Research article

Quadrature Rules for Solving Two-Dimensional Fredholm Integral Equations of Second Kind

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Abstract

In this paper, an iterative method of successive approximations based on the trapezoidal quadrature rule to solve two-dimensional Fredholm integral equations of second kind (2DFIE) is proposed. The error estimation of the proposed method is presented. The benefit of the method is that we do not have to solve a system of algebraic equations. Finally, a numerical example verify the theoretical results and show the accuracy of the method.

Keywords: Integral equations, Trapezoidal quadrature, Uniform modulus of continuity, Iterative method

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1 Introduction

The integral equations provide important tools for modeling a wide range of phenomena and processes [14], and many problems in engineering and physics give rise to two-dimensional integral equations [8, 13, 16]. There are many numerical methods for solving integral equations. The Galerkin and collocation methods are two commonly used methods for the numerical solutions of these equations [9, 10]. Several numerical methods for approximating the solutions of integral equations were presented. Here, we recall some published works on this subject. These include Gauss product quadrature rule [6], polynomial interpolation methods [24], discrete Galerkin and iterated discrete Galerkin methods [12], triangular functions method [15], Legender polynomial method [22], Nystrom method [11], meshless method [1], Haar wavelet method [4]. Analytic methods, analytic-numeric methods like Adomian decomposition, homotopy perturbation method and regularization-homotopy method have been studied by many authors [17]. The use of successive approximations method in such cases can be therefore useful [18–20]. In this paper, we introduce an iterative method based on 2D trapezoidal quadrature rule for solving 2DFIE as

$$F(s,t) = f(s,t) + \lambda \int_c^d \int_a^b K(s,t,x,y)F(x,y)dxdy,$$

where $(s,t) \in \Omega = [a,b] \times [c,d]$ and $f(s,t)$, $K(s,t,x,y)$ are the given analytical functions. In the most of numerical methods, the integral equation is transformed into a system of linear or nonlinear algebraic equations which has to be solved with iterative methods. It is cumbersome to solve these systems, or the solution may be unreliable. The proposed method does not lead to a nonlinear algebraic

equations system. This is a great advantage of this method. The rest of the paper is organized as follows: we begin by introducing some necessary definitions and mathematical preliminaries of the some quadrature rules for 2-D integral in Section 2. Section 3, devoted to prove of the existence and uniqueness of the solution of 2DFIE by the method of successive approximations. Also, a conclusion is given in Section 4.

2 Preliminaries

Definition 1. *Suppose that $f : \Omega \to \mathbb{R}$, be a bounded mapping, then the function $\omega_\Omega(f,.) : \mathbb{R}^+ \cup 0 \to \mathbb{R}^+$ defined by*

$$\omega_\Omega(f,\delta) = \sup_{x,s \in [a,b]; y,t \in [c,d]} \{|f(x,y) - f(s,t)|; \sqrt{(x-s)^2 + (y-t)^2} \le \delta\}, \tag{1}$$

is called the modulus of oscillation of f on Ω.

Also, if $f \in C(\Omega)$ (i.e. $f : \Omega \to \mathbb{R}$ is continuous on Ω, then $\omega_\Omega(f,\delta)$ is called uniform modulus of continuity of f.

Theorem 1. *The following properties holds [23]:*

(i) $|f(x,y) - f(s,t)| \le \omega_{[a,b] \times [c,d]}(f, \sqrt{(x-s)^2 + (y-t)^2})$ *for all $x, s \in [a,b]$ and $y, t \in [c,d]$,*

(ii) $\omega_\Omega(f,\delta)$ *is an non-decreasing mapping in δ,*

(iii) $\omega_\Omega(f,0) = 0$,

(iv) $\omega_\Omega(f,\delta_1 + \delta_2) \le \omega_\Omega(f,\delta_1) + \omega_\Omega(f,\delta_2)$ *for any $\delta_1, \delta_2 \ge 0$,*

(v) $\omega_\Omega(f,n\delta) \le n\omega_\Omega(f,\delta)$ *for any $\delta \ge 0$ and $n \in \mathbb{N}$,*

(vi) $\omega_\Omega(f,\lambda\delta) \le (\lambda+1)\omega_{[a,b] \times [c,d]}(f,\delta)$ *for any $\delta, \lambda \ge 0$,*

(vii) *If $[a,b] \times [c,d] \subseteq [e,f] \times [g,h]$, then $\omega_{[a,b] \times [c,d]}(f,\delta) \le \omega_{[e,f] \times [g,h]}(f,\delta)$ for all $\delta \ge 0$.*

Theorem 2. *[21] Let $f : [c,d] \times [c,d] \to \mathbb{R}$, be a integrable, bounded mappings. Then, for any divisions*

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and

$$c = y_0 < y_1 < \cdots < y_n = d,$$

and any points $\xi_i \in [x_{i-1}, x_i]$ and $\eta_j \in [y_{j-1}, y_j]$ we have

$$\left| \int_c^d \int_a^b f(s,t)dsdt - \sum_{j=1}^n \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})f(\xi_i, \eta_j) \right| \le \sum_{j=1}^n \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1})$$

$$\times \omega_{[x_{i-1}, x_i] \times [y_{j-1}, y_j]}(f, \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}).$$

Corollary 1. *Assume that $f : [a,b] \times [c,d] \to \mathbb{R}$, be a integrable, bounded mapping. Then with the following notation*

$$\omega_{xy \times zt} = \omega_{[x,y] \times [z,t]}\left(f, \sqrt{(y-x)^2 + (t-z)^2}\right),$$

we have

$$\left| \int_c^d \int_a^b f(s,t)dsdt - \left[(x-a)(y-c)f(u,\alpha) + (x-a)(d-y)f(u,\beta) + (b-x)(d-y)f(c,\beta) + (b-x)(d-y)f(v,\beta) \right] \right|$$

$$\le (x-a)(y-c)\omega_{ax \times cy} + (b-x)(y-c)\omega_{xb \times cy} + (x-a)(d-y)\omega_{ax \times yd} + (b-x)(d-y)\omega_{xb \times yd},$$

For all $x \in [a,b]$, $y \in [c,d]$, $u \in [a,x]$, $v \in [x,b]$, $\alpha \in [c,y]$, and $\beta \in [y,d]$.

Proof. Taking in the previous theorem $n = 2$, $x_1 = \xi_1 = \xi_2 = x$, and $y_1 = \eta_1 = \eta_2 = y$ we obtain the required inequality.   □

Corollary 2. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$, *be a two dimensional integrable, bounded mapping. Then the following inequalities holds:*

$$\left| \int_c^d \int_a^b f(s,t)dsdt - (b-a)(d-c)f(\frac{a+b}{2}, \frac{c+d}{2}) \right| \leq (b-a)(d-c)\omega_{[a,b]\times[c,d]}\left(f, \frac{(b-a)(d-c)}{4} \right).$$

Proof. If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Corollary (1), we obtain the required inequality.   □

3 Existence and Uniqueness of Fredholm Integral Equations

Here, we consider the two dimensional Fredholm integral equations as follows:

$$F(s,t) = f(s,t) + \lambda \int_c^d \int_a^b K(s,t,x,y)F(x,y)dxdy, \tag{2}$$

where $\lambda > 0$, $K(s,t,x,y)$ is an arbitrary kernel on Ω^2 and $f : \Omega \to \mathbb{R}$. We assume that K is continuous and therefore it is uniformly continuous with respect to (s,t). This property implies that there exists $M > 0$ such that

$$M = \max_{\substack{a \leq s, x \leq b \\ c \leq t, y \leq d}} \left| K(s,t,x,y) \right|.$$

Now, we shall prove the existence and uniqueness of the solution of equation (2) by the method of successive approximations.

Let $\mathbf{X} = \{ f : \Omega \to \mathbb{R}; f \text{ is continuous} \}$ be the space of two dimensional continuous functions with the metric

$$\|f - g\| = \sup_{\substack{a \leq s \leq b \\ c \leq t \leq d}} \left| f(s,t) - g(s,t) \right|. \tag{3}$$

We define the operator $A : \mathbf{X} \to \mathbf{X}$ by

$$A(F)(s,t) = f(s,t) + \lambda \int_c^d \int_a^b K(s,t,x,y)F(x,y)dxdy, \ \ \forall (s,t) \in \Omega, \ \ \forall f \in \mathbf{X}. \tag{4}$$

Sufficient conditions for the existence of an unique solution of equation (2) are given in the following result.

Theorem 3. *Let* $K(s,t,x,y)$ *be continuous for* $a \leq s$, $x \leq b$, $c \leq t$, $y \leq d$, *and* $f : \Omega \to \mathbb{R}$ *be continuous on* $[a,b] \times [c,d]$. *If* $\Lambda = \lambda M(b-a)(d-c) < 1$, *then the iterative procedure*

$$F_0(s,t) = f(s,t), \tag{5}$$

$$F_m(s,t) = f(s,t) + \lambda \int_c^d \int_a^b K(s,t,x,y)F_{m-1}(x,y)dxdy, \quad m \geq 1 \tag{6}$$

converges to the unique solution F^* *of equation (2). Moreover, the following error bound holds:*

$$\left\| F^* - F_m \right\| \leq \frac{\Lambda^{m+1}}{1 - \Lambda} \|f\|, \tag{7}$$

where

$$\|f\| = \sup_{\substack{a \leq s \leq b \\ c \leq t \leq d}} |f(s,t)|. \tag{8}$$

Proof. To proof this theorem we investigate the conditions of the Banach's fixed point principle. We first show that A maps $\mathbf{X}$ into $\mathbf{X}$ (i.e. $A(\mathbf{X}) \subset \mathbf{X}$). To the end, we show that the operator A is uniformly continuous. Since f is continuous on compact set of Ω, we deduce that it is uniformly continuous and hence for $\varepsilon_1 > 0$ exists $\delta_1 > 0$ such that

$$\left| f(s_1,t_1) - f(s_2,t_2) \right| < \varepsilon_1 \ \text{ whenever } \ \sqrt{(t_2-t_1)^2 + (s_2-s_1)^2} < \delta_1,$$

for all $s_1, s_2 \in [a,b]$ and $t_1, t_2 \in [c,d]$. As mentioned above, K also is uniformly continuous thus, for $\varepsilon_2 > 0$ exists $\delta_2 > 0$ such that

$$\left|K(s_1,t_1,x,y) - K(s_2,t_2,x,y)\right| < \varepsilon_2 \ \text{ whenever } \ \sqrt{(t_2 - t_1)^2 + (s_2 - s_1)^2} < \delta_2,$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and $\sqrt{(t_2 - t_1)^2 + (s_2 - s_1)^2} < \delta$. We obtain

$$\left|\bigl(A(F)\bigr)(s_1,t_1) - A(F)(s_2,t_2)\right| \leq \left|f(s_1,t_1) - f(s_2,t_2)\right| + \lambda \int_c^d \int_a^b \left|K(s_1,t_1,x,y) - K(s_2,t_2,x,y)\right| |F(x,y)| dxdy$$

$$\leq \varepsilon_1 + \lambda \varepsilon_2 \int_c^d \int_a^b |F(x,y)| dxdy$$

$$\leq \varepsilon_1 + \lambda(b-a)(d-c)\|F\|\varepsilon_2,$$

where

$$\|F\| = \sup_{\substack{a \leq x \leq b \\ c \leq y \leq d}} |F(x,y)|.$$

By choosing $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 = \frac{1}{2M_1\lambda(b-a)(d-c)}\varepsilon$ we derive

$$\left|A(F)(s_1,t_1) - A(F)(s_2,t_2)\right| \leq \varepsilon.$$

This shows that $A(F)$ is uniformly continuous for any $F \in \mathbf{X}$, and so continuous on Ω, and hence $A(\mathbf{X}) \subset \mathbf{X}$.

Now, we prove that the operator A is contraction map. So, for $H_1, H_2 \in \mathbf{X}$ and $s \in [a,b]$ and $t \in [c,d]$, we have

$$\left|A(H_1)(s,t) - A(H_2)(s,t)\right| \leq \lambda \left|K(s,t,x,y)\right| \int_c^d \int_a^b \left|H_1(x,y) - H_2(x,y)\right| dxdy$$

$$\leq \lambda M \int_c^d \int_a^b \left|H_1(x,y) - H_2(x,y)\right| dxdy$$

$$\leq \lambda M \int_c^d \int_a^b \|H_1 - H_2\| dxdy$$

$$= \lambda M(b-a)(d-c)\|H_1 - H_2\|$$

$$= \Lambda \|H_1 - H_2\|.$$

Therefore, we obtain

$$\left\|A(H_1)(s,t) - A(H_2)(s,t)\right\| \leq \Lambda \|H_1 - H_2\|.$$

Since $\Lambda < 1$, the operator A is a contraction. Consequently, the Banach's fixed point principle implies that equation (2) has a unique solution F^* in $\mathbf{X}$ and we also have

$$\left|F^*(s,t) - F_m(s,t)\right| \leq \|F^* - F_m\|$$

$$= \Lambda \|F^* - F_{m-1}\|$$

$$\leq \Lambda \|F^* - F_m\| + \Lambda \|F_{m-1} - F_m\|$$

$$\leq \Lambda \|F^* - F_m\| + \Lambda^m \|F_0 - F_1\|.$$

Therefore,

$$\|F^* - F_m\| \leq \frac{\Lambda^m}{1 - \Lambda} \|F_0 - F_1\|, \tag{9}$$

on the other hand,

$$\|F_0 - F_1\| = \sup_{\substack{a \leq s \leq b \\ c \leq t \leq d}} \left|f(s,t) - f(s,t) - \lambda \int_c^d \int_a^b K(s,t,x,y)F_0(x,y)dxdy\right|$$

$$\leq \sup_{\substack{a \leq s \leq b \\ c \leq t \leq d}} \lambda \int_c^d \int_a^b \left|K(s,t,x,y)F_0(x,y)\right| dxdy$$

$$\leq M\lambda \int_c^d \int_a^b \sup_{\substack{a\leq s\leq b \\ c\leq t\leq d}} |F_0(x,y)|dxdy$$

$$= \lambda M(b-a)(d-c)\|f\| = \|f\|\Lambda. \tag{10}$$

So, by (9) and (10) we obtain inequality (7), which completes the proof. $\qquad\square$

Now, we introduce a numerical method to solve equation (2). we consider equation (2) with continuous kernel $K(s,t,x,y)$ having positive sign on $\Omega \times \Omega$ and uniform partitions

$$D_x : a = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = b,$$

$$D_y : c = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = d,$$

with $s_i = a+ih$, $t_j = c+jh'$, where $h = \frac{b-a}{n}$, $h' = \frac{d-c}{n}$. Then the following iterative procedure gives the approximate solution of equation (2) in point (s,t),

$$u_0(s,t) = f(s,t)$$

$$u_m(s,t) = f(s,t) + \frac{\lambda hh'}{4} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Big(K(s,t,s_i,t_j)u_{m-1}(s_i,t_j) + K(s,t,s_i,t_{j+1})u_{m-1}(s_i,t_{j+1}) + K(s,t,s_{i+1},t_j)u_{m-1}(s_{i+1},t_j)$$

$$+ K(s,t,s_{i+1},t_{j+1})u_{m-1}(s_{i+1},t_{j+1}) \Big). \tag{11}$$

3.1 Error Estimation

Here, we obtain an error estimate between the exact solution and the approximate solution for the given Fredholm integral equation (2).

Theorem 4. *Consider the equation (2) with continuous kernel $K(s,t,x,y)$ on $\Omega \times \Omega$ and suppose that f is continuous on Ω. If $\Lambda < 1$, then the iterative procedure (11) converges to the unique solution of equation (2), F^*, and the following error estimate holds true,*

$$\|F^* - u_m\| \leq \left(\frac{\Lambda^{m+1}}{1-\Lambda}\right)\|f\| + \left(\frac{\Lambda}{4(1-\Lambda)}\right)\omega_{[a,b]\times[c,d]}(f,hh') + \left(\frac{\mu\Lambda^2+4\tau\Lambda}{4M(1-\Lambda)}\right)\omega_{st}(K,h+h'),$$

where

$$\omega_{st}(K,\delta) = \sup_{\substack{s_1,s_2\in[a,b] \\ t_1,t_2\in[c,d]}} \left\{ |K(s_1,t_1,x,y) - K(s_2,t_2,x,y)|; \sqrt{(s_2-s_1)^2 + (t_2-t_1)^2} \leq \delta \right\},$$

and

$$\begin{cases} M_k = \sup_{(s,t)\in\Omega} |u_k(s,t)|, \\ \Gamma_k = \sup_{(s,t)\in\Omega} |F_k(s,t)|, \\ \tau = \max_{i=0,1,\ldots,m-1} \{M_i\}, \\ \mu = \max_{i=0,1,\ldots,m-2} \{\Gamma_i\}. \end{cases} \tag{12}$$

Proof. Considering iterative procedure (11), we obtain

$$|F_1(s,t) - u_1(s,t)| \leq \lambda \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Bigg| \int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}} K(s,t,x,y)f(x,y)dxdy - \frac{hh'}{4}\Big[K(s,t,x,y)f(s_i,t_j) + K(s,t,x,y)f(s_i,t_{j+1})$$

$$+ K(s,t,x,y)f(s_{i+1},t_j) + K(s,t,x,y)f(s_{i+1},t_{j+1})\Big] \Bigg| + \lambda \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Bigg| \frac{hh'}{4}\Big[K(s,t,x,y)f(s_i,t_j) + K(s,t,x,y)f(s_i,t_{j+1})$$

$$+ K(s,t,x,y)f(s_{i+1},t_j) + K(s,t,x,y)f(s_{i+1},t_{j+1})\Big] - \frac{hh'}{4}\Big[K(s,t,s_i,t_j)f(s_i,t_j)) + K(s,t,s_i,t_{j+1})f(s_i,t_{j+1}$$

$$+ K(s,t,s_{i+1},t_j)f(s_{i+1},t_j) + K(s,t,s_{i+1},t_{j+1})f(s_{i+1},t_{j+1})\Big] \Bigg|$$

$$\leq \lambda \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left| \int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}} K(s,t,x,y)f(x,y)dxdy - \frac{hh'}{4}\left[K(s,t,x,y)f(s_i,t_j)+K(s,t,x,y)f(s_i,t_{j+1})\right.\right.$$

$$\left.+K(s,t,x,y)f(s_{i+1},t_j)+K(s,t,x,y)f(s_{i+1},t_{j+1})\right]\Bigg| + \lambda \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left| \frac{hh'}{4}\left[K(s,t,x,y)f(s_i,t_j)+K(s,t,x,y)f(s_i,t_{j+1})\right.\right.$$

$$K(s,t,x,y)f(s_{i+1},t_j)+K(s,t,x,y)f(s_{i+1},t_{j+1})\big] - \frac{hh'}{4}\left[K(s,t,s_i,t_j)f(s_i,t_j)+K(s,t,s_i,t_{j+1})f(s_i,t_{j+1})\right.$$

$$\left.+K(s,t,s_{i+1},t_j)f(s_{i+1},t_j)+K(s,t,s_{i+1},t_{j+1})f(s_{i+1},t_{j+1})\right]\Bigg|$$

$$\leq \lambda M \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left| \int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}} f(x,y)dxdy - \frac{hh'}{4}\left[f(s_i,t_j)+f(s_i,t_{j+1})+f(s_{i+1},t_j)+f(s_{i+1},t_{j+1})\right]\right|$$

$$+\frac{\lambda hh'}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left[\left|K(s,t,x,y)f(s_i,t_j)-K(s,t,s_i,t_j)f(s_i,t_j)\right|+\left|K(s,t,x,y)f(s_i,t_{j+1})-K(s,t,s_i,t_{j+1})f(s_i,t_{j+1})\right|\right.$$

$$\left.+\left|K(s,t,x,y)f(s_{i+1},t_j)-K(s,t,s_{i+1},t_j)f(s_{i+1},t_j)\right|+\left|K(s,t,x,y)f(s_{i+1},t_{j+1})-K(s,t,s_{i+1},t_{j+1})f(s_{i+1},t_{j+1})\right|\right].$$

Using Corollary 2 and part (vii) of Theorem 1 we deduce

$$\left|F_1(s,t)-u_1(s,t)\right| \leq \frac{\lambda M hh'}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left(4\omega_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}(f,\frac{hh'}{4})\right) + \frac{\lambda hh'}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left[\left|K(s,t,x,y)-K(s,t,s_i,t_j)\right|\left|f(s_i,t_j)\right|\right.$$

$$+\left|K(s,t,x,y)-K(s,t,s_i,t_{j+1})\right|\left|f(s_i,t_{j+1})\right|+\left|K(s,t,x,y)-K(s,t,s_{i+1},t_j)\right|\left|f(s_{i+1},t_j)\right|$$

$$\left.+\left|K(s,t,x,y)-K(s,t,s_{i+1},t_{j+1})\right|\left|f(s_{i+1},t_{j+1})\right|\right].$$

By part (ii) of Theorem 1 and direct computation, it follows that

$$\left|F_1(s,t)-u_1(s,t)\right| \leq \frac{\lambda M(b-a)(d-c)}{4}\omega(f,hh')+\lambda hh'\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left(\left|K(s,t,x,y)-K(s,t,s_i,t_j)\right|\right)\|f(s,t)\|$$

$$\leq \frac{\lambda M(b-a)(d-c)}{4}\omega(f,hh')+\lambda(b-a)(d-c)M_0\omega_{st}(K,h+h')$$

$$=\frac{\Lambda}{4}\omega(f,hh')+\frac{\Lambda}{M}M_0\omega_{st}(K,h+h'),$$

therefore we obtain

$$\left|F_1(s,t)-u_1(s,t)\right| \leq \frac{\Lambda}{4}\omega(f,hh')+\frac{\Lambda}{M}M_0\omega_{st}(K,h+h'). \tag{13}$$

Now, we have

$$\left|F_2(s,t)-u_2(s,t)\right| \leq \frac{\lambda M(b-a)(d-c)}{4}\omega_{[a,b]\times[c,d]}(F_1,hh')+\frac{\lambda M(b-a)(d-c)}{4}\left[\left|F_1(s_i,t_j)-u_1(s_i,t_j)\right|\right.$$

$$+\left|F_1(s_i,t_{j+1})-u_1(s_i,t_{j+1})\right|+\left|F_1(s_{i+1},t_j)-u_1(s_{i+1},t_j)\right|$$

$$\left.+\left|F_1(s_{i+1},t_{j+1})-u_1(s_{i+1},t_{j+1})\right|\right]+\lambda(b-a)(d-c)M_1\omega_{st}(K,h+h').$$

Therefore,

$$\left|F_2(s,t)-u_2(s,t)\right| \leq \frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_1,hh')+\frac{\Lambda}{4}\left[\left|F_1(s_i,t_j)-u_1(s_i,t_j)\right|+\left|F_1(s_i,t_{j+1})-u_1(s_i,t_{j+1})\right|+\left|F_1(s_{i+1},t_j)-u_1(s_{i+1},t_j)\right|\right.$$

$$\left.+\left|F_1(s_{i+1},t_{j+1})-u_1(s_{i+1},t_{j+1})\right|\right)\Bigg]+\lambda(b-a)(d-c)M_1\omega_{st}(K,h+h').$$

By induction for $m \geq 3$, using (5), (6), (7), (11), and (12), we see that

$$\left|F_m(s,t)-u_m(s,t)\right| \leq \frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_{m-1},hh')+\frac{\Lambda}{4}\left[\left|F_{m-1}(s_i,t_j)-u_{m-1}(s_i,t_j)\right|+\left|F_{m-1}(s_i,t_{j+1})-u_{m-1}(s_i,t_{j+1})\right|\right.$$

$$+ \left|F_{m-1}(s_{i+1},t_j)-u_{m-1}(s_{i+1},t_j)\right|+\left|F_{m-1}(s_{i+1},t_{j+1})-u_{m-1}(s_{i+1},t_{j+1})\right|\Big] + \frac{\Lambda}{M}M_{m-1}\omega_{st}(K,h+h'). \tag{14}$$

Taking supremum for $(t,s)\in\Omega$ from (14) we conclude that the following inequalites hold

$$\left\|F_m-u_m\right\|\leq\frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_{m-1},hh')+\Lambda\left\|F_{m-1}-u_{m-1}\right\|+\frac{\Lambda}{M}M_{m-1}\omega_{st}(K,h+h'),$$

$$\left\|F_{m-1}-u_{m-1}\right\|\leq\frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_{m-2},hh')+\Lambda\left\|F_{m-2}-u_{m-2}\right\|+\frac{\Lambda}{M}M_{m-2}\omega_{st}(K,h+h'),$$

$$\left\|F_{m-2}-u_{m-2}\right\|\leq\frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_{m-3},hh')+\Lambda\left\|F_{m-3}-u_{m-3}\right\|+\frac{\Lambda}{M}M_{m-3}\omega_{st}(K,h+h'),$$

$$\vdots$$

$$\left\|F_1-u_1\right\|\leq\frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_0,hh')+\Lambda\left\|F_0-u_0\right\|+\frac{\Lambda}{M}M_0\omega_{st}(K,h+h'),$$

multiplying the above inequalities by $1,\Lambda,\Lambda^2,\ldots,\Lambda^{m-1}$, respectively and summing them we obtain

$$\left\|F_m-u_m\right\|\leq\frac{\Lambda}{4}\Big(\omega_{[a,b]\times[c,d]}(F_{m-1},hh')+\Lambda\omega_{[a,b]\times[c,d]}(F_{m-2},hh')+\cdots+\Lambda^{m-1}\omega_{[a,b]\times[c,d]}(f,hh')\Big)$$
$$+\frac{\Lambda}{M}\omega_{st}(K,h+h')\Big(M_{m-1}+\Lambda M_{m-2}+\Lambda^2 M_{m-3}+\cdots+\Lambda^{m-1}M_0\Big). \tag{15}$$

Since, for $(s_1,t_1),(s_2,t_2)\in\Omega$ with $|s_1-s_2|\leq h,|t_1-t_2|\leq h'$, we have

$$\left|F_m(s_1,t_1)-F_m(s_2,t_2)\right|=\left|f(s_1,t_1)+\lambda\int_c^d\int_a^b K(s_1,t_1,x,y)F_{m-1}(x,y)dxdy-f(s_2,t_2)+\lambda\int_c^d\int_a^b K(s_2,t_2,x,y)F_{m-1}(x,y)dxdy\right|$$
$$\leq\left|f(s_1,t_1)-f(s_2,t_2)\right|+\frac{\Lambda}{M}\omega_{st}(K,h+h')\Gamma_{m-1},$$

therefore, we infer

$$\omega_{[a,b]\times[c,d]}(F_m,hh')\leq\omega_{[a,b]\times[c,d]}(f,hh')+\frac{\Lambda}{M}\omega_{st}(K,h+h')\Gamma_{m-1}. \tag{16}$$

By this inequality and (15), we see that

$$\left\|F_m-u_m\right\|\leq\frac{\Lambda}{4}\Big(1+\Lambda+\Lambda^2+\cdots+\Lambda^{m-1}\Big)\omega_{[a,b]\times[c,d]}(f,hh')+\frac{\Lambda}{4M}\omega_{st}(K,h+h')\Big(\Lambda\Gamma_{m-2}+\Lambda^2\Gamma_{m-3}+\cdots+\Lambda^{m-1}\Gamma_0\Big)$$
$$+\frac{\Lambda}{M}\omega_{st}(K,h+h')\Big(M_{m-1}+\Lambda M_{m-2}+\Lambda^2 M_{m-3}+\cdots+\Lambda^{m-1}M_0\Big)$$
$$=\frac{\Lambda}{4}\left(\frac{1-\Lambda^m}{1-\Lambda}\right)\omega_{[a,b]\times[c,d]}(f,hh')$$
$$+\frac{\Lambda}{4M}\omega_{st}(K,h+h')\Big[\Big(\Lambda\Gamma_{m-2}+\Lambda^2\Gamma_{m-3}+\cdots+\Lambda^{m-1}\Gamma_0\Big)+4\Big(M_{m-1}+\Lambda M_{m-2}+\Lambda^2 M_{m-3}+\cdots+\Lambda^{m-1}M_0\Big)\Big].$$

By (12) since $\Lambda<1$ we obtain

$$\left\|F_m-u_m\right\|\leq\frac{\Lambda}{4}\left(\frac{1-\Lambda^m}{1-\Lambda}\right)\omega_{[a,b]\times[c,d]}(f,hh')+\frac{\Lambda}{4M}\omega_{st}(K,h+h')\left(\frac{\Lambda(1-\Lambda^m)}{1-\Lambda}\mu+\frac{4(1-\Lambda^m)}{1-\Lambda}\tau\right)$$
$$\leq\frac{\Lambda}{4(1-\Lambda)}\omega_{[a,b]\times[c,d]}(f,hh')+\frac{\Lambda}{4M}\omega_{st}(K,h+h')\left(\frac{\mu\Lambda+4\tau}{1-\Lambda}\right).$$

Therefore, we obtain

$$\left\|F_m-u_m\right\|\leq\left(\frac{\Lambda}{4(1-\Lambda)}\right)\omega_{[a,b]\times[c,d]}(f,hh')+\left(\frac{\mu\Lambda^2+4\tau\Lambda}{4M(1-\Lambda)}\right)\omega_{st}(K,h+h'). \tag{17}$$

By inequalities (17) and (7) we deduce that

$$\|F^* - u_m\| \leq \|F^* - F_m\| + \|F_m - u_m\|$$

$$\leq \left(\frac{\Lambda^{m+1}}{1-\Lambda}\right)\|f\| + \left(\frac{\Lambda}{4(1-\Lambda)}\right)\omega_{[a,b]\times[c,d]}(f, hh') + \left(\frac{\mu\Lambda^2 + 4\tau\Lambda}{4M(1-\Lambda)}\right)\omega_{st}(K, h+h').$$

$\square$

Remark 1. *Since $\Lambda < 1$, it is easy to see that*

$$\lim_{\substack{m\to\infty \\ h,h'\to 0}} \|F^* - u_m\| = 0,$$

that shows the convergence of the method.

Example 1. *Consider the following two dimensional Fredholm integral equation*

$$F(s,t) = f(s,t) + \int_0^1 \int_0^1 K(s,t,x,y)F(x,y)dxdy, \tag{18}$$

where

$$f(s,t) = s\sin\frac{t}{2}, \qquad\qquad K(s,t,x,y) = s^2 tx,$$

with the exact solution

$$F(s,t) = s\sin\frac{t}{2} - \frac{16}{21}\left(\cos\frac{1}{2} - 1\right)s^2 t.$$

By using the proposed method, we can present the approximate solution for this example. To compare the numerical results with the exact solution for different values of m, n, see Table 1 .

Table 1. Numerical results of Example 1.

(s,t)	Exact	$m = 3, n = 10$ $\|F - u_m\|$	$m = 3, n = 50$ $\|F - u_m\|$	$m = 6, n = 10$ $\|F - u_m\|$	$m = 6, n = 50$ $\|F - u_m\|$
(0.2,0.2)	0.0207	4.456×10^{-7}	3.016×10^{-10}	2.362×10^{-7}	2.092×10^{-10}
(0.4,0.4)	0.0854	3.565×10^{-6}	2.500×10^{-9}	1.889×10^{-6}	1.674×10^{-9}
(0.6,0.6)	0.1974	1.203×10^{-5}	1.916×10^{-8}	1.015×10^{-5}	1.248×10^{-8}
(0.8,0.8)	0.3593	2.852×10^{-5}	2.811×10^{-8}	2.111×10^{-5}	2.339×10^{-8}
(1.0,1.0)	0.5727	6.570×10^{-5}	3.299×10^{-8}	4.252×10^{-5}	3.014×10^{-8}

4 Conclusions

To approximate the solution of 2DFIE of Fredholm type, we used an efficient iterative algorithm, based on the method of successive approximations. In the present paper, using an iterative method based on 2D Trapezoidal quadrature rule we have approximated the numerical solution of two-dimensional Fredholm integral equations. We established the theorem of existence of unique solution of these equations, and we have proved it by using Banach's fixed point principle. Moreover, the proof of convergence of quadrature formula is discussed in Theorem 4.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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References

[1] P. Assari, H. Adibi, M. Dehghan. A meshless method for solving nonlinear two-dimensional integral equations of the second kind on non-rectangular domains using radial basis functions with error analysis, Journal of Computational and Applied Mathematics, 239, 72–92 (2013).

[2] K. Atkinson. A survey of numerical methods for the solution of fredholm integral equations of the second kind, 1976.

[3] K. E. Atkinson. The numerical solution of integral equations of the second kind, volume 4 of Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 1997.

[4] I. Aziz, F. Khan, et al. A new method based on haar wavelet for the numerical solution of two-dimensional nonlinear integral equations, Journal of Computational and Applied Mathematics, 272, 70–80 (2014).

[5] C. T. Baker. The numerical treatment of integral equations, 1977.

[6] S. Bazm and E. Babolian. Numerical solution of nonlinear two-dimensional fredholm integral equations of the second kind using gauss product quadrature rules, Communications in Nonlinear Science and Numerical Simulation, 17(3), 1215–1223 (2012).

[7] L. M. Delves, J. Mohamed. Computational methods for integral equations, CUP Archive, 1988.

[8] I. G. Graham. Collocation methods for two dimensional weakly singular integral equations, The Journal of the Australian Mathematical Society. Series B. Applied Mathematics, 22(04), 456–473 (1981).

[9] K. Atkinson, F. Potra, Projection and iterated projection methods for nonlinear integral equations, SIAMJ. Numer. Anal., 24, 1352–1373 (1987).

[10] E. Hashemizadeh, M. Rostami, Numerical solution of Hammerstein integral equations of mixed type using the Sinc-collocation method, J. Comput. Appl. Math., 279, 31–39 (2015).

[11] H. Guoqiang, W. Jiong. Extrapolation of nystrom solution for two dimensional nonlinear fredholm integral equations, Journal of Computational and Applied Mathematics, 134(1), 259–268, (2001).

[12] G. Han, R. Wang. Richardson extrapolation of iterated discrete galerkin solution for two-dimensional fredholm integral equations, Journal of Computational and Applied Mathematics, 139(1), 49–63 (2002).

[13] R. J. Hanson, J. L. Phillips. Numerical solution of two-dimensional integral equations using linear elements, SIAM Journal on Numerical Analysis, 15(1), 113–121 (1978).

[14] A. Jerri. Introduction to integral equations with applications, John Wiley & Sons, 1999.

[15] M. Kazemi, Triangular functions for numerical solution of the nonlinear Volterra integral equations, J. Appl. Math. Comput., 1–24 (2021).

[16] S. McKee, T. Tang, T. Diogo. An euler-type method for two-dimensional volterra integral equations of the first kind. IMA Journal of Numerical Analysis, 20(3), 423-440 (2000).

[17] S. T. Mohyud-Din and M. A. Noor. Homotopy perturbation method for solving partial differential equations. Zeitschrift für Naturforschung A, 64(3-4), 157–170, (2009).

[18] A. M. Bica , Z. Satmari, A. M. Bica, Bernstein polynomials based iterative method for solving fractional integral equations, Math. Slovaca, 72, 1623–1640 (2022).

[19] M. Kazemi, Sinc approximation for numerical solutions of two-dimensional nonlinear Fredholm integral equations, Math. Commun., 29, 83–103 (2024).

[20] M. Kazemi, Approximating the solution of three-dimensional nonlinear Fredholm integral equations, J. Comput. Appl. Math., 395, 113590 (2021).

[21] M. Kazemi, R. Ezzati, Numerical solution of two-dimensional non- linear integral equations via quadrature rules and iterative method, Adv. Differ. Equat. Contr. Process., 17, 27–56 (2016).

[22] S. Nemati, P. M. Lima, and Y. Ordokhani. Numerical solution of a class of two-dimensional nonlinear volterra integral equations using legendre polynomials, Journal of Computational and Applied Mathematics, 242, 53–69 (2013).

[23] T. J. Rivlin. An introduction to the approximation of functions. Courier Corporation, 2003.

[24] W.-J. Xie and F.-R. Lin. A fast numerical solution method for two dimensional fredholm integral equations of the second kind, Applied Numerical Mathematics, 59(7), 1709–1719 (2009).