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An Extension of the Min-Max Method for Approximate Solutions of Multiobjective Optimization Problems

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Abstract It is a common characteristic of many multiobjective optimization problems that the efficient solution set can only be identified approximately. This study addresses scalarization techniques for solving multiobjective optimization problems. The min-max scalarization technique is considered, and efforts are made to overcome its weaknesses in studying approximate efficient solutions. To this end, two modifications of the min-max scalarization technique are proposed. First, an alternative form of the min-max method is introduced. Additionally, by using slack and surplus variables in the constraints and penalizing violations in the objective function, we obtain easy-to-check conditions for approximate efficiency. The established theorems clarify the relationship between ε-(weakly and properly) efficient solutions of the multiobjective optimization problem and ϵ-optimal solutions of the proposed scalarized problems, without requiring any assumptions of convexity.

Keywords Multiobjective programming · Scalarization · Min-max method · Approximate solutions.

Mathematics Subject Classification (2010) 90C20 · 90C29

1 Introduction

Multiobjective optimization problem (MOP) is a part of vector optimization that deals with mathematical programming involving more than one objective function to be minimized over a set of decisions. In recent years, MOPs have

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emerged in various fields such as engineering, economics, management and medicine (see, for example,[2–5, 12, 18, 27]). There has been a growing interest in engaging with approximately efficient solutions for MOPs. This interest can be attributed to several important reasons. Firstly, numerical algorithms that inherently generate approximate solutions have become increasingly common in the field of optimization. Furthermore, under specific conditions such as compactness or boundedness, the set of efficient solutions for an MOP may be empty. However, the set of approximately efficient solutions often remains nonempty, even when these conditions are not satisfied.

Kutateladze initially introduced the concept of approximate solutions in MOP [14]. Later, Loridan expanded this concept [17]. Subsequently, White introduced various types of approximate solutions within the framework of MOPs [26]. Since then, many researchers have investigated the properties of these approximate solutions and identified the necessary and sufficient conditions for ε-(weakly and properly) efficient solutions in MOP. For more details, readers are referred to [8–10, 13, 21, 22].

In addition to theoretical studies, the practical importance of approximate efficient solutions is emphasized by their use in different fields. A prominent example is the research conducted by Shao and Ehrgott [23, 24], who utilized approximate efficient solutions to optimize radiation therapy processes. These applications demonstrate how the investigation of approximate solutions in MOP extends beyond theoretical concepts to solve real-life problems. A classical method used for solving MOPs is the scalarization technique. This approach involves converting an MOP into a single-objective problem, possibly involving some parameters or additional constraints. Some of the well-known scalarization techniques include the weighted sum method[19], the min-max method [15], and the ε-constraint method[6].

An interesting research area in MOPs is exploring the relationship between the (approximate) efficient solutions of an MOP and the (approximate) optimal solutions of the corresponding scalarized problem, which enhances our understanding of the trade-offs in multiobjective decision-making. In the area of approximate solutions, Liu obtained necessary and sufficient conditions for ε-proper efficient solutions in convex MOP using the weighted sum method [16]. In [10, 11], Ghaznavi and Khorram achieved important results by utilizing the elastic ε-constraint method. Our approach aims to determine conditions that relate ε-(weakly and properly) efficient solutions of an MOP to ϵ-optimal solutions of the scalarized problem. In the following, we will introduce initial notions and definitions that will be used throughout the rest of the paper. To compare two vectors in $\mathbb{R}^p$, some common orders are as follows. Let $y^1, y^2 \in \mathbb{R}^p$. We say $y^1 \leqq y^2$ ($y^1 < y^2$) if and only if $y^1_i \leq y^2_i$ ($y^1_i < y^2_i$) for all $i = 1, \ldots, p$. Moreover, we write $y^1 \leq y^2$ if and only if $y^1 \leqq y^2$ and $y^1 \neq y^2$. An MOP can be expressed in the following way

$$\min_{x \in X} f(x) = (f_1(x), \ldots, f_p(x)), \tag{1}$$

where, $X \subseteq \mathbb{R}^n$ is the set of all feasible solutions or decisions and each f_i, for $1 \leqslant i \leqslant p$, is a real valued function on X. In this study, we will assume that the functions f_i are continuous and bounded above on X. Thus, $\max_{k=1,\dots,p} f_k(x)$ is guaranteed to exist.

Definition 1 ([25]) Let $\epsilon \geqslant 0$. Consider a real-valued function h defined on $X \subseteq \mathbb{R}^n$. A point $\hat{x} \in X$ is referred to as an ϵ-optimal solution for the problem $\min_{x\in X} h(x)$, if $h(\hat{x}) - \epsilon \leqslant h(x)$ for all $x \in X$.

Definition 2 ([21]) Consider $\varepsilon \in \mathbb{R}^p_{\geqq} = \{x \in \mathbb{R}^p \mid x \geqq 0\}$. A point $\hat{x} \in X$ for the MOP (1) is called

(1) ε-Weakly efficient solution if there is no other $x \in X$ such that $f(x) < f(\hat{x}) - \varepsilon$,
(2) ε-Efficient solution if there is no other $x \in X$ such that $f(x) \leq f(\hat{x}) - \varepsilon$.

Definition 3 ([21]) A point $\hat{x} \in X$ is called ε-properly efficient solution for the MOP (1) if it is ε-efficient solution and there exists a positive constant M such that for each $1 \leqslant i \leqslant p$ and for any $x \in X$ satisfying $f_i(x) < f_i(\hat{x}) - \varepsilon_i$, there exists an index $1 \leqslant j \leqslant p$ such that $f_j(x) > f_j(\hat{x}) - \varepsilon_j$ and the following inequality holds

$$\frac{f_i(\hat{x}) - f_i(x) - \varepsilon_i}{f_j(x) - f_j(\hat{x}) + \varepsilon_j} \leqslant M.$$

In a sequel to this paper, the sets of all ε-weakly efficient, ε-efficient, and properly ε-efficient solutions will be referred to as $X_{\varepsilon wE}, X_{\varepsilon E}$, and $X_{\varepsilon pE}$, respectively.

The remainder of this paper is organized as follows: In Section 2, an alternative form of the min-max method is introduced, and the relationship between the optimal solutions derived from the proposed alternative min-max method and the original min-max method is presented through a theorem. In Section 3, an extension of the proposed method is utilized to derive conditions for ε-(weakly and properly) efficient solutions in the context of MOPs. Finally, the paper concludes with some final remarks in the last section.

2 An alternative form of the min-max method

A conventional and widely used approach for addressing MOPs is scalarization. This technique transforms an MOP into a single-objective problem, which may include various parameters and additional constraints. One famous scalarization technique for tackling MOPs is the min-max method, which is defined as follows

$$\min_{x\in X} \max_{k=1,\dots,p} f_k(x). \tag{2}$$

In problem (2), if we assume that $x_{n+1} = \max_{k=1,\ldots,p} f_k(x)$, it can be reformulated as follows

$$\begin{aligned} &\min\ x_{n+1} \\ &s.t.\ \ f_k(x) \leqslant x_{n+1}, \qquad k = 1,\ldots,p, \\ &\qquad x \in X, \quad x_{n+1} \in \mathbb{R}. \end{aligned} \tag{3}$$

It is noted that problem (3) can be considered a special case of the *Pascoletti-Serafini* scalarization technique [7,20]. The next theorem shows that an optimal objective value of problem (3) is a lower bound to an optimal objective value of problem (2). To establish this, we prove the following lemma.

Lemma 1 *If $\bar{x}$ is a feasible solution for problem (2), then there exists an $\bar{x}_{n+1} \in \mathbb{R}$ such that the vector $(\bar{x}, \bar{x}_{n+1}) \in \mathbb{R}^{n+1}$ is a feasible solution for problem (3).*

Proof *Let $\bar{x}$ be a feasible solution for problem (2). Define $\bar{x}_{n+1} = \max_{k=1,\ldots,p} f_k(\bar{x})$. Obviously, $(\bar{x}, \bar{x}_{n+1})$ is a feasible solution for problem (3), which completes the proof.*

Theorem 1 *Assume that the set of optimal solutions for problem (3) is not empty and that $(\hat{x}, \hat{x}_{n+1})$ is an optimal solution of problem (3). Let $\bar{x}$ be an optimal solution of problem (2). Then, the following statements hold.*

(1) $\hat{x}_{n+1} \leqslant \max_{k=1,\ldots,p} f_k(\bar{x})$,
(2) If $\hat{x}$ is a feasible solution for problem (2), then $\hat{x}_{n+1} = \max_{k=1,\ldots,p} f_k(\bar{x})$.

Proof (1) Let $\bar{x}$ be a feasible solution for problem (2). Define

$$\bar{x}_{n+1} = \max_{k=1,\ldots,p} f_k(\bar{x}).$$

By Lemma 1, $(\bar{x}, \bar{x}_{n+1})$ is a feasible solution of problem (3). Now, by assumption, $(\hat{x}, \hat{x}_{n+1})$ is an optimal solution of problem (3), it follows that

$$\hat{x}_{n+1} \leqslant \max_{k=1,\ldots,p} f_k(\bar{x}). \tag{4}$$

(2) To prove the second part of the theorem, note that since $\hat{x}$ is a feasible solution for problem (2), it implies that

$$\max_{k=1,\ldots,p} f_k(\bar{x}) \leqslant \max_{k=1,\ldots,p} f_k(\hat{x}) = \hat{x}_{n+1}. \tag{5}$$

Combining (4) with (5), we infer that

$$\hat{x}_{n+1} \leqslant \max_{k=1,\ldots,p} f_k(\bar{x}) \leqslant \max_{k=1,\ldots,p} f_k(\hat{x}) = \hat{x}_{n+1}.$$

In continuation, we aim to characterize approximate efficient solutions of MOP (1) through problem (3). The following theorem illustrates the connection between ε-(weakly) efficient solutions of MOP (1) and the ϵ-optimal solutions of problem (3).

Theorem 2 *Let $\epsilon \geqslant 0$, and $\varepsilon \geqq 0$. Then, for problem (3) we have the following results.*

(1) If $(\hat{x}, \hat{x}_{n+1})$ is an ϵ-optimal solution of problem (3) with $\epsilon \leqslant \min_{i=1,\dots,p} \varepsilon_i$, then $\hat{x} \in X_{\varepsilon wE}$.
(2) If $(\hat{x}, \hat{x}_{n+1})$ is the unique ϵ-optimal solution of problem (3), then $\hat{x} \in X_{\varepsilon E}$.

Proof (1) Suppose $\hat{x}$ is not an ε-weakly efficient solution of MOP (1). Then, there exists $x \in X$ such that $f_i(x) < f_i(\hat{x}) - \varepsilon_i$ for all $1 \leqslant i \leqslant p$. Therefore, for all i, we can find $v_i > 0$ such that $f_i(x) + v_i = f_i(\hat{x}) - \varepsilon_i$. Define $\delta = \min_{i=1,\dots,p} v_i$. Thus,

$$f_i(x) + \delta \leqslant f_i(\hat{x}) - \varepsilon_i \leqslant \hat{x}_{n+1} - \epsilon,$$

for all $1 \leqslant i \leqslant p$. Therefore, $(x, \hat{x}_{n+1} - \delta - \epsilon)$ is a feasible solution of the problem (3) such that

$$\hat{x}_{n+1} - \delta - \epsilon < \hat{x}_{n+1} - \epsilon.$$

This contradicts the ϵ-optimality of $(\hat{x}, \hat{x}_{n+1})$.
(2) For the second part, let us assume there exists an $x \in X$ such that

$$f_i(x) \leqslant f_i(\hat{x}) - \varepsilon_i \leqslant \hat{x}_{n+1},$$

for all $1 \leqslant i \leqslant p$ and some $j \in \{1, \dots, p\}$, we have

$$f_j(x) < f_j(\hat{x}) - \varepsilon_j \leqslant \hat{x}_{n+1}.$$

Therefore, $(x, \hat{x}_{n+1})$ is a feasible solution to problem (3) with the same objective function value as $(\hat{x}, \hat{x}_{n+1})$. the uniqueness of the ϵ-optimal solution demonstrates that $x = \hat{x}$.

In the next section, we will formulate a new version of the scalarization problem (3) that aims to characterize ε-weakly, ε-properly, and ε-efficient solutions of MOP (1).

3 ε-(weakly and properly) efficient solutions

In this section, using slack and surplus variables, an extension of the scalarization problem (3) is proposed as follows

$$\begin{aligned} &\min\ x_{n+1} - \sum_{i=1}^{p} \mu_i s_i^+ + \sum_{i=1}^{p} \gamma_i s_i^- \\ &s.t.\ \ f_i(x) + s_i^+ - s_i^- \leqslant x_{n+1}, \qquad i = 1, \dots, p, \\ &\qquad x \in X,\ x_{n+1} \in \mathbb{R},\ s^+,\ s^- \geqq 0, \end{aligned} \tag{6}$$

where μ_i and γ_i for $1 \leqslant i \leqslant p$, are non-negative weights. According to the next lemma, we need to assume that $\gamma - \mu \geqq 0$.

Lemma 2 *Suppose there exists $1 \leqslant i \leqslant p$ such that $\gamma_i - \mu_i < 0$. Then problem (6) is unbounded, otherwise there exists a partition $I \cup \bar{I}$ of $\{1, \ldots, p\}$ such that $s_i^+ = 0$ for all $i \in I$, and $s_i^- = 0$ for all $i \in \bar{I}$.*

Proof The proof follows a similar approach to the proof of Lemma 5.1 in [6].

The following theorem establishes sufficient conditions for the ε-weakly efficient solution of MOP (1).

Theorem 3 *Assume that $\epsilon \geqslant 0$, and $\varepsilon \geqq 0$. Then for the scalarized model (6), we have the following statements*

(1) If $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$ is an ϵ-optimal solution of problem (6) with $\epsilon \leqslant \min\limits_{i=1,\ldots,p} \varepsilon_i$, then $\hat{x}$ is an ε-weakly efficient solution of MOP (1).
(2) If $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$ is an ϵ-optimal solution of problem (6) with $\epsilon \leqslant \sum\limits_{i=1}^{p} \mu_i \varepsilon_i$ and $\mu \geq 0$, then $\hat{x}$ is an ε-weakly efficient solution of MOP (1).
(3) If $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$ is an ϵ-optimal solution of problem (6) with $\epsilon \leqslant \sum\limits_{i=1}^{p} \gamma_i \varepsilon_i$, $\varepsilon \leqq \hat{s}^-$ and $\gamma \geq 0$, then $\hat{x}$ is an ε-weakly efficient solution of MOP (1).

Proof (1) The proof of this part follows the first part of Theorem 2.
(2) If $\hat{x}$ is not an ε-weakly efficient solution of MOP (1), then there exists an $x \in X$ such that $f_i(x) + v_i < f_i(\hat{x}) - \varepsilon_i$, where $v_i > 0$ for all $1 \leqslant i \leqslant p$. Thus, we have

$$f_i(x) + \hat{s}_i^+ - \hat{s}_i^- + v_i + \varepsilon_i < f_i(\hat{x}) + \hat{s}_i^+ - \hat{s}_i^- \leqslant \hat{x}_{n+1},$$

for all i. Setting $s_i^+ = \hat{s}_i^+ + v_i + \varepsilon_i$ for all $1 \leqslant i \leqslant p$. Therefore, $(x, \hat{x}_{n+1}, s^+, \hat{s}^-)$ is a feasible point of problem (6) such that

$$\begin{aligned}\hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i s_i^+ + \sum_{i=1}^{p} \gamma_i \hat{s}_i^- &= \hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i \hat{s}_i^+ + \sum_{i=1}^{p} \gamma_i \hat{s}_i^- - \sum_{i=1}^{p} \mu_i v_i - \sum_{i=1}^{p} \mu_i \varepsilon_i \\ &< \hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i \hat{s}_i^+ + \sum_{i=1}^{p} \gamma_i \hat{s}_i^- - \epsilon.\end{aligned}$$

This contradicts the fact that $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$ is an ϵ-optimal solution of problem (6). The proof for part (3) follows an analogous approach to that of part (2).

The following theorem provides sufficient conditions for the ε-efficient solution of MOP (1).

Theorem 4 *Suppose that $\epsilon \geqslant 0$, and $\varepsilon \geqq 0$. Then, the subsequent statements are valid.*

(1) If $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$ is the unique ϵ-optimal solution of problem (6), then $\hat{x}$ is an ε-efficient solution of MOP (1).

(2) If $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$ is an ϵ-optimal solution of problem (6) with $\mu > 0$ and $\epsilon \leqslant \sum_{i=1}^{p} \mu_i \varepsilon_i$, then $\hat{x}$ is an ε-efficient solution of MOP (1).

(3) If $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$ is an ϵ-optimal solution of problem (6) with $\gamma > 0,\ \varepsilon < \hat{s}^-$ and $\epsilon \leqslant \sum_{i=1}^{p} \gamma_i \varepsilon_i$, then $\hat{x}$ is an ε-efficient solution of MOP (1).

Proof (1) The proof is similar in spirit to the second part of Theorem 2.
(2) Assume that $\hat{x}$ is not an ε-efficient solution of MOP (1), then there exists an $x \in X$ such that $f_i(x) \leqslant f_i(\hat{x}) - \varepsilon_i$ for all $1 \leqslant i \leqslant p$, and $f_j(x) < f_j(\hat{x}) - \varepsilon_j$ for some $j \in \{1, \ldots, p\}$. Consequently,

$$f_i(x) + \hat{s}_i^+ - \hat{s}_i^- \leqslant f_i(\hat{x}) + \hat{s}_i^+ - \hat{s}_i^- - \varepsilon_i \leqslant \hat{x}_{n+1}, \quad 1 \leqslant i \leqslant p,$$

and

$$f_j(x) + \hat{s}_j^+ - \hat{s}_j^- < f_j(\hat{x}) + \hat{s}_j^+ - \hat{s}_j^- - \varepsilon_j \leqslant \hat{x}_{n+1},$$

for some j. Choosing $v_j > 0$ such that $v_j \leqslant f_j(\hat{x}) - f_j(x) - \varepsilon_j$ gives $f_j(x) + \hat{s}_j^+ - \hat{s}_j^- + v_j + \varepsilon_j \leqslant \hat{x}_{n+1}$. Putting

$$s_i^+ = \begin{cases} \hat{s}_i^+ + \varepsilon_i, & i \neq j, \\ \hat{s}_i^+ + v_i + \varepsilon_i, & i = j. \end{cases}$$

Thus, $(x, \hat{x}_{n+1}, s^+, \hat{s}^-)$ is a feasible point of problem (6). Furthermore,

$$\begin{aligned} \hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i s_i^+ + \sum_{i=1}^{p} \gamma_i \hat{s}_i^- &= \hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i \hat{s}_i^+ - \sum_{i=1}^{p} \mu_i \varepsilon_i - \mu_j v_j + \sum_{i=1}^{p} \gamma_i \hat{s}_i^- \\ &< \hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i \hat{s}_i^+ + \sum_{i=1}^{p} \gamma_i \hat{s}_i^- - \epsilon. \end{aligned}$$

This leads to a contradiction with the ϵ-optimality of $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$.
(3) By selecting $v_j > 0$ such that $v_j \leqslant \min\{f_j(\hat{x}) - f_j(x) - \varepsilon_j, \hat{s}_j - \varepsilon_j\}$, and defining the new variables

$$s_i^- = \begin{cases} \hat{s}_i^- - \varepsilon_i, & i \neq j, \\ \hat{s}_i - v_i - \varepsilon_i, & i = j, \end{cases}$$

we conclude that $(x, \hat{x}_{n+1}, \hat{s}^+, s^-)$ is a feasible point of problem (6). Moreover,

$$\hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i \hat{s}_i^+ + \sum_{i=1}^{p} \gamma_i s_i^- < \hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i \hat{s}_i^+ + \sum_{i=1}^{p} \gamma_i \hat{s}_i^- - \epsilon.$$

This contradicts the ϵ-optimality of $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$, leading to a contradiction.

Our later theorem provides more sufficient conditions for the ε-efficient solution of MOP (1).

Theorem 5 *Assume that $\epsilon \geqslant 0$, and $\varepsilon \geqq 0$. If $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$ is an ϵ-optimal solution of problem (6) and also*

(1) $\epsilon < \sum_{i=1}^{p} \mu_i \varepsilon_i$, then $\hat{x}$ is an ε-efficient solution of MOP (1).

(2) $\epsilon < \sum_{i=1}^{p} \gamma_i \varepsilon_i$ and $\varepsilon < \hat{s}^-$, then $\hat{x}$ is an ε-efficient solution of MOP (1).

Proof The proof follows a similar structure to the proof of Theorem 4.

In the following, under the assumptions cited in Theorem 4 (parts (2) and (3)), we prove that $\hat{x}$ is an ε-properly efficient solution of the MOP (1).

Theorem 6 *Let $\epsilon \geqslant 0$, and $\varepsilon \geqq 0$. If $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$ is an ϵ-optimal solution of problem (6) such that*

(1) $\mu > 0$ and $\epsilon \leqslant \sum_{i=1}^{p} \mu_i \varepsilon_i$, then $\hat{x}$ is an ε-properly efficient solution of MOP (1).

(2) $\gamma > 0$, $\varepsilon \leqq \hat{s}^-$ and $\epsilon \leqslant \sum_{i=1}^{p} \gamma_i \varepsilon_i$, then $\hat{x}$ is an ε-properly efficient solution of MOP (1).

Proof We provide the proof of the second part, proof of the first part is similar and will be omitted.
(2) Based on the second part of Theorem 4, $\hat{x}$ is an ε-efficient solution of the MOP (1). Assume that $\hat{x} \notin X_{\varepsilon pE}$. Thus, for all $M > 0$ there exists $l \in \{1, \ldots, p\}$ and $x \in X$ with $f_l(x) < f_l(\hat{x}) - \varepsilon_l$ such that

$$\frac{f_l(\hat{x}) - f_l(x) - \varepsilon_l}{f_j(x) - f_j(\hat{x}) + \varepsilon_j} > M, \tag{7}$$

for all j with $f_j(x) > f_j(\hat{x}) - \varepsilon_j$. For the index l, we have

$$f_l(x) + v = f_l(\hat{x}) - \varepsilon_l, \tag{8}$$

where $v > 0$. In this way,

$$f_l(x) + \hat{s}_l^+ + v - \hat{s}^- + \varepsilon_l = f_l(\hat{x}) + \hat{s}_l^+ - \hat{s}_l^- \leqslant \hat{x}_{n+1}.$$

Define $J = \{1 \leqslant j \leqslant p \mid f_j(x) > f_j(\hat{x}) - \varepsilon_j\}$, and let $M > 0$ such that $\sum_{i \in J} \gamma_i < \mu_l M$. From (7) and (8) we see that for all $j \in J$,

$$f_j(x) < f_j(\hat{x}) + \frac{v}{M} - \varepsilon_j.$$

Since $(x, \hat{x}_{n+1}, s^+, s^-)$ is a feasible point of problem (6), we acquire

$$\begin{aligned} f_j(x) + \hat{s}_j^+ - \hat{s}_j^- &< f_j(\hat{x}) + \hat{s}_j^+ - \hat{s}_j^- + \frac{v}{M} - \varepsilon_j \\ &\leqslant \hat{x}_{n+1} + \frac{v}{M} - \varepsilon_j, \end{aligned}$$

for all $j \in J$. Therefore,

$$f_j(x) + \hat{s}_j^+ - \hat{s}_j^- + \varepsilon_j - \frac{v}{M} \leqslant \hat{x}_{n+1}, \qquad \forall j \in J.$$

On the other hand, if $i \notin J \cup \{l\}$, then $f_i(x) \leqslant f_i(\hat{x}) - \varepsilon_i$. From this, by the feasibility of the point $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$, we infer that

$$f_i(x) + \hat{s}_i^+ - \hat{s}_i^- + \varepsilon_i \leqslant f_i(\hat{x}) + \hat{s}_i^+ - \hat{s}_i^- \leqslant \hat{x}_{n+1}, \qquad \forall i \notin J \cup \{l\}.$$

Define

$$s_i^+ = \begin{cases} \hat{s}_i^+ + v, & i = l, \\ \hat{s}_i^+, & i \neq l, \end{cases}$$

and

$$s_i^- = \begin{cases} \hat{s}_i^- - \varepsilon_i, & i \notin J, \\ \hat{s}_i^- - \varepsilon_i + \frac{v}{M}, & i \in J. \end{cases}$$

Thus, $(x, \hat{x}_{n+1}, s^+, s^-)$ is a feasible point of problem (6) and also

$$\begin{aligned} &\hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i s_i^+ + \sum_{i=1}^{p} \gamma_i s_i^- \\ &= \hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i \hat{s}_i^+ + \sum_{i=1}^{p} \gamma_i \hat{s}_i^- + \sum_{i \in J} \gamma_i \frac{v}{M} - \mu_l v - \sum_{i=1}^{p} \gamma_i \varepsilon_i \\ &< \hat{x}_{n+1} - \sum_{i=1}^{p} \mu_i \hat{s}_i^+ + \sum_{i=1}^{p} \gamma_i \hat{s}_i{}^- - \epsilon. \end{aligned}$$

The last inequality holds based on the assumptions $\sum_{i \in J} \gamma_i \frac{v}{M} - \mu_l v < 0$ and $\epsilon \leqslant \sum_{i=1}^{p} \gamma_i \varepsilon_i$. This is contrary to the ϵ-optimality of $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$.

Under the hypotheses of Theorem 5, we assert that $\hat{x}$ is an ε-properly efficient solution of the MOP (1).

Theorem 7 *Let $\epsilon \geqslant 0$, and $\varepsilon \geqq 0$. Assume that $(\hat{x}, \hat{x}_{n+1}, \hat{s}^+, \hat{s}^-)$ is an ϵ-optimal solution of problem (6). If*

(1) $\epsilon < \sum_{i=1}^{p} \mu_i \varepsilon_i$, then $\hat{x}$ is an ε-properly efficient solution of the MOP (1).

(2) $\epsilon < \sum_{i=1}^{p} \gamma_i \varepsilon_i$ *and* $\varepsilon \leqq \hat{s}^-$, *then* $\hat{x}$ *is an* ε*-properly efficient solution of the MOP (1).*

Proof The proof is similar to that of Theorem 6.

Table 1 Summary of results for an ϵ-optimal solution of problem (6).

Condition on parameters	Implication for $\hat{x}$	Reference
$\epsilon \leqslant \min_{i=1,\ldots,p} \varepsilon_i$	$\hat{x} \in X_{\varepsilon wE}$	Theorem 3(1)
$\epsilon \leqslant \sum_{i=1}^{p} \mu_i \varepsilon_i,\ \mu \geq 0$	$\hat{x} \in X_{\varepsilon wE}$	Theorem 3(2)
$\epsilon \leqslant \sum_{i=1}^{p} \gamma_i \varepsilon_i,\ \varepsilon \leqq \hat{s}^-,\ \gamma \geq 0$	$\hat{x} \in X_{\varepsilon wE}$	Theorem 3(3)
Unique ϵ-optimal solution	$\hat{x} \in X_{\varepsilon E}$	Theorem 4(1)
$\epsilon \leqslant \sum_{i=1}^{p} \mu_i \varepsilon_i,\ \mu > 0$	$\hat{x} \in X_{\varepsilon E}$	Theorem 4(2)
$\epsilon \leqslant \sum_{i=1}^{p} \gamma_i \varepsilon_i,\ \varepsilon < \hat{s}^-,\ \ \gamma > 0$	$\hat{x} \in X_{\varepsilon E}$	Theorem 4 (3)
$0 < \epsilon < \sum_{i=1}^{p} \mu_i \varepsilon_i$	$\hat{x} \in X_{\varepsilon E}$	Theorem 5 (1)
$0 < \epsilon < \sum_{i=1}^{p} \gamma_i \varepsilon_i,\ \varepsilon \leqq \hat{s}^-$	$\hat{x} \in X_{\varepsilon E}$	Theorem 5(2)
$\epsilon \leqslant \sum_{i=1}^{p} \mu_i \varepsilon_i,\ \mu > 0$	$\hat{x} \in X_{\varepsilon pE}$	Theorem 6(1)
$\epsilon \leqslant \sum_{i=1}^{p} \gamma_i \varepsilon_i,\ \varepsilon < \hat{s}^-,\ \ \gamma > 0$	$\hat{x} \in X_{\varepsilon pE}$	Theorem 6(2)
$0 < \epsilon < \sum_{i=1}^{p} \mu_i \varepsilon_i$	$\hat{x} \in X_{\varepsilon pE}$	Theorem 7(1)
$0 < \epsilon < \sum_{i=1}^{p} \gamma_i \varepsilon_i,\ \varepsilon \leqq \hat{s}^-$	$\hat{x} \in X_{\varepsilon pE}$	Theorem 7(2)

4 Conclusion and future works

In this research, we considered a novel variant of the min-max method to find approximate efficient solutions in the context of MOPs. Specifically, we investigate the relationship between the optimal solution of the min-max method and that of the novel variant. We then introduced an extension of this method that incorporates flexible constraints, enabling us to study conditions that illustrate a deep relationship between ε-(weakly and properly) efficient solutions within the MOP and ϵ-optimal solutions in the extensive scalarization problem. In Table 1, we summarize some results obtained from the previous sections for the proposed model. As a future research plan, we intend to focus on investigating ε-(weakly and properly) efficient solutions using Pascoletti–Serafini scalarization technique.

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