F(z^-, t_2) = z_2^- = (x_2, y_2) \\ \lim z_k^+ \to z, \quad as \quad k \to \infty & \lim z_k^- \to z^-, \quad as \quad k \to \infty \end{array}$$

$$\Rightarrow z^- \leq z.$$

Therefore, one can see if $B > 0$, then the dynamical system has a stable periodic orbit. Indeed, Stable periodic orbit exists around the unstable fixed point.

Therefore, the proof of theorem is done.

3 Conclusion

The equilibrium points and periodic orbits explain the equilibrium populations and oscillation populations respectively. The stability of equilibrium point indeed is main concept in this area. In this work, we concentrated on Gause model and found out some the results about equilibrium points and their stability which are analyzed in presented theorem. In fact, by adding some conditions on existing parameters, we are able to make stable the said model.

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