Analytical and Numerical Solutions for Nonlinear Equations
Volume 8, Issue 1, pp. 81–94

Research Article

Quadrature Rules for Solving Two-Dimensional Fredholm Integral Equations of Second Kind

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Received: 27 July 2024 / Accepted: 20 October 2024

Abstract In this paper, an iterative method of successive approximations based on the trapezoidal quadrature rule to solve two-dimensional Fredholm integral equations of second kind (2DFIE) is proposed. The error estimation of the proposed method is presented. The benefit of the method is that we do not have to solve a system of algebraic equations. Finally, a numerical example verify the theoretical results and show the accuracy of the method.

Keywords Integral equations · Trapezoidal quadrature · Uniform modulus of continuity · Iterative method

Mathematics Subject Classification (2010) 47H09 · 47H10

1 Introduction

The integral equations provide important tools for modeling a wide range of phenomena and processes [14], and many problems in engineering and physics give rise to two-dimensional integral equations [16,13,8]. There are many numerical methods for solving integral equations. The Galerkin and collocation methods are two commonly used methods for the numerical solutions of these equations [9,10]. Several numerical methods for approximating the solutions of integral equations were presented. Here, we recall some published works on this subject. These include Gauss product quadrature rule [6], polynomial interpolation methods [24], discrete Galerkin and iterated discrete Galerkin methods

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[12], triangular functions method [15], Legender polynomial method [22], Nystrom method [11], meshless method [1], Haar wavelet method [4]. Analytic methods, analytic-numeric methods like Adomian decomposition, homotopy perturbation method and regularization-homotopy method have been studied by many authors [17]. The use of successive approximations method in such cases can be therefore useful [18–20]. In this paper, we introduce an iterative method based on 2D trapezoidal quadrature rule for solving 2DFIE as

$$F(s,t) = f(s,t) + \lambda \int_c^d \int_a^b K(s,t,x,y)F(x,y)dxdy,$$

where $(s,t) \in \Omega = [a,b] \times [c,d]$ and $f(s,t)$, $K(s,t,x,y)$ are the given analytical functions. In the most of numerical methods, the integral equation is transformed into a system of linear or nonlinear algebraic equations which has to be solved with iterative methods. It is cumbersome to solve these systems, or the solution may be unreliable. The proposed method does not lead to a nonlinear algebraic equations system. This is a great advantage of this method. The rest of the paper is organizied as follows: we begin by introducing some necessary definitions and mathematical preliminaries of the some quadrature rules for 2-D integral in Section 2. Section 3, devoted to prove of the existence and uniqueness of the solution of 2DFIE by the method of successive approximations. Also, a conclusion is given in Section 4.

2 Preliminaries

Definition 1 Suppose that $f : \Omega \to \mathbb{R}$, be a bounded mapping, then the function $\omega_\Omega(f,.) : \mathbb{R}^+ \cup 0 \to \mathbb{R}^+$ defined by

$$\omega_\Omega(f,\delta) = \sup_{x,s\in[a,b];y,t\in[c,d]} \{|f(x,y) - f(s,t)|; \sqrt{(x-s)^2 + (y-t)^2} \leq \delta\}, \quad (1)$$

is called the modulus of oscillation of f on Ω.
Also, if $f \in C\big(\Omega\big)$ $(i.e.$ $f : \Omega \to \mathbb{R}$ is continuous on Ω, then $\omega_\Omega(f,\delta)$ is called uniform modulus of continuity of f.

Theorem 1 *The following properties holds [23]:*

(i) $|f(x,y) - f(s,t)| \leq \omega_{[a,b]\times[c,d]}(f, \sqrt{(x-s)^2 + (y-t)^2})$ *for all* $x,s \in [a,b]$ *and* $y,t \in [c,d]$,

(ii) $\omega_\Omega(f,\delta)$ *is an non-decreasing mapping in* δ,

(iii) $\omega_\Omega(f,0) = 0$,

(iv) $\omega_\Omega(f,\delta_1+\delta_2) \leq \omega_\Omega(f,\delta_1) + \omega_\Omega(f,\delta_2)$ *for any* $\delta_1,\delta_2 \geq 0$,

(v) $\omega_\Omega(f,n\delta) \leq n\omega_\Omega(f,\delta)$ *for any* $\delta \geq 0$ *and* $n \in \mathbb{N}$,

(vi) $\omega_\Omega(f,\lambda\delta) \leq (\lambda+1)\omega_{[a,b]\times[c,d]}(f,\delta)$ *for any* $\delta,\lambda \geq 0$,

(vii) *If* $[a,b]\times[c,d]\subseteq[e,f]\times[g,h]$, *then* $\omega_{[a,b]\times[c,d]}(f,\delta)\leq\omega_{[e,f]\times[g,h]}(f,\delta)$ *for all* $\delta\geq 0$.

Theorem 2 *[21] Let* $f:[c,d]\times[c,d]\to\mathbb{R}$, *be a integrable, bounded mappings. Then, for any divisions*

$$a=x_0<x_1<\cdots<x_n=b,$$

and

$$c=y_0<y_1<\cdots<y_n=d,$$

and any points $\xi_i\in[x_{i-1},x_i]$ *and* $\eta_j\in[y_{j-1},y_j]$ *we have*

$$\left|\int_c^d\int_a^b f(s,t)dsdt-\sum_{j=1}^{n}\sum_{i=1}^{n}(x_i-x_{i-1})(y_j-y_{j-1})f(\xi_i,\eta_j)\right|$$
$$\leq\sum_{j=1}^{n}\sum_{i=1}^{n}(x_i-x_{i-1})(y_j-y_{j-1})$$
$$\times\,\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(f,\sqrt{(x_i-x_{i-1})^2+(y_j-y_{j-1})^2}).$$

Corollary 1 *Assume that* $f:[a,b]\times[c,d]\to\mathbb{R}$, *be a integrable, bounded mapping. Then with the following notation*

$$\omega_{xy\times zt}=\omega_{[x,y]\times[z,t]}\left(f,\sqrt{(y-x)^2+(t-z)^2}\right),$$

we have

$$\left|\int_c^d\int_a^b f(s,t)dsdt\ -\Big[(x-a)(y-c)f(u,\alpha)+(x-a)(d-y)f(u,\beta)\right.$$
$$\left.+(b-x)(d-y)f(c,\beta)+(b-x)(d-y)f(v,\beta)\Big]\right|$$
$$\leq(x-a)(y-c)\omega_{ax\times cy}+(b-x)(y-c)\omega_{xb\times cy}+(x-a)(d-y)\omega_{ax\times yd}$$
$$+(b-x)(d-y)\omega_{xb\times yd},$$

For all $x\in[a,b]$, $y\in[c,d]$, $u\in[a,x]$, $v\in[x,b]$, $\alpha\in[c,y]$, *and* $\beta\in[y,d]$.

Proof Taking in the previous theorem $n=2$, $x_1=\xi_1=\xi_2=x$, and $y_1=\eta_1=\eta_2=y$ we obtain the required inequality.

Corollary 2 *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$, *be a two dimensional integrable, bounded mapping. Then the following inequalities holds:*

$$\left|\int_c^d\int_a^b f(s,t)dsdt\ -(b-a)(d-c)f(\frac{a+b}{2},\frac{c+d}{2})\right|$$
$$\leq(b-a)(d-c)\omega_{[a,b]\times[c,d]}\left(f,\frac{(b-a)(d-c)}{4})\right).$$

Proof If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in Corollary (1), we obtain the required inequality.

3 Existence and uniqueness of Fredholm integral equations

Here, we consider the two dimensional Fredholm integral equations as follows:

$$F(s,t) = f(s,t) + \lambda \int_c^d \int_a^b K(s,t,x,y)F(x,y)dxdy, \tag{2}$$

where $\lambda > 0$, $K(s,t,x,y)$ is an arbitrary kernel on Ω^2 and $f : \Omega \to \mathbb{R}$. We assume that K is continuous and therefore it is uniformly continuous with respect to (s,t). This property implies that there exists $M > 0$ such that

$$M = \max_{\substack{a \le s,x \le b \\ c \le t,y \le d}} \Big| K(s,t,x,y) \Big|.$$

Now, we shall prove the existence and uniqueness of the solution of equation (2) by the method of successive approximations.
Let $\mathbf{X} = \{f : \Omega \to \mathbb{R}; f \text{ is continuous}\}$ be the space of two dimensional continuous functions with the metric

$$\big\| f - g \big\| = \sup_{\substack{a \le s \le b \\ c \le t \le d}} \big| f(s,t) - g(s,t) \big|. \tag{3}$$

We define the operator $A : \mathbf{X} \to \mathbf{X}$ by

$$A(F)(s,t) = f(s,t) + \lambda \int_c^d \int_a^b K(s,t,x,y)F(x,y)dxdy, \ \ \forall\, (s,t) \in \Omega, \ \ \forall\, f \in \mathbf{X}. \tag{4}$$

Sufficient conditions for the existence of an unique solution of equation (2) are given in the following result.

Theorem 3 *Let $K(s,t,x,y)$ be continuous for $a \le s,\ x \le b,\ c \le t,\ y \le d$, and $f : \Omega \to \mathbb{R}$ be continuous on $[a,b] \times [c,d]$. If $\Lambda = \lambda M(b-a)(d-c) < 1$, then the iterative procedure*

$$F_0(s,t) = f(s,t), \tag{5}$$

$$F_m(s,t) = f(s,t) + \lambda \int_c^d \int_a^b K(s,t,x,y)F_{m-1}(x,y)dxdy, \qquad m \ge 1 \tag{6}$$

converges to the unique solution F^* of equation (2). Moreover, the following error bound holds:

$$\big\| F^* - F_m \| \le \frac{\Lambda^{m+1}}{1-\Lambda} \| f \|, \tag{7}$$

where

$$\| f \| = \sup_{\substack{a \le s \le b \\ c \le t \le d}} | f(s,t) |. \tag{8}$$

Proof To proof this theorem we investigate the conditions of the Banach's fixed point principle. We first show that A maps $\mathbf{X}$ into $\mathbf{X}$ (i.e. $A(\mathbf{X}) \subset \mathbf{X}$). To the end, we show that the operator A is uniformly continuous. Since f is continuous on compact set of Ω, we deduce that it is uniformly continuous and hence for $\varepsilon_1 > 0$ exists $\delta_1 > 0$ such that

$$\big|f(s_1,t_1) - f(s_2,t_2)\big| < \varepsilon_1 \quad \text{whenever} \quad \sqrt{(t_2-t_1)^2 + (s_2-s_1)^2} < \delta_1,$$

for all $s_1, s_2 \in [a,b]$ and $t_1, t_2 \in [c,d]$. As mentioned above, K also is uniformly continuous thus, for $\varepsilon_2 > 0$ exists $\delta_2 > 0$ such that

$$\big|K(s_1,t_1,x,y) - K(s_2,t_2,x,y)\big| < \varepsilon_2 \quad \text{whenever} \quad \sqrt{(t_2-t_1)^2 + (s_2-s_1)^2} < \delta_2,$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and $\sqrt{(t_2-t_1)^2 + (s_2-s_1)^2} < \delta$. We obtain

$$\begin{aligned} |\big(A(F)(s_1,t_1) - A(F)(s_2,t_2)\big| &\le \big|f(s_1,t_1) - f(s_2,t_2)\big| \\ &+ \lambda \int_c^d \int_a^b \Big|K(s_1,t_1,x,y) - K(s_2,t_2,x,y)\Big| |F(x,y)| dxdy \\ &\le \varepsilon_1 + \lambda\varepsilon_2 \int_c^d \int_a^b \big|F(x,y)\big| dxdy \\ &\le \varepsilon_1 + \lambda(b-a)(d-c)\|F\|\varepsilon_2, \end{aligned}$$

where

$$\|F\| = \sup_{\substack{a \le x \le b \\ c \le y \le d}} |F(x,y)|.$$

By choosing $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 = \frac{1}{2M_1\lambda(b-a)(d-c)}\varepsilon$ we derive

$$\big|A(F)(s_1,t_1) - A(F)(s_2,t_2)\big| \le \varepsilon.$$

This shows that $A(F)$ is uniformly continuous for any $F \in \mathbf{X}$, and so continuous on Ω, and hence $A(\mathbf{X}) \subset \mathbf{X}$.
Now, we prove that the operator A is contraction map. So, for $H_1, H_2 \in \mathbf{X}$ and $s \in [a,b]$ and $t \in [c,d]$, we have

$$\begin{aligned} \big|A(H_1)(s,t) - A(H_2)(s,t)\big| &\le \lambda\big|K(s,t,x,y)\big| \int_c^d \int_a^b \big|H_1(x,y) - H_2(x,y)\big| dxdy \\ &\le \lambda M \int_c^d \int_a^b \big|H_1(x,y) - H_2(x,y)\big| dxdy \\ &\le \lambda M \int_c^d \int_a^b \big\|H_1 - H_2\| dxdy \\ &= \lambda M(b-a)(d-c)\big\|H_1 - H_2\big\| \\ &= \Lambda\big\|H_1 - H_2\big\|. \end{aligned}$$

Therefore, we obtain

$$\big\|A(H_1)(s,t) - A(H_2)(s,t)\big\| \le \Lambda\big\|H_1 - H_2\big\|.$$

Since $\Lambda < 1$, the operator A is a contraction. Consequently, the Banach's fixed point principle implies that equation (2) has a unique solution F^* in $\mathbf{X}$ and we also have

$$\begin{aligned}\big|F^*(s,t) - F_m(s,t)\big| &\leq \big\|F^* - F_m\big\| \\ &= \Lambda\big\|F^* - F_{m-1}\big\| \\ &\leq \Lambda\|F^* - F_m\| + \Lambda\|F_{m-1} - F_m\| \\ &\leq \Lambda\|F^* - F_m\| + \Lambda^m\|F_0 - F_1\|.\end{aligned}$$

Therefore,

$$\big\|F^* - F_m\big\| \leq \frac{\Lambda^m}{1-\Lambda}\ \|F_0 - F_1\|, \tag{9}$$

on the other hand,

$$\begin{aligned}\big\|F_0 - F_1\| &= \sup_{\substack{a\leq s\leq b\\ c\leq t\leq d}} \Big|f(s,t) - f(s,t) - \lambda\int_c^d\int_a^b K(s,t,x,y)F_0(x,y)dxdy\Big| \\ &\leq \sup_{\substack{a\leq s\leq b\\ c\leq t\leq d}} \lambda\int_c^d\int_a^b \big|K(s,t,x,y)F_0(x,y)\big|dxdy \\ &\leq M\lambda\int_c^d\int_a^b \sup_{a\leq s\leq bc\leq t\leq d}\big|F_0(x,y)\big|dxdy \\ &= \lambda M(b-a)(d-c)\|f\| = \|f\|\Lambda.\end{aligned} \tag{10}$$

So, by (9) and (10) we obtain inequality (7), which completes the proof.

Now, we introduce a numerical method to solve equation (2). we consider equation (2) with continuous kernel $K(s,t,x,y)$ having positive sign on $\Omega\times\Omega$ and uniform partitions

$$\begin{aligned}D_x &: a = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = b, \\ D_y &: c = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = d,\end{aligned}$$

with $s_i = a + ih$, $t_j = c + jh'$, where $h = \frac{b-a}{n}$, $h' = \frac{d-c}{n}$. Then the following iterative procedure gives the approximate solution of equation (2) in point (s,t),

$$\begin{aligned}u_0(s,t) &= f(s,t) \\ u_m(s,t) &= f(s,t) + \frac{\lambda hh'}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\Big(K(s,t,s_i,t_j)u_{m-1}(s_i,t_j) \\ &\quad + K(s,t,s_i,t_{j+1})u_{m-1}(s_i,t_{j+1}) + K(s,t,s_{i+1},t_j)u_{m-1}(s_{i+1},t_j) \\ &\quad + K(s,t,s_{i+1},t_{j+1})u_{m-1}(s_{i+1},t_{j+1})\Big).\end{aligned} \tag{11}$$

3.1 Error estimation

Here, we obtain an error estimate between the exact solution and the approximate solution for the given Fredholm integral equation (2).

Theorem 4 *Consider the equation (2) with continuous kernel $K(s,t,x,y)$ on $\Omega \times \Omega$ and suppose that f is continuous on Ω. If $\Lambda < 1$, then the iterative procedure (11) converges to the unique solution of equation (2), F^*, and the following error estimate holds true,*

$$\begin{aligned}\|F^* - u_m\| \leq & \left(\frac{\Lambda^{m+1}}{1-\Lambda}\right)\|f\| + \left(\frac{\Lambda}{4(1-\Lambda)}\right)\omega_{[a,b]\times[c,d]}\big(f, hh'\big) \\ & + \big(\frac{\mu\Lambda^2 + 4\tau\Lambda}{4M(1-\Lambda)}\big)\omega_{st}\big(K, h+h^{'}\big),\end{aligned}$$

where

$$\begin{aligned}\omega_{st}\big(K,\delta\big) = \sup_{\substack{s_1,s_2\in[a,b]\\ t_1,t_2[c,d]}} & \Big\{\big|K(s_1,t_1,x,y) - K(s_2,t_2,x,y)\big|; \\ & \sqrt{(s_2-s_1)^2 + (t_2-t_1)^2} \leq \delta\Big\},\end{aligned}$$

and

$$\begin{cases} M_k = \sup\limits_{(s,t)\in\Omega} |u_k(s,t)|, \\ \Gamma_k = \sup\limits_{(s,t)\in\Omega} |F_k(s,t)|, \\ \tau = \max\limits_{i=0,1,\ldots,m-1}\{M_i\}, \\ \mu = \max\limits_{i=0,1,\ldots,m-2}\{\Gamma_i\}. \end{cases} \tag{12}$$

Proof Considering iterative procedure (11), we obtain

$$\begin{aligned}\big|F_1(s,t) - u_1(s,t)\big| \leq & \lambda \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\bigg|\int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}} K(s,t,x,y)f(x,y)dxdy \\ & - \frac{hh'}{4}\big[K(s,t,x,y)f(s_i,t_j) + K(s,t,x,y)f(s_i,t_{j+1}) \\ & + K(s,t,x,y)f(s_{i+1},t_j) + K(s,t,x,y)f(s_{i+1},t_{j+1})\big]\bigg| \\ & + \lambda \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\bigg|\frac{hh'}{4}\big[K(s,t,x,y)f(s_i,t_j) + K(s,t,x,y)f(s_i,t_{j+1}) \\ & + K(s,t,x,y)f(s_{i+1},t_j) + K(s,t,x,y)f(s_{i+1},t_{j+1})\big] \\ & - \frac{hh'}{4}\big[K(s,t,s_i,t_j)f(s_i,t_j)) + K(s,t,s_i,t_{j+1})f(s_i,t_{j+1} \\ & + K(s,t,s_{i+1},t_j)f(s_{i+1},t_j) + K(s,t,s_{i+1},t_{j+1})f(s_{i+1},t_{j+1})\big]\bigg|\end{aligned}$$

$$\begin{aligned}
&\leq \lambda \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Big| \int_{s_i}^{s_{i+1}} \int_{t_j}^{t_{j+1}} K(s,t,x,y)f(x,y)dxdy \\
&- \frac{hh'}{4}\big[K(s,t,x,y)f(s_i,t_j) + K(s,t,x,y)f(s_i,t_{j+1}) \\
&+ K(s,t,x,y)f(s_{i+1},t_j) + K(s,t,x,y)f(s_{i+1},t_{j+1})\big]\Big| \\
&+ \lambda \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Big| \frac{hh'}{4}\big[K(s,t,x,y)f(s_i,t_j) + K(s,t,x,y)f(s_i,t_{j+1}) \\
&K(s,t,x,y)f(s_{i+1},t_j) + K(s,t,x,y)f(s_{i+1},t_{j+1})\big] \\
&- \frac{hh'}{4}\big[K(s,t,s_i,t_j)f(s_i,t_j) + K(s,t,s_i,t_{j+1})f(s_i,t_{j+1}) \\
&+ K(s,t,s_{i+1},t_j)f(s_{i+1},t_j) + K(s,t,s_{i+1},t_{j+1})f(s_{i+1},t_{j+1})\big]\Big| \\
&\leq \lambda M \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} | \int_{s_i}^{s_{i+1}} \int_{t_j}^{t_{j+1}} f(x,y)dxdy \\
&- \frac{hh'}{4}\big[f(s_i,t_j) + f(s_i,t_{j+1}) + f(s_{i+1},t_j) + f(s_{i+1},t_{j+1})\big]\Big| \\
&+ \frac{\lambda hh'}{4} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Big[\Big|K(s,t,x,y)f(s_i,t_j) - K(s,t,s_i,t_j)f(s_i,t_j)\Big| \\
&+ \Big|K(s,t,x,y)f(s_i,t_{j+1}) - K(s,t,s_i,t_{j+1})f(s_i,t_{j+1})\Big| \\
&+ \Big|K(s,t,x,y)f(s_{i+1},t_j) - K(s,t,s_{i+1},t_j)f(s_{i+1},t_j)\Big| \\
&+ \Big|K(s,t,x,y)f(s_{i+1},t_{j+1}) - K(s,t,s_{i+1},t_{j+1})f(s_{i+1},t_{j+1})\Big|\Big].
\end{aligned}$$

Using Corollary 2 and part (vii) of Theorem 1 we deduce

$$\begin{aligned}
\big|F_1(s,t) - u_1(s,t)\big| \leq{}& \frac{\lambda M hh'}{4} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \big(4\omega_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}(f, \frac{hh'}{4})\big) \\
&+ \frac{\lambda hh'}{4} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \Big[\big|K(s,t,x,y) - K(s,t,s_i,t_j)\big|\big|f(s_i,t_j)\big| \\
&+ \big|K(s,t,x,y) - K(s,t,s_i,t_{j+1})\big|\big|f(s_i,t_{j+1})\big| \\
&+ \big|K(s,t,x,y) - K(s,t,s_{i+1},t_j)\big|\big|f(s_{i+1},t_j)\big| \\
&+ \big|K(s,t,x,y) - K(s,t,s_{i+1},t_{j+1})\big|\big|f(s_{i+1},t_{j+1})\big|\Big].
\end{aligned}$$

By part (*ii*) of Theorem 1 and direct computation, it follows that

$$\begin{aligned}\big|F_1(s,t)-u_1(s,t)\big| &\le \frac{\lambda M(b-a)(d-c)}{4}\omega(f,hh') \\ &+ \lambda hh' \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\big(\big|K(s,t,x,y)-K(s,t,s_i,t_j)\big|\big)\|f(s,t)\| \\ &\le \frac{\lambda M(b-a)(d-c)}{4}\omega(f,hh') \\ &+ \lambda(b-a)(d-c)M_0\omega_{st}(K,h+h') \\ &= \frac{\Lambda}{4}\omega(f,hh') + \frac{\Lambda}{M}M_0\omega_{st}(K,h+h'),\end{aligned}$$

therefore we obtain

$$\big|F_1(s,t)-u_1(s,t)\big| \le \frac{\Lambda}{4}\omega(f,hh') + \frac{\Lambda}{M}M_0\omega_{st}(K,h+h'). \tag{13}$$

Now, we have

$$\begin{aligned}\big|F_2(s,t)-u_2(s,t)\big| &\le \frac{\lambda M(b-a)(d-c)}{4}\omega_{[a,b]\times[c,d]}(F_1,hh') \\ &+ \frac{\lambda M(b-a)(d-c)}{4}\Big[\big|F_1(s_i,t_j)-u_1(s_i,t_j)\big| \\ &+ \big|F_1(s_i,t_{j+1})-u_1(s_i,t_{j+1})\big| + \big|F_1(s_{i+1},t_j)-u_1(s_{i+1},t_j)\big| \\ &+ \big|F_1(s_{i+1},t_{j+1})-u_1(s_{i+1},t_{j+1})\big|\Big] \\ &+ \lambda(b-a)(d-c)M_1\omega_{st}(K,h+h').\end{aligned}$$

Therefore,

$$\begin{aligned}\big|F_2(s,t)-u_2(s,t)\big| &\le \frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_1,hh') + \frac{\Lambda}{4}\Big[\big|F_1(s_i,t_j)-u_1(s_i,t_j)\big| \\ &+ \big|F_1(s_i,t_{j+1})-u_1(s_i,t_{j+1})\big| + \big|F_1(s_{i+1},t_j)-u_1(s_{i+1},t_j)\big| \\ &+ \big|F_1(s_{i+1},t_{j+1})-u_1(s_{i+1},t_{j+1})\big|\big)\Big] \\ &+ \lambda(b-a)(d-c)M_1\omega_{st}(K,h+h').\end{aligned}$$

By induction for $m \geq 3$, using (5), (6), (7), (11), and (12), we see that

$$\begin{aligned}\big|F_m(s,t) - u_m(s,t)\big| \leq{}& \frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_{m-1}, hh') \\ &+ \frac{\Lambda}{4}\Big[\big|F_{m-1}(s_i,t_j) - u_{m-1}(s_i,t_j)\big| \\ &+ \big|F_{m-1}(s_i,t_{j+1}) - u_{m-1}(s_i,t_{j+1})\big| \\ &+ \big|F_{m-1}(s_{i+1},t_j) - u_{m-1}(s_{i+1},t_j) \\ &+ \big|F_{m-1}(s_{i+1},t_{j+1}) - u_{m-1}(s_{i+1},t_{j+1})\big|\Big] \\ &+ \frac{\Lambda}{M}M_{m-1}\omega_{st}(K, h+h'). \qquad (14)\end{aligned}$$

Taking supremum for $(t,s) \in \Omega$ from (14) we conclude that the following inequalites hold

$$\begin{aligned}\big\|F_m - u_m\big\| \leq{}& \frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_{m-1}, hh') + \Lambda\big\|F_{m-1} - u_{m-1}\big\| \\ &+ \frac{\Lambda}{M}M_{m-1}\omega_{st}(K, h+h'),\end{aligned}$$

$$\begin{aligned}\big\|F_{m-1} - u_{m-1}\big\| \leq{}& \frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_{m-2}, hh') + \Lambda\big\|F_{m-2} - u_{m-2}\big\| \\ &+ \frac{\Lambda}{M}M_{m-2}\omega_{st}(K, h+h'),\end{aligned}$$

$$\begin{aligned}\big\|F_{m-2} - u_{m-2}\big\| \leq{}& \frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_{m-3}, hh') + \Lambda\big\|F_{m-3} - u_{m-3}\big\| \\ &+ \frac{\Lambda}{M}M_{m-3}\omega_{st}(K, h+h'),\end{aligned}$$

$$\vdots$$

$$\begin{aligned}\big\|F_1 - u_1\big\| \leq{}& \frac{\Lambda}{4}\omega_{[a,b]\times[c,d]}(F_0, hh') \\ &+ \Lambda\big\|F_0 - u_0\big\| + \frac{\Lambda}{M}M_0\omega_{st}(K, h+h'),\end{aligned}$$

multiplying the above inequalities by $1, \Lambda, \Lambda^2, \ldots, \Lambda^{m-1}$, respectively and summing them we obtain

$$\begin{aligned}\big\|F_m - u_m\big\| \leq{}& \frac{\Lambda}{4}\bigg(\omega_{[a,b]\times[c,d]}(F_{m-1}, hh') \\ &+ \Lambda\omega_{[a,b]\times[c,d]}(F_{m-2}, hh') + \cdots + \Lambda^{m-1}\omega_{[a,b]\times[c,d]}(f, hh')\bigg) \\ &+ \frac{\Lambda}{M}\omega_{st}(K, h+h')\bigg(M_{m-1} + \Lambda M_{m-2} + \Lambda^2 M_{m-3} + \cdots + \Lambda^{m-1}M_0\bigg). \qquad (15)\end{aligned}$$

Since, for $(s_1,t_1),(s_2,t_2)\in\Omega$ with $|s_1-s_2|\le h$, $|t_1-t_2|\le h'$, we have

$$\begin{aligned}\big|F_m(s_1,t_1)-F_m(s_2,t_2)\big| &= \Big|f(s_1,t_1)+\lambda\int_c^d\int_a^b K(s_1,t_1,x,y)F_{m-1}(x,y)dxdy\\ &\quad - f(s_2,t_2)+\lambda\int_c^d\int_a^b K(s_2,t_2,x,y)F_{m-1}(x,y)dxdy\Big|\\ &\le \big|f(s_1,t_1)-f(s_2,t_2)\big|+\frac{\Lambda}{M}\omega_{st}(K,h+h')\Gamma_{m-1},\end{aligned}$$

therefore, we infer

$$\omega_{[a,b]\times[c,d]}(F_m,hh')\le\omega_{[a,b]\times[c,d]}(f,hh')+\frac{\Lambda}{M}\omega_{st}(K,h+h')\Gamma_{m-1}. \tag{16}$$

By this inequality and (15), we see that

$$\begin{aligned}\|F_m-u_m\| &\le \frac{\Lambda}{4}\Big(1+\Lambda+\Lambda^2+\cdots+\Lambda^{m-1}\Big)\omega_{[a,b]\times[c,d]}(f,hh')\\ &\quad+\frac{\Lambda}{4M}\omega_{st}(K,h+h')\Big(\Lambda\Gamma_{m-2}+\Lambda^2\Gamma_{m-3}+\cdots+\Lambda^{m-1}\Gamma_0\Big)\\ &\quad+\frac{\Lambda}{M}\omega_{st}(K,h+h')\Big(M_{m-1}+\Lambda M_{m-2}+\Lambda^2M_{m-3}+\cdots+\Lambda^{m-1}M_0\Big)\\ &=\frac{\Lambda}{4}\Big(\frac{1-\Lambda^m}{1-\Lambda}\Big)\omega_{[a,b]\times[c,d]}(f,hh')\\ &\quad+\frac{\Lambda}{4M}\omega_{st}(K,h+h')\Big[\big(\Lambda\Gamma_{m-2}+\Lambda^2\Gamma_{m-3}\\ &\quad+\cdots+\Lambda^{m-1}\Gamma_0\big)+4\big(M_{m-1}+\Lambda M_{m-2}+\Lambda^2M_{m-3}+\cdots+\Lambda^{m-1}M_0\big)\Big].\end{aligned}$$

By (12) since $\Lambda<1$ we obtain

$$\begin{aligned}\|F_m-u_m\| &\le \frac{\Lambda}{4}\Big(\frac{1-\Lambda^m}{1-\Lambda}\Big)\omega_{[a,b]\times[c,d]}(f,hh')\\ &\quad+\frac{\Lambda}{4M}\omega_{st}(K,h+h')\Big(\frac{\Lambda(1-\Lambda^m)}{1-\Lambda}\mu+\frac{4(1-\Lambda^m)}{1-\Lambda}\tau\Big)\\ &\le\frac{\Lambda}{4(1-\Lambda)}\omega_{[a,b]\times[c,d]}(f,hh')+\frac{\Lambda}{4M}\omega_{st}(K,h+h')\Big(\frac{\mu\Lambda+4\tau}{1-\Lambda}\Big).\end{aligned}$$

Therefore, we obtain

$$\|F_m-u_m\|\le\Big(\frac{\Lambda}{4(1-\Lambda)}\Big)\omega_{[a,b]\times[c,d]}(f,hh')+\Big(\frac{\mu\Lambda^2+4\tau\Lambda}{4M(1-\Lambda)}\Big)\omega_{st}(K,h+h'). \tag{17}$$

By inequalities (17) and (7) we deduce that

$$\begin{aligned}\|F^* - u_m\| &\le \|F^* - F_m\| + \|F_m - u_m\| \\ &\le \left(\frac{\Lambda^{m+1}}{1-\Lambda}\right)\|f\| + \left(\frac{\Lambda}{4(1-\Lambda)}\right)\omega_{[a,b]\times[c,d]}(f, hh') \\ &+ \left(\frac{\mu\Lambda^2 + 4\tau\Lambda}{4M(1-\Lambda)}\right)\omega_{st}(K, h+h').\end{aligned}$$

Remark 1 Since $\Lambda < 1$, it is easy to see that

$$\lim_{\substack{m\to\infty \\ h,h'\to 0}} \|F^* - u_m\| = 0,$$

that shows the convergence of the method.

Example 1 Consider the following two dimensional Fredholm integral equation

$$F(s,t) = f(s,t) + \int_0^1 \int_0^1 K(s,t,x,y)F(x,y)dxdy, \tag{18}$$

where

$$f(s,t) = s\sin\frac{t}{2}, \qquad K(s,t,x,y) = s^2tx,$$

with the exact solution

$$F(s,t) = s\sin\frac{t}{2} - \frac{16}{21}(\cos\frac{1}{2} - 1)s^2t.$$

By using the proposed method, we can present the approximate solution for this example. To compare the numerical results with the exact solution for different values of m, n, *see Table 1* .

Table 1 Numerical results of Example 1.

		$m=3$, $n=10$	$m=3$, $n=50$	$m=6$, $n=10$	$m=6$, $n=50$
(s,t)	Exact	$\lvert F-u_m\rvert$	$\lvert F-u_m\rvert$	$\lvert F-u_m\rvert$	$\lvert F-u_m\rvert$
(0.2,0.2)	0.0207	4.456×10^{-7}	3.016×10^{-10}	2.362×10^{-7}	2.092×10^{-10}
(0.4,0.4)	0.0854	3.565×10^{-6}	2.500×10^{-9}	1.889×10^{-6}	1.674×10^{-9}
(0.6,0.6)	0.1974	1.203×10^{-5}	1.916×10^{-8}	1.015×10^{-5}	1.248×10^{-8}
(0.8,0.8)	0.3593	2.852×10^{-5}	2.811×10^{-8}	2.111×10^{-5}	2.339×10^{-8}
(1.0,1.0)	0.5727	6.570×10^{-5}	3.299×10^{-8}	4.252×10^{-5}	3.014×10^{-8}

4 Conclusions

To approximate the solution of 2DFIE of Fredholm type, we used an efficient iterative algorithm, based on the method of successive approximations. In the present paper, using an iterative method based on 2D Trapezoidal quadrature rule we have approximated the numerical solution of two-dimensional Fredholm integral equations. We established the theorem of existence of unique solution of these equations, and we have proved it by using Banach's fixed point principle. Moreover, the proof of convergence of quadrature formula is discussed in Theorem 4.

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