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An Extension of Order Bounded Operators

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Abstract Let E be a normed lattice and an g-order dense majorizing sublattice of a vector lattice E^t. We extend the norm of E to E^t, denoted by $\|.\|_t$. The pair $(E^t, \|.\|_t)$ forms a normed lattice and preserves certain lattices and topological properties whenever these properties hold in E. As a consequence, every positive linear operator defined on a normed lattice E has a linear extension to E^t. This manuscript provides an explicit formula for these extensions. The extended operator T^t is a lattice homomorphism from E^t into F, and it belongs to $\mathcal{L}_n(E^t, F)$ whenever $0 \leq T \in \mathcal{L}_n(E, F)$ and $T(x \wedge y) = Tx \wedge Ty$ for all $0 \leq x, y \in E$. Furthermore, if $T \in \mathcal{L}_b(E, F)$ and certain lattice and topological properties hold for T, then $T^t \in \mathcal{L}_b(E^t, F)$ will also preserve these properties.

Keywords Riesz space · Order convergence · Unbounded order convergence

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1 Introduction

A vector sublattice E of vector lattice G is said to be order dense in G whenever for each $0 < x \in G$ there exists some $y \in E$ with $0 < y \leq x$ and E is generalized

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order dense (g-order dense) in G whenever for each $0 < x < z$ in G there exists some $y \in E$ with $0 < x \leq y \leq z$. It is clear that each g-order dense subspace is order dense, but the converse not holds. For example, c_0 is order dense in ℓ^∞, but is not g-order dense. Let us say that a vector subspace E of an ordered vector space G is majorizing of G whenever for each $x \in G$ there exists some $y \in E$ with $x \leq y$. Let E be a normed lattice that is both g-order dense and majorizing in a vector lattice E^t. It is possible to extend the norm from E to E^t. In this paper, we investigate the method of this norm extension and demonstrate that certain lattice and topological properties can be carried over from E to E^t. Now, suppose T is a positive order bounded operator from a normed lattice E to a Dedekind complete normed lattice F. Then, there exists a linear operator T^t from E^t to F that extends T, and furthermore, we have $\|T\| = \|T^t\|$. In Section 1.2 of [2], the authors studied some new extensions of operators on vector lattices. In [3], Onno van Gaans introduced and studied a generalization of the notion of a seminorm on a directed partially ordered vector space. In this paper, we investigate this problem in a different way and extend some results to the general case.

Let E be a normed lattice and a sublattice of G, and assume that E is order dense and majorizing in a vector lattice E^t that is a subset of G. The motivations of this manuscript are as follows:

1. We can extend the norm from E to E^t as follows: For any $x \in E^t$, we define $\|x\|_t = \inf\{\|y\| : y \in E, y \geq |x|\}$, where $|x| = x \vee (-x)$, which is the supremum of x and its additive inverse $-x$. Then, $(E^t, \|\cdot\|_t)$ is a normed lattice.
2. Suppose T is an order-bounded operator from E to a Dedekind complete normed lattice F. We can define a linear extension $T^t : E^t \to F$ of T to E^t as follows:
   For any $x \in E^t$, we define $T^t(x) = \sup\{T(y) : y \in E, y \leq x\}$, where the supremum is taken in F. Then, T^t is well-defined and order-bounded.
3. Moreover, T^t is the unique linear extension of T from E^t to F in the sense that if $S : E^t \to F$ is any extension of T using the same method, then $T^t = S$.
   If certain lattice and topological properties hold for $T \in \mathcal{L}_b(E, F)$, then $T^t \in \mathcal{L}_b(E^t, F)$ will also preserve these properties.

To state our result, we need to fix some notation and recall some definitions. A Banach lattice E has order continuous norm if $\|x_\alpha\| \to 0$ for every decreasing net $(x_\alpha)_\alpha$ with $\inf_\alpha x_\alpha = 0$. A Banach lattice E is said to be an AL-space if we have $\|x + y\| = \|x\| + \|y\|$ for each $x, y \in E$ such that $|x| \wedge |y| = 0$. A Banach lattice E is said to be KB-space whenever each increasing norm bounded sequence of E^+ is norm convergent. A Riesz space that is at the same time Dedekind complete and laterally complete is referred to as a universally complete Riesz space. Let E and F be Riesz spaces. An operator $T : E \to F$ is said to be order bounded if it maps each order bounded subset of E into order bounded subset of F. The collection of all order bounded operators from a Riesz space E into a Riesz space F will be denoted by $\mathcal{L}_b(E, F)$. A linear operator between two Riesz spaces is order continuous (resp. σ-order continuous) if it maps order null nets (resp. sequences) to order null nets (resp. sequences). The collection of all order continuous (resp. σ-order continuous) linear operators from vector lattice E into vector lattice F will be denoted by $\mathcal{L}_n(E,F)$ (resp. $\mathcal{L}_c(E,F)$).

A Dedekind complete vector lattice G is said to be a Dedekind completion of the vector lattice E whenever E is lattice isomorphism to a majorizing order dense sublattice of G. A subset A of a vector lattice E is said to be order closed whenever $(x_\alpha)_\alpha \subseteq A$ and $x_\alpha \xrightarrow{o} x$ in E imply $x \in A$. A lattice norm $\|.\|$ on a vector lattice E is said to be a Fatou norm (or that $\|.\|$ satisfies the Fatou property) if $0 \le x_\alpha \uparrow x$ in E implies $\|x_\alpha\| \uparrow \|x\|$. σ-Fatou norm has similar definition. An operator $T : E \to E$ on a vector lattice is said to be band preserving whenever T leaves all bands of E invariant, i.e., whenever $T(B) \subseteq B$ holds for each band B of E. An operator $T : E \to F$ between two vector lattices is said to be preserve disjointness whenever $x \perp y$ in E implies $Tx \perp Ty$ in F. For a normed lattice E, E' is the its order dual and $\sigma(E,E')$ is the weak topology for E. For unexplained terminology and facts on Banach lattices and positive operators, we refer the reader to [1,2].

2 An extension of the norms

Let E be an Archimedean vector lattice. Then there exists a Dedekind complete vector lattice E^δ that contains a majorizing, order dense vector subspace that is Riesz isomorphic to E, which we will identify as E. E^δ is called the Dedekind completion of E. Throughout this manuscript, we assume that the vector lattices under consideration are Archimedean. Let E and G be a normed lattice and a vector lattice, respectively, such that E is order dense and majorizing in G. The universal completion of a vector lattice E will be denoted by E^u. According to [[1], Theorem 7.21], every Archimedean vector lattice has a unique universal completion. In all parts of this manuscript, we assume that E is g-order dense and majorizing in G. Throughout this paper, $(E, \|\cdot\|)$ denotes a normed space that serves as a vector sublattice of G.

Theorem 1 *For each $x \in G$, let $\rho(x) = \sup\{\|z\| : \ z \le |x|, \ z \in E^+\}$. Then $\rho(x)$ is a norm on G, and moreover, $(G, \rho(x))$ is a normed lattice.*

Proof It is clear that $\rho(x) = 0$ if and only if $x = 0$, and $\rho(\lambda x) = |\lambda|\rho(x)$ for each real number λ and $x \in G$. Now we prove that $\rho(x+y) \le \rho(x) + \rho(y)$ whenever $x, y \in G$.

Let $x, y \in G$. Fix $z \in E^+$ such that $z \le |x+y|$. By Riesz Decomposition property, [[1], Theorem 1.10], there are $z_1, z_2 \in G$ such that $|z_1| \le |x|$, $|z_2| \le |y|$ and $z = z_1 + z_2$. Since E is order dense in G, there are $w_1, w_2 \in E^+$ such that $|z_1| \le w_1 \le |x|$ and $|z_2| \le w_2 \le |y|$. It follows that

$$z = z_1 + z_2 \le |z_1| + |z_2| \le w_1 + w_2 \le |x| + |y|.$$

Then we have

$$\|z\| \leq \|w_1 + w_2\| \leq \|w_1\| + \|w_2\| \leq \rho(x) + \rho(y).$$

Consequently, we have $\sup\{\|z\| : \ z \leq |x+y| \text{ and } z \in E^+\} \leq \rho(x) + \rho(y)$, which implies that $\rho(x+y) \leq \rho(x) + \rho(y)$.

For a normed lattice $(E, \|.\|)$, assume that E^ρ is the set of all $x \in G$ such that satisfies in the following equality,

$$\begin{aligned} \rho(x) &= \inf\{\|y\| : \ |x| \leq y, \ y \in E^+\} && (1) \\ &= \sup\{\|z\| : \ z \leq |x|, \ z \in E^+\}. && (2) \end{aligned}$$

Then E is subspace of E^ρ and ρ is a real function from E^ρ into $[0, +\infty)$ and satisfies in the following properties:

1. $\rho(x) = 0$ iff $x = 0$
2. $\rho(\lambda x) = \lambda \rho(x)$ for each $\lambda \in \mathbb{R}^+$ and $x \in E^\rho$.
3. $\rho(x+y) \leq \rho(x) + \rho(y)$, for $x, y \in E^\rho$.

(E^ρ, ρ) is an extension of $(E, \| \cdot \|)$, meaning that E is a sublattice of E^ρ and $\|x\| = \rho(x)$ for all $x \in E$.
To see why this is true, note that by Theorem 1, we can extend the norm on E to a complete lattice norm ρ on E^ρ, such that $\|x\| = \rho(x)$ for all $x \in E$. Therefore, (E^ρ, ρ) is indeed an extension of $(E, \| \cdot \|)$.
An example that illustrates this point is as follows.

Example 1 Let c be the collection of all real number sequences which are convergence in $\mathbb{R}$ with ℓ^∞-norm. It is obvious that c is order dense majorizing of ℓ^∞. By easy calculation, we can prove that $c^\rho = \ell^\infty$.

Definition 1 Assume that $E \subseteq E^t$ is a vector sublattice of G in which every element of E^t satisfies the equalities (1) and (2), we can define a new norm in E^t called the t-norm, denoted by $\|x\|_t = \rho(x)$.

It is evident that $(E^t, \|.\|_t)$ is a normed lattice. However, E^t is not necessarily unique, and in general, we have $E \subseteq E^t \subseteq G$. The objective of this manuscript is to identify vector lattices E^t that are distinct from E. Therefore, in this manuscript, E is a proper sublattice of E^t.
In Theorem 2, we will demonstrate that $E^t = G$ whenever E is a Dedekind complete or has an order-continuous norm.

Theorem 2 *By one of the following conditions, the equality* (1) *holds for each $x \in G$, that is, $E^t = G$, $(G, \|.\|_t)$ is normed lattice and $\|y\| = \|y\|_t$ for each $y \in E$.*

i) E is a Dedekind complete.
ii) E has order continuous norm.

Proof i) According to Theorem 1, the function

$$\rho(x) = \sup\{\|z\| : z \leq |x|,\, z \in E^+\},$$

defines a norm for the vector lattice G. By contradiction, assume that

$$\rho(x) < \inf\{\|y\| :\ |x| \leq y,\ y \in E^+\}.$$

Let $A = \{y \in E^+ :\ |x| \leq y\}$. Since E is order dense in G, A is bounded below, and so A has infimum in E, by Dedekind completeness of E. Take $\inf A = y_0$ where $y_0 \in E$. It is clear that $y_0 < |x|$ and $\rho(x) \leq \|y_0\|$. Then $\|y_0\| = \rho(y_0) = \rho(x)$. Let the natural number n be enough large such that

$$\rho(x) < \|y_0\| + \frac{1}{n}\|y_0\| < \inf\{\|y\| :\ |x| \leq y,\ y \in E^+\}.$$

Put $z_0 = (1 + \frac{1}{n})y_0$. Consequently we have $z_0 \in A$, then

$$\inf\{\|y\| :\ |x| \leq y,\ y \in E^+\} < \|z_0\|,$$

which is impossible.

ii) First we show that

$$\inf\{\|y\| :\ |x| \leq y,\ y \in E\} = \sup\{\|z\| :\ z \leq |x|,\ z \in E\},$$

holds whenever $x \in G$. Set

$$A = \{z \leq |x| :\ z \in E^+\},$$

and

$$B = \{y \geq |x| :\ y \in E\}.$$

Since E is order dense and majorizing of G, it follows that A and B are not empty and they are directed sets. We consider the set A as a net $\{z_\alpha\}$, where $z_\alpha = \alpha$ for each $\alpha \in A$. In the same way we consider $B = \{y_\beta\}$, and by using [[2], Theorem 1.34], we write $z_\alpha \uparrow |x|$ and $y_\beta \downarrow |x|$. Since $z_\alpha \leq |x| \leq y_\beta$ for each α and β, it follows that $y_\beta - z_\alpha \downarrow 0$, and so

$$0 \leq \|y_\beta\| - \|z_\alpha\| \leq \|y_\beta - z_\alpha\| \to 0.$$

It follows that $\|x\|_t = \inf\|y_\beta\| = \sup\|z_\alpha\|$. Obviously that $\|.\|_t$ is a norm for G and $(G, \|.\|_t)$ is a normed lattice.

In Example 1, we note that c is neither Dedekind complete nor equipped with an order-continuous norm, yet we observe that $c^t = \ell^\infty$. However, Theorem 2 provides justification for extending the norm of E to a vector lattice E^t in various other cases.

It is also important to determine when $(E^t)^t = E^t$. In the following example, we demonstrate that E^t exists whenever E satisfies the Fatou property. It is worth noting that according to Example 4.3 and 4.4 from [1], every normed lattice with the Fatou property, in a general sense, is neither order-continuous nor Dedekind complete.

Example 2 By [[1], Theorem 4.12], if $(E, \|.\|)$ satisfies the Fatou property, the Dedekind completion of E, E^{δ} is a normed space with $\delta - norm$. Let E be the vector lattice of all real-valued functions defined on an infinite set X whose range is finite, with the pointwise ordering and satisfies the Fatou property. It can be seen that E is not Dedekind complete and $E^{\delta} = \ell^{\infty}(X)$.

We now present an important lemma that plays a crucial role throughout this manuscript.

Lemma 1 *Let E has order continuous norm. For each $0 \leq x \in E^{t}$, there are sequences $\{x_n\} \subseteq E^{+}$ and $\{y_n\} \subseteq E^{+}$ such that $x_n \uparrow x$, $x_n \xrightarrow{\|\cdot\|_t} x$, $y_n \downarrow x$ and $y_n \xrightarrow{\|\cdot\|_t} x$.*

Proof Choose $\{r_n\} \subseteq \mathbb{R}^{+}$ and $\{x_n\} \subseteq E^{+}$ satisfies in the following conditions:

1. $r_n \downarrow 0$,
2. $x_n \in \{z \in E : \ z \leq x \text{ and } \|x - z\|_t < r_n\}$, for each $n \in \mathbb{N}$,
3. $x_n \uparrow x$.

The justification for the above statement is as follows:
By [[2], Theorem 1.34], set

$$A = \{z \leq x : \ z \in E^{+}\} = \{z_{\alpha}\},$$

and

$$B = \{y \geq x : \ y \in E\} = \{y_{\beta}\},$$

such that $z_{\alpha} \uparrow x$ and $y_{\beta} \downarrow x$. Then $z_{\alpha} \leq x \leq y_{\beta}$ holds for each α and β. Thus

$$\|x - y_{\beta}\|_t, \|x - z_{\alpha}\|_t \leq \|z_{\alpha} - y_{\beta}\|_t = \|z_{\alpha} - y_{\beta}\| \to 0.$$

Let $0 < r_1 \in \mathbb{R}$. Then there exist

$$z_1 \in \{z \in A : \ \|x - z\|_t \leq r_1\},$$

and

$$0 < r_2 < \min\{r_1, \ \|z_1 - x\|_t\}.$$

We choose $z_2, z_3, \ldots, z_n$ and $z_{n+1} \in \{z \in A : \ \|x - z_n\|_t \leq r_n\}$ where

$$0 < r_n < \min\{r_{n-1}, \ \|z_{n-1} - x\|_t\}.$$

We define $x_n = \vee_{i=1}^{n} z_i$. Now, if $x_n \leqslant w \leqslant x$ for each $n \in \mathbb{N}$, then

$$0 \leqslant x - w \leqslant x - x_n \leq x - z_n.$$

It follows that

$$\|x - w\|_t \leqslant \|x - x_n\|_t \leqslant \|x - z_n\|_t \leqslant r_n \downarrow 0.$$

Thus $x = w$, and so $\sup x_n = x$. Therefore $x_n \uparrow x$ and $\|x_n - x\| \to 0$.
The existence of $\{y_n\}$ follows the same argument.

Theorem 3 *Suppose E is a normed lattice. If E is a KB-space or an AL-space, then E^t is also a KB-space or an AL-space, respectively.*

Proof Assume that $\{x_n\} \subseteq (E^t)^+$ is increasing sequence such that

$$\sup \|x_n\|_t < +\infty.$$

By using Lemma 1, for each $n \in \mathbb{N}$, there is increasing sequences

$$\{x_{n,m}\}_m \subseteq E^+,$$

such that $x_{n,m} \uparrow_m x_n$ and $\|x_n - x_{n,m}\|_t \xrightarrow{m} 0$. Take $y_n = \bigvee_{i,j=1}^n x_{i,j}$. It follows that $0 \leq y_n \uparrow$ and $\sup \|y_n\| \leq \sup_{i,j} \|x_{i,j}\| \leq \sup \|x_n\| < +\infty$. Since E is a KB-space, it follows that there exists $x \in E$ such that $\|y_n - x\|_t \to 0$. On the other hand, the inequalities $y_n \leq x_n \leq x$ implies that $\|x_n - x\|_t \leq \|y_n - x\|_t$ for each $n \in \mathbb{N}$. It follows that $\|x_n - x\|_t \to 0$ holds in E^t. Now, if E is an $AL-$space, then E has order continuous norm. Now, let $0 < x, y \in E^t$ with $x \wedge y = 0$. By using Lemma 1, there are $\{x_n\}$ and $\{y_n\}$ in E^+ such that $x_n \uparrow x$, $y_n \uparrow y$, $\|x - x_n\|_t \to 0$ and $\|y - y_n\|_t \to 0$. It follows that $0 \leqslant x_n \wedge y_n \uparrow x \wedge y = 0$ implies that $x_n \wedge y_n = 0$ for each $n \in \mathbb{N}$. Hence

$$\|x_n + y_n\| = \|x_n\| + \|y_n\|,$$

for each $n \in \mathbb{N}$. Then

$$\|x + y\|_t = \lim_n \|x_n + y_n\| = \lim_n \|x_n\| + \lim_n \|y_n\| = \|x_n\|_t + \|y\|_t.$$

Consequently, E^t is an $AL-$space.

Theorem 4 *For a normed lattice E with order continuous norm, we have the following assertions*

1. *If $\hat{E}$ is a norm completion of E, then $E^t \subseteq \hat{E} = E^u$, and if E is norm complete, then $E^t = E^u = E$.*
2. *For each $x \in E^t$ and $A \subseteq E$ with $\sup A = x$, we have $\|x\|_t = \sup_{z \in A} \|z\|$.*
3. *For each $x \in E^t$ and $A \subseteq E$ with $\inf A = x$, we have $\|x\|_t = \inf_{z \in A} \|z\|$.*
4. *$(E^t, \|.\|_t)$ has Fatou property and $B_{E^t} = \{x \in E^t : \|x\|_t \leq 1\}$ is order closed.*
5. *If E is an ideal in E^t, then $\hat{E} = E^t$.*

Proof 1. According to [[1], Theorem 2.40], $(\hat{E}, \|\hat{.}\|)$ is a normed lattice, where $\|\hat{.}\|$ is the unique extension of the norm from E to $\hat{E}$. Let $x \in E^t$. Then by Lemma 1, there exists $\{x_n\}$ in E^+ such that $x_n \uparrow x^+$ and $\|x^+ - x_n\|_t \to 0$. Thus $\{x_n\}$ is a norm Cauchy sequence in E, and so convergence in $\hat{E}$. It follows that $x^+ \in \hat{E}$. In the similar way $x^- \in \hat{E}$, which implies that $x \in \hat{E}$. Now by Theorem 7.51 of [1], we conclude that $E^t \subseteq \hat{E} = E^u$ and $\|.\|_t = \|\hat{.}\|$. On the other hand if E is norm complete, it is obvious that $E^t = E^u = E$ and $\|.\| = \|.\|_t = \|\hat{.}\|$.

2. By [[1], Theorem 7.54], E^u has order continuous norm. Since by part (1), we have $E^t \subseteq E^u$, it follows that E^t has order continuous norm. Consider $A = (x_\alpha)$ with $\sup A = x$. It follows that $x - x_\alpha \downarrow 0$ which implies that $\|x - x_\alpha\|_t \to 0$. Then by using inequalities $0 \leq \|x\|_t - \|x_\alpha\| \leq \|x - x_\alpha\|_t$, we have $\sup_\alpha \|x_\alpha\| = \|x\|_t$.
3. The proof follows a similar argument as that of (2).
4. By [[1], Lemma 4.2], $(E, \|.\|)$ has Fatou property. The proof of the first statement follows a similar argument to that of Theorem 3(1), and we omit the details. The second part follows by [[1], Theorem 4.6].
5. The proof follows by [[1], Theorem 3.8].

Note that a linear subspace E of a partially ordered vector space G is said to be order dense if $x = \inf\{y \in E : \ x \leq y\}$ for every $x \in G$. Based on our earlier discussion, we can pose the following question:

Problem 1 If E^t is a partially ordered vector space and E is order dense and majorizing in E^t, is there a norm extension from $(E, \| \cdot \|)$ to E^t?

3 The extension of order bounded operators

In this section, we explore the extension properties of order-bounded operators. Specifically, we consider T to be an order-bounded operator from a normed lattice E into a Dedekind complete normed lattice F, and we aim to introduce an operator T^t from E^t to F as an extension of T. We investigate various lattice and topological properties of T^t that hold when these properties are satisfied by T. Our analysis provides insights into the behavior of order-bounded operators under extensions of normed lattices, which has important applications in the positive operators studying and related fields.

Theorem 5 *Let T be an order bounded operator from normed lattice E into Dedekind complete normed lattice F. We have the following assertions.*

1. *There exists an extension order bounded operator T^t from E^t into F satisfying $T^t(y) = Ty$ for each $y \in E$.*
2. *For each positive continuous operator T, we have $\|T\| = \|T^t\|$, and if T is norm continuous, then so is T^t.*
3. *$|T|^t = |T^t|$.*
4. *For each $T, S \in \mathcal{L}_b(E, F)$, we have $(T \vee S)^t = T^t \vee S^t$.*
5. *If $S : E^t \to F$ is an order bounded and norm continuous operator, then $T^t = S$.*
6. *Each order interval of E^t is $\sigma(E^t, (E^t)')$-compact.*

Proof 1. Since T is an order bounded operator and F Dedekind complete, we have $T = T^+ - T^-$. So first we assume that T is a positive operator from E into F. According to [[2], Theorem 1.32], the mapping $p : E^t \to F$ defined via the formula

$$p(x) = \inf\{Ty : \ y \in E, \ x \leqslant y\}, \ x \in E^t.$$

is a monotone sublinear and $Ty = p(y)$ for each $y \in E$. So by [[3], Theorem 1.5.7], there is an extension T^t from E^t into F satisfying $T^t x \leq p(x^+)$ for all $x \in E^t$, and $T^t y = Ty$ for all $y \in E$. Now we define $T^t = (T^+)^t - (T^-)^t$, and so for all $y \in E$, we have

$$T^t y = (T^+)^t(y) - (T^-)^t(y) = T^+y - T^-y = Ty.$$

2. Assume that T is a positive operator and $x \in E^t$. According part (1), we have $T^t x \leq p(x^+) \leq Ty$ for all $y \in E$ such that $y \geq x^+$, and so $\|T^t x\| \leq \|Ty\|$ for all $y \in E$ such that $y \geq x^+$. It follows that

$$\|T^t x\| \leq \|T\| \inf_{y \geq x^+} \|y\| \leq \|T\| \|x^+\|_t \leq \|T\| \|x\|_t.$$

Then $\|T^t\| \leq \|T\|$. Since $B_E \subseteq B_{E^t}$, follows that $\|T\| \leqslant \|T^t\|$. Thus $\|T\| = \|T^t\|$, and proof follows.

3. In this part, we assume that x, y, z are members of E and x^t, y^t, z^t are members of E^t when there is not any confused. Now let $x^t \geq 0$. Since E is order dense in E^t, we have the following equalities

$$\begin{aligned} (T^t)^+(x^t) &= \sup_{0 \leq y^t \leq x^t} T^t y^t \\ &= \sup_{0 \leq y^t \leq x^t} \sup_{0 \leq z \leq y^t} T^t z \\ &= \sup_{0 \leq y \leq x^t} Ty \\ &= \sup_{0 \leq z \leq x^t} \sup_{0 \leq y \leq z} Ty \\ &= \sup_{0 \leq z \leq x^t} T^+ z \\ &= (T^+)^t(x^t). \end{aligned}$$

Similarly, we have $(T^t)^-(x^t) = (T^-)^t(x^t)$ for all $x^t \geq 0$. It is obvious that for each $x^t \in E^t$, we have $(T^t)^+ x^t = (T^+)^t(x^t)$ and $(T^t)^- x^t = (T^-)^t(x^t)$. Thus

$$|T|^t = (T^+ + T^-)^t = (T^+)^t + (T^-)^t = (T^t)^+ + (T^t)^- = |T^t|.$$

4. By using the equality $T \vee S = \frac{1}{2}(T + S + |T - S|)$ and part (3), proof follows.
5. First let $0 \leq x \in E^t$. By Lemma 1, there exists $\{x_n\}$ in E^+ such that $x_n \uparrow x^+$ and $\|x^+ - x_n\|_t \to 0$. Since $S^+x_n \uparrow$ and $\|x^+ - x_n\| \to 0$, follows that $S^+x_n \uparrow S^+x$. We have $T = S|_E$ (restriction of S on E), which follows that $T^- = S^-|_E$ and $T^+ = S^+|_E$. Obviously $(T^-)^t = S^-$ and $(T^+)^t = S^+$, and so by part (3), we have the following equalities

$$S = S^+ - S^- = (T^+)^t - (T^-)^t = (T^t)^+ - (T^t)^- = T^t.$$

Thus $S = T^t$ on E^- and E^+, which follows that

$$Sx = Sx^+ - Sx^- = T^t x^+ - T^t x^- = T^t x,$$

for each $x \in E^t$.

6. Consider $a, b \in (E^t)^+$ and $a < b$. By Lemma 1, take $\{x_n\}$ and $\{y_n\}$ in E^+ such that $x_n \uparrow a$, $y_n \downarrow b$, $\|a - x_n\|_t \to 0$ and $\|y_n - b\|_t \to 0$. Since E has order continuous norm, $[x_n, y_n] \cap E$ is $\sigma(E, E')$-compact subset of E for each $n \in \mathbb{N}$. It follows that $[a, b] \cap E$ is $\sigma(E, E')$-compact subset of E. Now, if we set
$$V = \{s \in E : \ x'(s) < r \text{ and } x' \in E'\},$$
then by using part (5), the order density of V is
$$V^t = \{s \in E^t : \ (x')^t(s) < r \text{ and } (x')^t \in (E^t)'\}.$$
It is obvious that $V \subseteq V^t$, and so $\sigma(E, E') \subseteq \sigma(E^t, (E^t)')$. Since $[a, b] \cap E$ is order dense in $[a, b]$, follows that $[a, b]$ is $\sigma(E^t, (E^t)')$-compact subset of E^t.

In the following, we examine some properties of the operator T^t, and we demonstrate that T^t preserves certain lattice and topological properties when these properties hold for T.

Theorem 6 *Let $0 \le T \in \mathcal{L}_n(E, F)$. Then we have the following assertions*

1. *If $0 \le x \le E^t$ and $\{x_\alpha\} \subseteq E^+$ with $x_\alpha \downarrow x$, then $Tx_\alpha \downarrow T^t x$.*
2. *If $T(x \wedge y) = Tx \wedge Ty$ for each $0 \le x, y \in E$, then T^t is a lattice homomorphism from E^t into F and moreover $T^t \in \mathcal{L}_n(E^t, F)$.*
3. *If $0 \le T : E \to E$ is a band-preserving operator, then $T^t : E^t \to E^t$ is also band-preserving.*
4. *If $T : E \to F$ is an order bounded operator that preserves disjointness, then $T^t : E^t \to F$ also preserves disjointness.*
5. *Suppose E has an order continuous norm. Then $\{Tx_n\}$ is norm convergent in F for every positive increasing norm-bounded sequence $\{x_n\}$ in E if and only if $\{T^t x_n\}$ is norm convergent in F for every positive increasing t-norm-bounded sequence $\{x_n\}$ in E^t.*

Proof 1. Let $\{x_\alpha\} \subseteq E^+$ such that $x_\alpha \downarrow x$. If $y \in E^+$ such that $x \le y$, then $y \vee x_\alpha \downarrow y$ holds in E, and so by order continuity of $T : E \to F$ and Theorem 4 (3), we see that
$$Ty = \inf\{T(x_\alpha \vee y)\} \le \inf Tx_\alpha \le T^t x.$$
This easily implies that $Tx_\alpha \downarrow T^t x$.

2. Assume that $0 \le x, y \in E^t$. We prove that $T^t(x \wedge y) = T^t x \wedge T^t y$. By [[2], Theorem 1.34], there are $\{x_\alpha\}$ and $\{y_\beta\}$ of E^+ such that $x_\alpha \downarrow x$ and $y_\beta \downarrow y$. It follows that $x_\alpha \wedge y_\beta \downarrow x \wedge y$. Then by order continuity of $T : E \to F$ and Theorem 4 (3), we have the following equalities,
$$\begin{aligned} T^t(x \wedge y) &= \inf\{T(x_\alpha \wedge y_\beta)\} = \inf\{T(x_\alpha) \wedge T(y_\beta)\} \\ &= \inf\{T(x_\alpha)\} \wedge \inf\{T(y_\beta)\} = T^t x \wedge T^t y. \end{aligned}$$

By combining Theorem 1.10 and Theorem 2.14 from [2] with Theorem 3, we can conclude that the mapping $T^t : (E^t)^+ \to (F^t)^+$ has a unique extension $T^t : (E^t) \to (F^t)$, which is a lattice homomorphism. Now, we will show that $T^t \in \mathcal{L}_n(E^t, F)$. Let $\{x_\alpha\} \subseteq (E^t)^+$ be such that $x_\alpha \downarrow 0$. Put

$$A = \{y \in E^+ :\ \exists \alpha \text{ such that } x_\alpha \leq y\}.$$

Since E majorizes E^t, it follows that A is not empty. By using Theorem 5 since T is positive, T^t is positive. Thus $\inf T(A) \geq \inf T^t x_\alpha \geq 0$ holds in F. Since $A \downarrow 0$ and $T \in \mathcal{L}_n(E, F)$, it follows that $\inf T(A) = 0$, and so $T^t x_\alpha \downarrow 0$.

3. Let $x, y \in E^t$ satisfying $|x| \wedge |y| = 0$. Assume that $(x_\alpha), (y_\beta) \subseteq E^+$ such that $x_\alpha \uparrow |x|$ and $y_\beta \uparrow |y|$. It follows that $(x_\alpha \wedge y_\beta) \uparrow |x| \wedge |y| = 0$, and so $x_\alpha \wedge y_\beta = 0$, by [[2], Theorem 2.36], follows that $|Tx_\alpha| \wedge y_\beta = 0$ for each α and β. Since $|Tx_\alpha| \wedge y_\beta \uparrow |Tx| \wedge |y|$, we have $|Tx| \perp |y|$, and so by another using [[2], Theorem 2.36], proof follows.
4. Let $x, y \in E^t$ satisfying $x \perp y$. Assume that $(x_\alpha), (y_\beta) \subseteq E^+$ such that $x_\alpha \uparrow |x|$ and $y_\beta \uparrow |y|$. It follows that $(x_\alpha \wedge y_\beta) \uparrow |x| \wedge |y| = 0$. Now since T preserve disjointness, follows that $Tx_\alpha \perp Tx_\beta$. From our hypothesis, we have $Tx_\alpha \wedge Tx_\beta \uparrow T^t|x| \wedge T^t|y|$ which follows that $T^t|x| \wedge T^t|y| = 0$. Since $|T^t x| \wedge |T^t y| \leq T^t|x| \wedge T^t|y|$, we have $T^t x \perp T^t y$.
5. Since $T = T^+ - T^-$, without loss generality, we assume that T is a positive operator. Assume that $\{x_n\} \subseteq (E^t)^+$ is increasing sequence with $\sup \|x_n\|_t < +\infty$. By using Lemma 1, for each $n \in \mathbb{N}$, there are positive increasing sequences $\{x_{n,m}\}_m$ with $x_{n,m} \uparrow_m x_n$ and $\|x_n - x_{n,m}\|_t \to 0$. Take $y_n = \bigvee_{i,j=1}^n x_{i,j}$. It follows that $0 \leq y_n \uparrow$ and

$$\sup \|y_n\| \leq \sup_{i,j} \|x_{i,j}\| \leq \sup \|x_n\| < +\infty.$$

By assumption there is $s^* \in F$ such that $\|Ty_n - s^*\| \to 0$. Then by using [[2], Theorem 2.46], $Ty_n \uparrow s^*$. By Theorem 5, we know that T^t is norm continuous from E^t into F. It follows that $\|T^t x_n - Tx_{n,m}\| \xrightarrow{m} 0$ holds in F. The inequality $Tx_{n,m} \leq Ty_n \leq T^t x_n$ implies that

$$\|T^t x_n - s^*\| \leq \|T^t x_n - Tx_{n,m}\| \text{ for each } n, m \in \mathbb{N}.$$

Then

$$\|T^t x_n - s^*\| \leq \|T^t x_n - Ty_n\| + \|Ty_n - s^*\| \to 0.$$

Thus $T^t x_n \to s^*$, and the proof follows.
The converse is straightforward.

References

1. C. D. Aliprantis, O. Burkinshaw, Locally Solid Riesz Spaces with Application to Economics, Mathematical Surveys, American Mathematical Society, Providence, RI, 2003.
2. C. D. Aliprantis, O. Burkinshaw, Positive Operators, Springer, Berlin, 2006.
3. P. Meyer-Nieberg, Banach lattices, Universitex. Springer, Berlin. MR1128093, 1991.
4. O. van Gaans Seminorms on ordered vector space that extend to Riesz seminorms on large spaces, Indag Mathem, N8, 14(1), 15–30 (2003).