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Analysis of Stability for Time-Invariant Linear Systems with Interval Coefficients

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Abstract In this paper, the stability of time-invariant (continuous-time) free linear system with interval coefficients is researched. After the introduction of parametric representation for intervals and subsequently the extension of this representation to interval matrices, stability with the concept of Lyapunov is discussed and investigated. The most important result of this idea, is the ability of checking stability without considering some constraints on the system. By presenting several examples, the stability of these systems, is researched by using the expressed approach.

Keywords Asymptotically stability · Marginal stability · Lyapunov function · Sylvester criterion

1 Introduction

System stability is a very important feature in the system and is considered an effective and decisive factor in the design of a system. In a stable system, by disturbing the system components, the changes created do not disrupt the system's performance, and therefore, examining this factor is very vital for any system. Stability in control theory, especially in optimal control problems, is a crucial topic of discussion. System instability is not considered a

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desirable feature for the system [1]-[6]. In most research on the stability of linear systems, researchers typically focus on specifying the coefficients of system equations. The presence of indeterminate coefficients and the use of real number intervals in these equations have prompted discussions among researchers regarding the behavioral characteristics of these systems. This is due to the extensive applications and broad scope of scientific issues in this field. Xian et al [7] examined the stability of large scale linear systems with interval coefficients. They initially decomposed these systems to several subsystems, and then studied the stability of them using the similarity principle and the vector Lyapunov function. Kaczorek [8] researched on positive linear systems that variables take positive values for positive initial conditions and positive inputs. He has conducted research on the positivity and stability of linear and nonlinear systems [9]-[13] and in [14], examined the stability of positive time-invariant (continuous-time) free linear systems with interval coefficients. Xian et al. [7] focused on the stability analysis of large-scale linear systems with interval coefficients by decomposing them into subsystems and applying the similarity principle and vector Lyapunov function. On the other hand, Kaczorek [8] studied positive linear systems where variables are constrained to positive values for positive initial conditions and inputs. He has also delved into the positivity and stability of both linear and nonlinear systems in various works [9]-[13], and specifically investigated the stability of positive time-invariant (continuous-time) free linear systems with interval coefficients in [14]. These research efforts contribute significantly to understanding the behavior and stability of systems with specific characteristics.

The motivation behind this paper lies in the necessity to study systems where state variables may not be strictly positive values, as in the case of positive linear systems. The primary objective is to analyze the stability of time-invariant (continuous-time) free linear systems with interval coefficients, without decomposing them into related subsystems.

The structure of the paper is organized as follows: In the second section, the parametric representation of interval numbers is introduced, and the formulation of time-invariant (continuous-time) linear systems with interval coefficients is presented. Section 3 defines the equilibrium point of these systems and explores their stability using this concept. Section 4 introduces a Lyapunov function with interval coefficients and investigates stability by utilizing this function in conjunction with the Sylvester criterion.

2 Initial concepts and statement of problem

In this section, we begin by presenting the parametric representation of interval numbers, initially introduced by Bhurjee and Panda in [15]. Let us regard the interval number as $\mathsf{X} = [x^1, x^2]$, so that $x^1, x^2 \in \mathbb{R}$. Each real number $x \in \mathsf{X} = [x^1, x^2]$, can be shown in a parametric form as ([15])

$$x(\lambda) = x^1 + \lambda(x^2 - x^1),\ 0 \le \lambda \le 1.$$

Suppose $\mathsf{Y} = [y^1, y^2] = \{y(\lambda_2)|\lambda_2 \in [0,1]\}$ is another interval number. The algebraic operations on interval numbers are expressed using their parametric representation as follows:

- $\mathsf{X} \oplus \mathsf{Y} = \{x(\lambda_1) + y(\lambda_2)|\lambda_1, \lambda_2 \in [0,1]\}$,
- $\mathsf{X} \ominus \mathsf{Y} = \{x(\lambda_1) - y(\lambda_2)|\lambda_1, \lambda_2 \in [0,1]\}$,
- $\mathsf{X} \odot \mathsf{Y} = \{x(\lambda_1).y(\lambda_2)|\lambda_1, \lambda_2 \in [0,1]\}$,
- $\mathsf{X} = \{kx(\lambda)|\lambda \in [0,1]\}$,
- $\mathsf{X} \oslash \mathsf{Y} = \{x(\lambda_1) \diagup y(\lambda_2)|\lambda_1, \lambda_2 \in [0,1], y(\lambda_2) \neq 0\}$.

Definition 1 Let $\mathcal{S}$ be a matrix. If all elements of the matrix are interval numbers then $\mathcal{S}$ is called an interval matrix.

Consider the following notations:

- $I(\mathbb{R})$ = The set of all interval numbers,
- $I(\mathbb{R})^n$ = The product space $I(\mathbb{R}) \times I(\mathbb{R}) \times \cdots \times I(\mathbb{R})$,
- $I(\mathbb{R})^{m\times n}$ = The set of all interval matrices $\mathcal{S}$ with m rows and n columns.
- $J[0,1]^{m\times n}$ = The set of all real matrices with m rows and n columns such that all elements of these matrices belong to $[0,1]$.

Definition 2 Let $S = [s_{ij}]_{m\times n}$ and $\mathcal{S} = [\mathsf{S}_{ij}]_{m\times n} \in I(\mathbb{R})^{m\times n}$ are respectively real and interval matrices, where $\mathsf{S}_{ij} = [s^0_{ij}, s^1_{ij}]$. $S \in \mathcal{S}$ if and only if $s_{ij} \in \mathsf{S}_{ij}$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Proposition 1 *An interval matrix $\mathcal{S}$ is presented by the infinite set of real matrices, i.e.*

$$\mathcal{S} = \{S_\Lambda | S_\Lambda = [s_{ij}(\lambda_{ij})]_{m\times n}\ ,\ \Lambda = [\lambda_{ij}]_{m\times n} \in J[0,1]^{m\times n},$$
$$s_{ij}(\lambda_{ij}) = s^0_{ij} + \lambda_{ij}(s^1_{ij} - s^0_{ij})\ \ i = 1, \ldots, m\ ,\ j = 1, \ldots, n\}.$$

Proof Suppose $S \in \mathcal{S}$ that, $S = [s_{ij}]_{m\times n}$, then

$$s_{ij} \in \mathsf{S}_{ij}\ ,\ i = 1, \ldots, m,\ j = 1, \ldots, n.$$

There exists $\lambda_{ij} \in [0,1]$ such that $s_{ij} = s^0_{ij} + \lambda_{ij}(s^1_{ij} - s^0_{ij})$. In this case, for each element of the matrix S, there exists a real number

$$\lambda_{ij} \in [0,1]\ ,\ i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}.$$

Consider the matrix $\Lambda = [\lambda_{ij}]_{m\times n}\ ,\ \lambda_{ij} \in [0,1]$ and suppose

$$s_{ij} = s^0_{ij} + \lambda_{ij}(s^1_{ij} - s^0_{ij}) = s_{ij}(\lambda_{ij}),$$

for $i = 1, \ldots, m\ ,\ j = 1, \ldots, n$. Then, $S = S_\Lambda$. Therefore, the real matrix S is a member of the infinite set of real matrices.
Now, if a real matrix $S = [s_{ij}]_{m\times n}$ belongs to the infinite set of real matrices, then, there is a real matrix $\Lambda = [\lambda_{ij}]_{m\times n}\ ,\ \lambda_{ij} \in [0,1]$ such that $S = S_\Lambda$ and

$$s_{ij} = s_{ij}(\lambda_{ij}) = s^0_{ij} + \lambda_{ij}(s^1_{ij} - s^0_{ij})\ ,\ \lambda_{ij} \in [0,1]\ ,\ i = 1, \ldots, m\ ,\ j = 1, \ldots, n.$$

Then $s_{ij} \in \mathsf{S}_{ij}\ ,\ i = 1, \cdots, m\ ,\ j = 1, \cdots, n$, and so $S \in \mathcal{S}$.

The key benefit of defining an interval matrix in the above proposition is that it illustrates all real matrices that fall within the interval matrix.

Definition 3 Let $\mathcal{S}$ be an interval matrix. If the real matrix $S_\Lambda \in \mathcal{S}$ is positive definite for all real matrices $\Lambda \in J[0,1]^{n\times n}$, then interval matrix $\mathcal{S}$ is called positive definite.

Definition 4 Let in the interval matrix $\mathcal{S}$, there is a real matrix $\Lambda_1 \in J[0,1]^{n\times n}$ such that the real matrix S_{Λ_1} is semi-positive definite and the real matrix S_Λ is positive definite or semi-positive definite for all real matrices $\Lambda \in J[0,1]^{n\times n}$, $\Lambda \neq \Lambda_1$, then the interval matrix $\mathcal{S}$ is called semi-positive definite.

Consider the time-invariant (continuous-time) free linear system with interval coefficients and by the initial state $x(t_0)$ as follows:

$$\dot{x}(t) = \mathcal{A}x(t), \tag{1}$$

where $\mathcal{A} = \{A_\Lambda \| \Lambda \in J[0,1]^{n\times n}\}$ is an interval matrix in $(I(\mathbb{R}))^{n\times n}$. Now, consider the following system

$$\dot{x}(t) = A_\Lambda x(t) \tag{2}$$

The system (2) has the similar structure with the system (1).

$$\phi(t,t_0) = e^{A_\Lambda(t-t_0)},$$

is the transition matrix of the system (2) and also this system has a unique solution as $x(t) = \phi(t,t_0)x(t_0)$, [16].

3 Equilibrium point and Stability

In this section, the equilibrium point for the system (1) is defined and then its stability is examined.
Let the initial-value problem $\dot{x}(t) = A_\Lambda x(t), x(t_0) = x_1,\ x_1 \in \mathbb{R}^n$ for all values $t \geq t_0$ has a unique solution as $x(t) = x_1$, x_1 is the equilibrium point of the system (2). In other words, whenever for all values $t \geq t_0$, $(I - e^{A_\Lambda(t-t_0)})x_1 = 0$ then, x_1 is an equilibrium point of the system (2), [16].

Definition 5 If x_1 is the equilibrium point of the system (2) for all real matrices $A_\Lambda \in \mathcal{A}$, $\Lambda \in J[0,1]^{n\times n}$ then x_1 is called the equilibrium point of the system (1).

Example 1 Consider the interval matrix $\mathcal{A}$ in the system (1) as

$$\mathcal{A} = \begin{bmatrix} [-1,2] & 0 \\ 0 & [-5,3] \end{bmatrix}.$$

For each real matrix $\Lambda \in J[0,1]^{2\times 2}$,

$$A_\Lambda = \begin{bmatrix} -1+3\lambda_{11} & 0 \\ 0 & -5+8\lambda_{22} \end{bmatrix} \in \mathcal{A},$$

and

$$e^{A_\Lambda(t-t_0)} = \begin{bmatrix} e^{(-1+3\lambda_{11})(t-t_0)} & 0 \\ 0 & e^{(-5+8\lambda_{22})(t-t_0)} \end{bmatrix}.$$

The transition matrix, $e^{A_\Lambda(t-t_0)}$ is nonsingular for each real matrix $\Lambda \in J[0,1]^{2\times 2}$ and all values $t > t_0$. Therefore, origin is unique equilibrium point for the system (1).

Definition 6 Let x_1 be equilibrium point of the system (2). Whenever

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall t > t_0 :\ \|x(t_0) - x_1\| < \delta \implies \|e^{A_\Lambda(t-t_0)}x(t_0) - x_1\| < \varepsilon,$$

that $\|\ \|$ is a norm, the system (2) is called marginal stable about an equilibrium point x_1. If

$$\lim_{t\to\infty} e^{A_\Lambda(t-t_0)}x(t_0) = x_1,$$

then the system (2) is called asymptotically stable about equilibrium point x_1.

Definition 7 The system (2) is called unstable about the point x_1, if it is not marginal stable about this point.

Definition 8 Let x_1 be an equilibrium point of the system (1). The system (2) is asymptotically stable about x_1 that A_Λ is the coefficient matrix of the system for all real matrices $\Lambda \in J[0,1]^{n\times n}$, then the system (1) is called asymptotically stable about the equilibrium point x_1.

Definition 9 Let x_1 be an equilibrium point of the system (1). The system (2) is marginal stable about the point x_1 if there is a real matrix $\Lambda_1 \in J[0,1]^{n\times n}$ that A_{Λ_1} is the coefficient matrix of the system (2) and the system (2) is marginal or asymptotically stable about x_1 that the real matrix A_Λ is the coefficient matrix of the system (2) for all real matrices $\Lambda \in J[0,1]^{n\times n}$, $\Lambda \neq \Lambda_1$, then the system (1) is marginal stable about the equilibrium point x_1.

Definition 10 Let x_1 be an equilibrium point of the system (1). The system (2) is unstable about the point x_1 that the real matrix A_{Λ_2} , $\Lambda_2 \in J[0,1]^{n\times n}$ is the coefficient matrix of the system (2) then the system (1) is unstable about the point x_1.

Using proportional changes and for each real matrix A_Λ that $\Lambda \in J[0,1]^{n\times n}$, origin is the equilibrium point of the system (2) and then it is the equilibrium point of the system (1), [16].

Example 2 In example 1, origin is a unique equilibrium point of the system. Let position $x(t_0) = (\delta_1, \delta_2)^T \in \mathbb{R}^2$ be initial state. For each real matrix $\varLambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \in J[0,1]^{2\times 2}$,

$$x(t) = e^{A_\varLambda(t-t_0)}x(t_0) = (\delta_1 e^{(-1+3\lambda_{11})(t-t_0)}, \delta_2 e^{(-5+8\lambda_{22})(t-t_0)})^T.$$

If $\lambda_{11} < \frac{1}{3}$ and $\lambda_{22} < \frac{5}{8}$ then

$$\lim_{t\to+\infty} \parallel x \parallel = 0.$$

The system (2) is asymptotically stable for real matrix $A_\varLambda$ that

$$\varLambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \in J[0,1]^{2\times 2},$$

and $\lambda_{11} < \frac{1}{3}$ and $\lambda_{22} < \frac{5}{8}$. If $\lambda_{11} = \frac{1}{3}$ and $\lambda_{22} = \frac{5}{8}$ then $x(t) = (\delta_1, \delta_2)^T$. Let $\delta \leqslant \varepsilon$, $\|x(t)\| < \varepsilon$ for all sufficiently large t whenever $\|x(t_0)\| < \delta$. The system (2) is only marginal stable for real matrix $A_\varLambda$ that

$$\varLambda = \begin{bmatrix} \frac{1}{2} & \lambda_{12} \\ \lambda_{21} & \frac{5}{8} \end{bmatrix} \in J[0,1]^{2\times 2},$$

and it is not asymptotically stable. The system (2) is unstable for real matrix $A_\varLambda$ that

$$\varLambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \in J[0,1]^{2\times 2},$$

and $\lambda_{11} > \frac{1}{3}$ or $\lambda_{22} > \frac{5}{8}$. Then, there is real matrix $A_\varLambda$ such that the system (2) is unstable. Therefore, the system (1) is unstable that the interval matrix is

$$\mathcal{A} = \begin{bmatrix} [-1,2] & 0 \\ 0 & [-5,3] \end{bmatrix}.$$

4 Examining stability using Lyapunov function

In this section, a Lyapunov function with interval coefficients is introduced, followed by an investigation into the stability of the system (1) using this function.

Definition 11 $\mathcal{V}(x)$ is a multivariate polynomial with interval coefficients that $\mathcal{V}(x) = x\mathcal{S}x^{'}$ and $\mathcal{V}(x) = \{v_\varLambda(x) = xS_\varLambda x^{'} | v_\varLambda : \mathbb{R}^n \longrightarrow \mathbb{R}, \varLambda \in J[0,1]^{n\times n}\}$, then $\mathcal{V}(x)$ is called a Quadratic function with interval coefficients.

Definition 12 $\dot{\mathcal{V}}(x) = \{\dot{v_\varLambda}(x) | \varLambda \in J[0,1]^{n\times n}\}$ is called the derivative of the Quadratic function with interval coefficients.

Definition 13 If $\mathcal{V}(x)$, the Quadratic function with interval coefficients, is positive definite and the derivative of it, is negative definite, then $\mathcal{V}(x)$ is a Lyapunov function with interval coefficients.

Proposition 2 *Let origin is an equilibrium point of the system (1). If a Lyapunov function with interval coefficients is in the neighborhood of origin, then origin is asymptotically stable. If the derivative of the Lyapunov function is only semi-negative definite, then origin is marginal stable.*

Proof There is a Lyapunov function with interval coefficients such as $\mathcal{V}(x)$ in the neighborhood of origin. Let $\mathcal{V}(x)$ be a quadratic Lyapunov function with interval coefficients. For each real matrix $\Lambda \in J[0,1]^{n\times n}$, $v_\Lambda(x)$ is a quadratic Lyapunov function in the neighborhood of origin and origin has asymptotically stable. Then origin is an equilibrium point with asymptotically stable for the system (2) that A_Λ is coefficient matrix for all real matrices $\Lambda \in J[0,1]^{n\times n}$ [17]. Therefore, origin is a equilibrium point with asymptotically stable for the system(1). Let $\dot{\mathcal{V}}(x)$ be semi-negative definite. There is a real matrix $\Lambda_1 \in J[0,1]^{n\times n}$ that $\dot{v_{\Lambda_1}}(x)$ is semi-negative definite and for each real matrix $\Lambda \in J[0,1]^{n\times n}$, $\Lambda \neq \Lambda_1$, $\dot{v_\Lambda}(x)$ is negative definite or semi-negative definite. Origin is marginal stable for the system $\dot{x} = A_{\Lambda_1}(x)$ and is marginal or asymptotically stable for the system $\dot{x} = A_\Lambda(x)$ that $\Lambda \in J[0,1]^{n\times n}$, $\Lambda \neq \Lambda_1$[17]. So origin is marginal stable for the system (1).

For being asymptotically stable of the system (1), should take a interval symmetric matrix $\mathcal{P}$ such that $\mathcal{V}(x) = x^{'}\mathcal{P}x$ is a positive definite quadratic function with interval coefficients and $\dot{V}(x)$ be negative definite. For each real matrix $\Lambda \in J[0,1]^{n\times n}$ that $P_\Lambda \in \mathcal{P}$, should take a positive definite quadratic function with real coefficients as $v_\Lambda(x) = x^{'}P_\Lambda x$ that $\dot{v}_\Lambda(x)$ be negative definite. To differentiate from quadratic function $v_\Lambda(x) = x^{'}P_\Lambda x$ and $\dot{v}_\Lambda(x)$ should be negative definite and I is positive definite matrix then the matrix equation

$$A_\Lambda^{'} P_\Lambda + P_\Lambda A_\Lambda = -I, \tag{3}$$

is obtained.

Example 3 In example 2, the stability of the system (1) with the interval coefficients

$$\mathcal{A} = \begin{bmatrix} [-1,2] & 0 \\ 0 & [-5,3] \end{bmatrix},$$

was examined. Now the stability is investigated using Lyapunov function. Let

$$P_\Lambda = \begin{bmatrix} p_{11} \; p_{12} \\ p_{12} \; p_{22} \end{bmatrix},$$

and from (3),

$$p_{11} = \frac{-1}{2(-1+3\lambda_{11})}, \quad p_{22} = \frac{-1}{2(-5+8\lambda_{22})}, \quad p_{12} = 0.$$

For being positive definite of the matrix P_Λ, it uses of Sylvester criterion [17] and then for this purpose $0 < \lambda_{11} < \frac{1}{3}$, $0 < \lambda_{22} < \frac{5}{8}$. There are some matrices that $\Lambda \in J[0,1]^{2\times 2}$, the system (2) is not asymptotically stable. So, the system (1) is not asymptotically stable. For $\lambda_{11} \geqslant \frac{1}{3}$ or $\lambda_{22} \geqslant \frac{5}{8}$, the nonzero elements of the matrix P_Λ are either unlimited or negative, and in this state, the matrix P_Λ is not semi-negative definite. The system (1) is not marginal stable and so it is unstable.

Example 4 In the system (1), let the coefficient matrix be

$$\mathcal{A} = \begin{bmatrix} [0,1] & 0 \\ [0,2] & 5 \end{bmatrix}.$$

The real matrix

$$A_\Lambda = \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 5 \end{bmatrix},$$

belongs to $\mathcal{A}$ for each real matrix $\Lambda \in J[0,1]^{2\times 2}$.

$$A_\Lambda^n = \begin{bmatrix} \lambda_{11}^n & 0 \\ a_{21} & 5^n \end{bmatrix},$$

such that

$$a_{21} = 2\lambda_{21}(\lambda_{11}^{n-2}(\lambda_{11}+5) + \sum_{k=1}^{n-3} \lambda_{11}^{n-3-k} 5^{k+1} + 5^{n-1}).$$

$e^{A_\Lambda(t-t_0)}$ is the transition matrix of the system (2) and the solution of the system is $x(t) = e^{A_\Lambda(t-t_0)}x(t_0)$. Regarding the exponent of the matrix A_Λ,

$$\lim_{x\to+\infty} \|x(t)\| = +\infty,$$

and the system (2) is unstable for each real matrix $\Lambda \in J[0,1]^{2\times 2}$ and so the system (1) is unstable. For examining stability of the system (1) using Lyapunov function from (3), the element of the matrix P_Λ are

$$p_{11} = -\frac{2\lambda_{21}^2}{5\lambda_{11}(5+\lambda_{11})} - \frac{1}{2\lambda_{11}}, \quad p_{12} = \frac{\lambda_{21}}{5(5+\lambda_{11})}, \quad p_{22} = -\frac{1}{10}.$$

The real matrix P_Λ is not positive definite for each real matrix $\Lambda \in J[0,1]^{2\times 2}$ and the system (2) is not asymptotically stable that each real matrix $A_\Lambda, \Lambda \in J[0,1]^{2\times 2}$ is the coefficient matrix. Then the system (1) is not asymptotically stable. No positive definite matrix as P_Λ is obtained from (3) for each real matrix $\Lambda \in J[0,1]^{2\times 2}$ and so the system (1) is not asymptotically stable.

Example 5 Let the coefficient matrix of the system (1) be

$$\mathcal{A} = \begin{bmatrix} [-3,-2] & [1,2] \\ [2,4] & [-4,-3] \end{bmatrix}.$$

If the system is positive, Kaczorek illustrated that the system is asymptotically stable [14]. Now there is no constraint on the system and the stability of the system is examined. There exists the real matrix

$$A_\Lambda = \begin{bmatrix} -3+\lambda_{11} & 1+\lambda_{12} \\ 2+2\lambda_{21} & -4+\lambda_{22} \end{bmatrix} \in \mathcal{A}.$$

The system of equation that is obtained from (3), is as following:

$$\begin{cases} 2\,(-3+\lambda_{11})\,p_{11}+2\,(2+2\lambda_{21})\,p_{12}=-1 \\ (1+\lambda_{12})\,p_{11}+(-7+\lambda_{11}+\lambda_{22})\,p_{12}+(2+2\lambda_{21})\,p_{22}=0 \\ 2\,(1+\lambda_{12})\,p_{12}+2\,(-4+\lambda_{22})\,p_{22}=-1 \end{cases} \tag{4}$$

If the system of equations (4) possesses a unique solution while satisfying the Sylvester criterion, then the system is asymptotically stable. The determinant of the coefficient matrix ranges from negative to positive values, the system of equations may have multiple solutions for certain parameter values, $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$. So the system is not asymptotically stable.

Example 6 Consider the free linear system with interval coefficients as:

$$\begin{cases} \dot{x}_1 = [-7,-5]x_1 \\ \dot{x}_2 = [2,3]x_2 \\ \dot{x}_3 = [-2,-1]x_3 \\ \dot{x}_4 = [-15,-11]x_1 + [-1,1]x_2 + [-9,-6]x_3 + [-7,-4]x_4 \end{cases} \tag{5}$$

The coefficient matrix of the system is

$$\mathcal{A} = \begin{bmatrix} [-7,-5] & 0 & 0 & 0 \\ 0 & [2,3] & 0 & 0 \\ 0 & 0 & [-2,-1] & 0 \end{bmatrix},$$

$$A_\Lambda = \begin{bmatrix} -7+2\lambda_{11} & 0 & 0 & 0 \\ 0 & 2+\lambda_{22} & 0 & 0 \\ 0 & 0 & -2+\lambda_{33} & 0 \\ -15+4\lambda_{41} & -1+2\lambda_{42} & -9+3\lambda_{43} & -7+3\lambda_{44} \end{bmatrix} \in \mathcal{A},$$

for all real matrices $\Lambda \in J[0,1]^{4\times 4}$. The system of equation that obtain from (3), has a coefficient matrix and its determinant is as following:

$$8(-7+2\lambda_{11})(-5+2\lambda_{11}+\lambda_{22})(-9+2\lambda_{11}+\lambda_{33})(-14+2\lambda_{11}+3\lambda_{44})$$
$$(2+\lambda_{22})(\lambda_{22}+\lambda_{33})(-5+\lambda_{22}+3\lambda_{44})(-2+\lambda_{33})(-9+\lambda_{33}+3\lambda_{44})(-7+3\lambda_{44}).$$

If λ_{22} and λ_{33} are not equal to zero then the determinant is not equal to zero. So, the system of equation that obtains from (3), has a unique solution and it is as following:

$$p_{11} = -\frac{(-15+4\lambda_{41})^2}{2(-7+2\lambda_{11})(-14+2\lambda_{11}+3\lambda_{44})(-7+3\lambda_{44})} - \frac{1}{2(-7+2\lambda_{11})},$$

$$p_{12} = \frac{-(-1+2\lambda_{42})(-15+4\lambda_{41})}{2(-5+2\lambda_{11}+\lambda_{22})(-14+2\lambda_{11}+3\lambda_{44})(-7+3\lambda_{44})} - \frac{(-15+4\lambda_{41})(-1+2\lambda_{42})}{2(-5+2\lambda_{11}+\lambda_{22})(-7+3\lambda_{44})(-5+\lambda_{22}+3\lambda_{44})},$$

$$p_{13} = \frac{-(-9+3\lambda_{43})(-15+4\lambda_{41})}{2(-9+2\lambda_{11}+\lambda_{33})(-14+2\lambda_{11}+3\lambda_{44})(-7+3\lambda_{44})} + \frac{(-15+4\lambda_{41})(-9+3\lambda_{43})}{2(-9+2\lambda_{11}+\lambda_{33})(-7+3\lambda_{44})(-9+\lambda_{33}+3\lambda_{44})},$$

$$p_{14} = \frac{-15+4\lambda_{41}}{2(-14+2\lambda_{11}+3\lambda_{44})(-7+3\lambda_{44})},$$

$$p_{22} = -\frac{(-1+2\lambda_{42})^2+(-5+\lambda_{22}+3\lambda_{44})(-7+3\lambda_{44})}{2(2+\lambda_{22})(-5+\lambda_{22}+3\lambda_{44})(-7+3\lambda_{44})},$$

$$p_{23} = -\frac{(-9+3\lambda_{43})(-1+2\lambda_{42})}{2(\lambda_{22}+\lambda_{33})(-5+\lambda_{22}+3\lambda_{44})(-7+3\lambda_{44})} - \frac{(-1+2\lambda_{42})(-9+3\lambda_{43})}{2(\lambda_{22}+\lambda_{33})(-9+\lambda_{33}+3\lambda_{44})(-7+3\lambda_{44})},$$

$$p_{24} = \frac{-1+2\lambda_{42}}{2(-5+\lambda_{22}+3\lambda_{44})(-7+3\lambda_{44})},$$

$$p_{33} = \frac{-(-9+3\lambda_{43})^2-(-9+\lambda_{33}+3\lambda_{44})(-7+3\lambda_{44})}{2(-2+\lambda_{33})(-9+\lambda_{33}+3\lambda_{44})(-7+3\lambda_{44})},$$

$$p_{34} = \frac{-9+3\lambda_{43}}{2(-9+\lambda_{33}+3\lambda_{44})(-7+3\lambda_{44})},$$

$$p_{44} = \frac{-1}{2(-7+3\lambda_{44})}.$$

$p_{11} > 0$ and

$$\begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} = p_{11}p_{22} - p_{12}^2 < 0,$$

for all real matrices $\Lambda \in J[0,1]^{4\times 4}$ then the real matrix P_Λ is not positive definite for all real matrices $\Lambda \in J[0,1]^{4\times 4}$. So, the system is not asymptotically stable.

5 Conclusion

In this paper, we investigate the asymptotic and marginal stability of continuous-time free linear systems with interval coefficients using parameterization of interval numbers. Stability is analyzed through definitions and the application of a Lyapunov function with interval coefficients. If a Lyapunov function with interval coefficients cannot be constructed, the system cannot be deemed asymptotically stable, and its instability cannot be determined. If the matrix equation (3) is not consistent for all positive definite matrices, stability cannot be assessed using a Lyapunov function. Obtaining a Lyapunov function with interval coefficients allows for determining the asymptotic stability of the system (1), which yields more reliable results compared to other approaches. The investigation of the stability of non-free system with interval coefficients is left for future research endeavors.

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