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Formulas for the Drazin Inverse of Anti-Triangular Matrices

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Abstract Let $\mathcal{A}$ be a Banach algebra. In this paper, for two Drazin invertible elements $a, b \in \mathcal{A}$, explicit formulas for the Drazin inverse $(a+b)$ are given in the cases of $a^2ba = 0$, $(ba)^2 = 0$ and $ab^2 = 0$. By using these formulas, the representations for the Drazin inverse of the anti-triangular operator matrices over Banach algebras are obtained, which also extend some existing results.

Keywords Banach algebra · Block matrix · Drazin inverse · Index · Operator matrix · Schur complement

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1 Introduction

Let $\mathcal{A}$ be a Banach algebra with an identity. As is well known, in 1958, Drazin [12] defined, an element $a \in \mathcal{A}$ has Drazin inverse if there is the element $x \in \mathcal{A}$ which satisfies

$$x \in comm(a),\ x = xax, \quad \text{and} \quad a - a^2x \in N(\mathcal{A}),$$

or

$$x \in comm(a),\ x = xax, \quad \text{and} \quad a^k = a^{k+1}x.$$

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The symbol $N(\mathcal{A})$ is the set of all nilpotent elements in $\mathcal{A}$ and the commutant of $a \in \mathcal{A}$ is defined by $comm(a) = \{x \in \mathcal{A}\ ;\ xa = ax\}$. The element x above is uniqe if it exists and is denoted by a^d and called the Drazin inverse of a. The smallest such nonnegetive integer k is called the Drazin index of a, denoted $ind(a)$. When $ind(a) = 1$, a^d is called the group inverse of a, denoted by $a^{\#}$ [13]. Clearly, if $ind(a) = 0$, then a is invertible and $a^d = a^{-1}$.

The Drazin inverse has many applications in singular differential equations and singular difference equations [3,6], Markov chains [16,21] and iterative methods [7]. In 1958, Drazin [12] gave the explicit formula of $(P+Q)^d$ in the case of $PQ = QP = 0$ for $P, Q \in C_{n\times n}$. In 2001, Hartwig et al. [17] gave a result of $(P+Q)^d$ when $PQ = 0$. In recent years, many papers focus on the problem under some weaker conditions. In 2011-2021, Yang and Liu [23], Bu et al. [2], Chen and Sheibani [9,11] and some other researchers gave the representation of $(P+Q)^d$ under the different conditions. This result was extended to bounded linear operators on an arbitrary complex Banach space by Djordjevic and Wei [14].

In section 2, the invertibility of the sum of two Drazin invertible elements in a Banach algebra under some conditions will be presented. We prove that for any $a, b \in \mathcal{A}^{\mathbf{D}}$, if $a^2ba = 0$, $(ba)^2 = 0$ and $ab^2 = 0$, then $a + b \in \mathcal{A}^{\mathbf{D}}$ and we give the explicit formula of $(a+b)^d$.

On the other hand, a problem of great interest in this algebras is concerned with the Drazin inverse of matrices partitioned as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.1}$$

where $A \in \mathcal{L}(X)^{\mathbf{D}}$, $B \in \mathcal{L}(X,Y)$, $C \in \mathcal{L}(Y,X)$ and $D \in \mathcal{L}(Y)^{\mathbf{D}}$ (A and D are square complex matrices but need not to be of the same size). Here, M is a bounded operator on $X \oplus Y$. It was posed as an open problem by Campbell and Meyer [6] in 1979, and it has received great attention. The most relevant case is concerned with block triangular matrices (either $B = 0$ or $C = 0$), solved by Meyer and Rose [20].

Otherwise, the representation of the Drazin inverse of an anti-triangular matrix M, where $A = 0$ or $D = 0$, was posed as an open problem by Campbell [3] in 1983, in relation with the solution of singular second-order differential equations. Furthermore, these structured matrices appear in applications like graph theory, saddle-point problems, and optimization problems [1,8,13]. In recent years, the problem has become an important issue and some results have been given under some conditions, but it still remains open.

In section 3, we focus on deriving formulas for the Drazin inverse of an anti-triangular matrises

$$N = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{N} = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix},$$

under some conditions. Then numerical examples are given to illustrate our results.

In section 4, we present some Drazin inverses for a 2×2 operator matrix M under a number of different conditions, which generalize [10], [24]. If $a \in \mathcal{A}$ has Drazin inverse a^d. The element $a^\pi = 1 - aa^d$ is called the spectral idempotent of a. In this section, we consider the Drazin inverse of a 2×2 operator matrix M under the perturbations on spectral idempotents. These also extends [15] to wider cases.

2 Main Results

In this section, we first present some lemmas without proof, then we give the formula of the Drazin inverse of $a + b$ under the conditions that $a^2ba = 0$, $(ba)^2 = 0$ and $ab^2 = 0$ which will be the main tool in our following development.

Lemma 1 *[5] Let $a, b \in \mathcal{A}$ and $ab \in \mathcal{A}^{\mathbf{D}}$. Then $(ba)^d = b\big((ab)^d\big)^2 a$.*

Lemma 2 *[17] Let $a, b \in \mathcal{A}^{\mathbf{D}}$ be such that $ind(a) = m$ and $ind(b) = n$. If $ab = 0$, then*

$$(a+b)^d = \sum_{i=0}^{n-1} b^\pi b^i (a^d)^{i+1} + \sum_{i=0}^{m-1} (b^d)^{i+1} a^i a^\pi.$$

Theorem 3 *Let $a, b \in \mathcal{A}^{\mathbf{D}}$. If $a^2ba = 0$, $(ba)^2 = 0$ and $ab^2 = 0$, then $a + b \in \mathcal{A}^{\mathbf{D}}$ and*

$$\begin{aligned}(a+b)^d = {} & b^d + (b^d)^2 a + \big[ab(a^d)^4 - (b^d)^2 aa^d - b^d a^d\big](a+b) \\ & + \Big[\sum_{i=0}^{k-1} b^\pi b^{2i}(I + ba^d)(a^d)^{2(i+1)} \\ & + \sum_{i=0}^{k-2} (b^d)^{2i+3}(a + b^d a^2) a^{2i} a^\pi\Big](a+b),\end{aligned}$$

where $k = \max\{ind(a^2) + 1, ind(b^2)\}$.

Proof From the definition of the Drazin inverse, we have that

$$(a+b)^d = (a+b)[(a+b)^d]^2 = (a+b)[a(a+b) + b(a+b)].$$

Denote by $F = a(a+b)$ and $G = b(a+b)$. Since $FG = 0$, matrices F and G satisfy the condition of Lemma 2 and therefore

$$(a+b)^d = (a+b)\Big[\sum_{i=0}^{ind(G)} G^\pi G^i (F^d)^{i+1} + \sum_{i=0}^{ind(F)} (G^d)^{i+1} F^i F^\pi\Big].$$

Applying Lemma 1, we have

$$F^d = [a(a+b)]^d = a[((a+b)a)^d]^2(a+b),$$

$$G^d = [b(a+b)]^d = b[((a+b)b)^d]^2(a+b).$$

If we denote by $F_1 = (a+b)a$ and $G_1 = (a+b)b$, we get

$$F^d = a(F_1^d)^2(a+b),$$

$$G^d = b(G_1^d)^2(a+b).$$

In addition,

$$(F^d)^2 = a(F_1^d)^2(a+b)a(F_1^d)^2(a+b) = a(F_1^d)^3(a+b),$$

$$(G^d)^2 = b(G_1^d)^2(a+b)b(G_1^d)^2(a+b) = b(G_1^d)^3(a+b).$$

and

$$(F^d)^i = a(F_1^d)^{i+1}(a+b),$$

$$(G^d)^i = b(G_1^d)^{i+1}(a+b),$$

for every $i \in \mathbb{N}$. After some computations we get

$$(a+b)^d = \Big[\sum_{i=0}^{ind(G_1)} G_1^{\pi} G_1^i (F_1^d)^{i+1} + \sum_{i=0}^{ind(F_1)} (G_1^d)^{i+1} F_1^i (F_1^d)^{\pi}\Big](a+b). \tag{2.1}$$

Now, we calculate F_1^d. Note that F_1 can be rewritten as the sum $F_1 = F_2 + F_3$, where $F_2 = a^2$ and $F_3 = ba$. We see that $F_3^2 = 0$ and $F_2F_3 = 0$. By Lemma 2,

$$(F_1^d)^i = (a^d)^{2i} + b(a^d)^{2i+1}, \tag{2.2}$$

for every $i \in \mathbb{N}$. Similarly, G_1 can be rewritten as the sum $G_1 = G_2 + G_3$, where $G_2 = ab$ and $G_3 = b^2$. We see that $G_2^3 = 0$ and $G_2G_3 = 0$. By Lemma 2,

$$(G_1^d)^i = (b^d)^{2i} + (b^d)^{2i+2}ab, \tag{2.3}$$

for every $i \in \mathbb{N}$. After computation we get

$$(F_1)^i = ((a+b)a)^i = a^{2i} + ba^{2i-1},$$

and

$$(G_1)^i = \begin{cases} (ab+b^2)^i, & i = 1, 2, \\ b^{2i-2}ab + b^{2i}, & i \geq 3, \end{cases}$$

for every $i \in \mathbb{N}$. By substituting equation (2.2) and equation (2.3) into equation (2.1), we complete the proof.

The next Theorem is a symmetrical formulation of Theorem 3.

Theorem 4 *Let $a, b \in \mathcal{A}^{\mathbf{D}}$. If $aba^2 = 0$, $(ab)^2 = 0$ and $b^2a = 0$, then $a + b \in \mathcal{A}^{\mathbf{D}}$ and*

$$\begin{aligned}(a+b)^d &= b^d + a(b^d)^2 + (a+b)\big[(a^d)^4ba - aa^d(b^d)^2 - a^db^d\big] \\ &\quad + (a+b)\big[\sum_{i=0}^{k-2} a^\pi a^{2i}(a + a^2b^d)(b^d)^{2i+3} \\ &\quad + \sum_{i=0}^{k-1}(a^d)^{2(i+1)}(I + a^db)b^{2i}b^\pi\big],\end{aligned}$$

where $k = \max\{ind(a^2) + 1, ind(b^2)\}$.

Corollary 5 *Let $a, b \in \mathcal{A}^{\mathbf{D}}$. If $aba = 0$ and $ab^2 = 0$, then*

$$(a+b)^d = \big[-a^db^d + \sum_{i=0}^{k-1} b^\pi b^i (a^d)^{i+2} + \sum_{i=0}^{k-1}(b^d)^{i+2}a^ia^\pi\big](a+b),$$

where $k = \max\{ind(a), ind(b)\}$.

Corollary 6 *Let $a, b \in \mathcal{A}^{\mathbf{D}}$. If $ab = 0$, then*

$$(a+b)^d = \sum_{i=0}^{k-1} b^\pi b^i (a^d)^{i+1} + \sum_{i=0}^{k-1}(b^d)^{i+1}a^ia^\pi,$$

where $k = \max\{ind(a), ind(b)\}$.

Corollary 7 *Let $a, b \in \mathcal{A}^{\mathbf{D}}$. If $ab = 0$ and $ba = 0$, then*

$$(a+b)^d = a^d + b^d.$$

3 Operator matrices

To illustrate the preceding results, we are concerned with the Drazin inverse for an operator matrix. Throughout this section, consider the anti-triangular operator matrices

$$N = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{N} = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}.$$

In following Theorems we focus our attention to give the expression of block matrices of N^d and $\mathcal{N}^d$ under some conditions, then using different splitting approach and Theorem 3, we will apply the computational formula to give the computational formulas for N^d and $\mathcal{N}^d$ and we give some examples.

Lemma 8 *[12] Let $A \in \mathcal{L}(X)^{\mathbf{D}}$ and $D \in \mathcal{L}(Y)^{\mathbf{D}}$. Then*

$$K = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \ L = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \ \text{and} \ \ G = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

have Drazin inverse and

$$K^d = \begin{pmatrix} A^d & 0 \\ 0 & 0 \end{pmatrix}, \ L^d = \begin{pmatrix} 0 & 0 \\ 0 & D^d \end{pmatrix}, \ \text{and} \ \ G^d = \begin{pmatrix} A^d & 0 \\ 0 & D^d \end{pmatrix}.$$

Lemma 9 *[13] Let $B \in \mathcal{L}(X, Y)$ and $C \in \mathcal{L}(Y, X)$. If $BC \in \mathcal{L}(X)^{\mathbf{D}}$ or $CB \in \mathcal{L}(Y)^{\mathbf{D}}$, then*

$$H = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

has Drazin inverse and

$$H^d = \begin{pmatrix} 0 & (BC)^d B \\ C(BC)^d & 0 \end{pmatrix} = \begin{pmatrix} 0 & B(CB)^d \\ (CB)^d C & 0 \end{pmatrix}.$$

Theorem 10 *Let $A, BC \in \mathcal{L}(X)^{\mathbf{D}}$. If $ABC = 0$ and $BCB = 0$, then N has Drazin inverse and*

$$N^d = \begin{pmatrix} A^d + BC(A^d)^3 & (A^d)^2 B + BC(A^d)^4 B \\ C(A^d)^2 + CBC(A^d)^4 & C(A^d)^3 B + CBC(A^d)^5 B \end{pmatrix}.$$

Proof Consider the splitting of N

$$N = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = p + q.$$

According to the assumptions, we have $p^2qp = 0$, $(qp)^2 = 0$ and $pq^2 = 0$. From $BCB = 0$, we can see that $q^4 = 0$. Applying Theorem 3, we have

$$N^d = \left[pq(p^d)^4 + (p^d)^2 + q(p^d)^3 + q^2(p^d)^4 + q^3(p^d)^5\right](p + q).$$

So, the statement of the theorem is valid.

Corollary 11 *Let $A, BC \in \mathcal{L}(X)^{\mathbf{D}}$. If $ABC = 0$ and $CBC = 0$, then N has Drazin inverse and*

$$N^d = \begin{pmatrix} A^d + BC(A^d)^3 & (A^d)^2 B + BC(A^d)^4 B \\ C(A^d)^2 & C(A^d)^3 B \end{pmatrix}.$$

Proof Using similar method as in Theorem 10, we get that the statement of the theorem is true.

Corollary 12 *Let $A \in \mathcal{L}(X)^{\mathbf{D}}$.*

(1) If $AB = 0$ and $BC = 0$, then $N^d = \begin{pmatrix} A^d & 0 \\ C(A^d)^2 & 0 \end{pmatrix}.$

(2) If $AB = 0$ and $CB = 0$, then $N^d = \begin{pmatrix} A^d + BC(A^d)^3 & 0 \\ C(A^d)^2 & 0 \end{pmatrix}.$

Example 13 *Consider matrix* $N = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, *where*

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix}.$$

After calculating, we get that $ABC = 0$ *and* $BCB = 0$. *Hence, the conditions of Theorem 10 are satisfied. By computing we obtain*

$$ind(A) = 2 \quad , \quad A^d = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then according to the formula in Theorem 10, we get

$$N^d = \begin{pmatrix} 2 & 2 & -4 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 \end{pmatrix}.$$

Theorem 14 *Let* $A, BC \in \mathcal{L}(X)^{\mathbf{D}}$. *If* $BCA = 0$ *and* $BCB = 0$, *then* N *has Drazin inverse and*

$$N^d = \begin{pmatrix} A^d + (A^d)^3BC & (A^d)^2B \\ C(A^d)^2 + C(A^d)^4BC & C(A^d)^3B \end{pmatrix}.$$

Proof Using similar method as in Theorem 10 and conditions of Theorem 4, that is $pqp^2 = 0$, $(pq)^2 = 0$ and $q^2p = 0$, we have

$$N^d = (p+q)\left[(p^d)^4qp + (p^d)^2 + (p^d)^3q + (p^d)^4q^2 + (p^d)^5q^3\right].$$

So, the statement of the theorem is valid.

Corollary 15 *Let* $A, BC \in \mathcal{L}(X)^{\mathbf{D}}$. *If* $BCA = 0$ *and* $CBC = 0$, *then* N *has Drazin inverse and*

$$N^d = \begin{pmatrix} A^d + (A^d)^3BC & (A^d)^2B + (A^d)^4BCB \\ C(A^d)^2 + C(A^d)^4BC & C(A^d)^3B + C(A^d)^5BCB \end{pmatrix}.$$

Proof Using similar method as in Theorem 14, we get that the statement of the theorem is true.

Corollary 16 *Let* $A \in \mathcal{L}(X)^{\mathbf{D}}$.

(1) If $CA = 0$ *and* $BC = 0$, *then* $N^d = \begin{pmatrix} A^d & (A^d)^2B \\ 0 & 0 \end{pmatrix}$.

(2) If $CA = 0$ and $CB = 0$, then $N^d = \begin{pmatrix} A^d + (A^d)^3BC & (A^d)^2B \\ 0 & 0 \end{pmatrix}$.

Corollary 17 *Let $A \in \mathcal{L}(X)^{\mathbf{D}}$. In Theorems 10 and 14, if $BC = 0$, then*

$$N^d = \begin{pmatrix} A^d & (A^d)^2B \\ C(A^d)^2 & C(A^d)^3B \end{pmatrix}.$$

Example 18 *Let*

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & -1 & 1 \end{pmatrix},$$

set the 2×2 block matrix $N = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$. *Then*

$$ind(A) = 3 \quad , \quad A^d = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that $BCA = 0$ and $BCB = 0$. Therefore we can apply Theorem 14 and we get

$$N^d = \begin{pmatrix} 0 & 2 & -1 & 1 & 1 \\ 0 & 2 & -1 & 1 & 1 \\ 0 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 19 *Let $D, CB \in \mathcal{L}(Y)^{\mathbf{D}}$. If $DCB = 0$ and $BCB = 0$, then $\mathcal{N}$ has Drazin inverse and*

$$\mathcal{N}^d = \begin{pmatrix} B(D^d)^3C & B(D^d)^2 \\ (D^d)^2C + CB(D^d)^4C & D^d + CB(D^d)^3 \end{pmatrix}.$$

Proof Consider the splitting of $\mathcal{N}$

$$\mathcal{N} = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = p + q.$$

The remaining proof follows directly from Theorem 10. Here, we omit the details.

Corollary 20 *Let $D, CB \in \mathcal{L}(Y)^{\mathbf{D}}$. If $DCB = 0$ and $CBC = 0$, then $\mathcal{N}$ has Drazin inverse and*

$$\mathcal{N}^d = \begin{pmatrix} B(D^d)^3C + BCB(D^d)^5C & B(D^d)^2 + BCB(D^d)^4 \\ (D^d)^2C + CB(D^d)^4C & D^d + CB(D^d)^3 \end{pmatrix}.$$

Corollary 21 *Let $D \in \mathcal{L}(Y)^{\mathbf{D}}$.*

(1) If $DC = 0$ and $BC = 0$, then $\mathcal{N}^d = \begin{pmatrix} 0 & B(D^d)^2 \\ 0 & D^d + CB(D^d)^3 \end{pmatrix}$.

(2) If $DC = 0$ and $CB = 0$, then $\mathcal{N}^d = \begin{pmatrix} 0 & B(D^d)^2 \\ 0 & D^d \end{pmatrix}$.

Example 22 *Consider the 2×2 block matrix* $\mathcal{N} = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$, *where*

$$D = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

By computing we obtain

$$ind(D) = 1, \quad D^d = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be checked that $DCB = 0$ and $BCB = 0$. Hence, the conditions of Theorem 19 are satisfied. Then according to the formula in Theorem 19, we get

$$\mathcal{N}^d = \begin{pmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 23 *Let $D, CB \in \mathcal{L}(Y)^{\mathbf{D}}$. If $CBD = 0$ and $BCB = 0$, then $\mathcal{N}$ has Drazin inverse and*

$$\mathcal{N}^d = \begin{pmatrix} B(D^d)^3C + B(D^d)^5CBC & B(D^d)^2 + B(D^d)^4CB \\ (D^d)^2C + (D^d)^4CBC & D^d + (D^d)^3CB \end{pmatrix}.$$

Proof Using similar method as in Theorem 19 and conditions of Theorem 4, the statement of the theorem is valid. The remaining proof follows directly from Theorem 14.

Corollary 24 *Let $D, CB \in \mathcal{L}(Y)^{\mathbf{D}}$. If $CBD = 0$ and $CBC = 0$, then $\mathcal{N}$ has Drazin inverse and*

$$\mathcal{N}^d = \begin{pmatrix} B(D^d)^3C & B(D^d)^2 + B(D^d)^4CB \\ (D^d)^2C & D^d + (D^d)^3CB \end{pmatrix}.$$

Corollary 25 *Let $D \in \mathcal{L}(Y)^{\mathbf{D}}$.*

(1) If $BD = 0$ and $BC = 0$, then $\mathcal{N}^d = \begin{pmatrix} 0 & 0 \\ (D^d)^2C & D^d + (D^d)^3CB \end{pmatrix}$.

(2) If $BD = 0$ *and* $CB = 0$*, then* $\mathcal{N}^d = \begin{pmatrix} 0 & 0 \\ (D^d)^2 C & D^d \end{pmatrix}$.

Corollary 26 *Let* $D \in \mathcal{L}(Y)^{\mathbf{D}}$*. In Theorems 19 and 23, if* $CB = 0$*, then*

$$\mathcal{N}^d = \begin{pmatrix} B(D^d)^3 C & B(D^d)^2 \\ (D^d)^2 C & D^d \end{pmatrix}.$$

Example 27 *For the* 2×2 *block matrix* $\mathcal{N} = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$*, where*

$$D = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix},$$

we have that

$$ind(D) = 2, \quad D^d = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

After calculating, we get that $CBD = 0$ *and* $BCB = 0$*, in Theorem 23. We have*

$$\mathcal{N}^d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ -2 & 2 & 1 & 3 \end{pmatrix}.$$

4 Applications to the Drazin Inverse of Block Matrix

In this section, we use the formulas in "Main Results" section and lemma 28 to give some representations for the Drazin inverse of some block matrices that the following Theorems 29 and 31 generalizes the result.

Lemma 28 *[19] Let* $A \in \mathcal{L}(X)^{\mathbf{D}}$*,* $D \in \mathcal{L}(Y)^{\mathbf{D}}$ *and* M *be given by* (1.1)*. If generalized Schur complement* $S = D - CA^dB$ *is zero,* $A^\pi B = 0$*,* $CA^\pi = 0$ *and* $AW = A^2A^d + AA^dBCA^d$ *has Drazin inverse, then* $M \in \mathcal{L}(X \oplus Y)^{\mathbf{D}}$ *and*

$$M^d = \begin{pmatrix} I \\ CA^d \end{pmatrix} \left((AW)^d\right)^2 A \begin{pmatrix} I & A^dB \end{pmatrix}.$$

Theorem 29 *Let* $A \in \mathcal{L}(X)^{\mathbf{D}}$*,* $D \in \mathcal{L}(Y)^{\mathbf{D}}$ *and* M *be given by* (1.1)*. If* $CA^\pi A = 0$*,* $CA^\pi BCA = 0$*,* $BCA^\pi BC = 0$ *and* $D = CA^dB$*. If* $AW = A^2A^d + AA^dBCA^d$ *has Drazin inverse, then* $M \in \mathcal{L}(X \oplus Y)^{\mathbf{D}}$ *and*

$$M^d = Q^d + (Q^d)^2 \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix} + (Q^d)^3 \begin{pmatrix} 0 & 0 \\ CA^\pi A & CA^\pi B \end{pmatrix},$$

where

$$(Q^d)^n = \sum_{i=0}^{k} \begin{pmatrix} A^\pi A & A^\pi B \\ 0 & 0 \end{pmatrix}^i (Q_1^d)^{i+n},$$

$$(Q_1^d)^i = \begin{pmatrix} I \\ CA^d \end{pmatrix} \left((AW)^d\right)^{i+1} A \begin{pmatrix} I & A^dB \end{pmatrix},$$

for every $i \in \mathbb{N}$ and $k = ind(A)$.

Proof We easily see that

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}, \text{ and } Q = \begin{pmatrix} A & B \\ CA^dA & CA^dB \end{pmatrix}.$$

By assumption, we verify that $P^2 = 0$, $(QP)^2 = 0$ and $PQ^2 = 0$. Clearly, P has Drazin inverse, we get $P^d = 0$. Moreover, according to Theorem 3, we have

$$M^d = Q^d + (Q^d)^2P + (Q^d)^3PQ. \tag{4.1}$$

Let $Q_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}$ and $Q_2 = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}$, then we have $Q = Q_1 + Q_2$, Q_2 is nilpotent and $Q_1Q_2 = 0$, so $Q_2^d = 0$. We easily check that

$$Q_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A & AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A & AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = A^2A^d + AA^dBCA^d = AW,$$

has Drazin inverse. Therefore Q_1 has Drazin inverse. By virtue of Theorem 3, $M \in \mathcal{L}(X \oplus Y)^{\mathbf{D}}$, as required. Now, according to Corollary 6, we get

$$Q^d = (Q_1 + Q_2)^d = \sum_{i=0}^{k} Q_2^i (Q_1^d)^{i+1}, \tag{4.2}$$

where $k = ind(A)$ and for every $n \in \mathbb{N}$,

$$(Q^d)^n = \sum_{i=0}^{k} Q_2^i (Q_1^d)^{i+n}. \tag{4.3}$$

The generalized Schur complement of Q_1 is equal to zero, and the matrix Q_1 satisfies

$$(A^2A^d)^\pi AA^dB = 0, \quad \text{and} \quad CAA^d(A^2A^d)^\pi = 0,$$

so according to Lemma 28, we get

$$((Q_1)^d)^i = \begin{pmatrix} I \\ CA^d \end{pmatrix} \left((AW)^d\right)^{i+1} A \begin{pmatrix} I & A^dB \end{pmatrix}.$$

for every $i \in \mathbb{N}$. By substituting Equ.(4.3) into Equ.(4.2), we obtain the expression of Q^d, substituting Q^d into equation (4.1), the expression of M^d is obtained.

Example 30 *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where*

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have that $ind(A) = 3$ *and*

$$A^d = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ D^d = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A^\pi = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \ (AW)^d = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

We easily check that $CAA^\pi = 0$, $CA^\pi BCA = 0$, $BCA^\pi BC = 0$ *and* $D = CA^dB$. *Then by Theorem 29, we obtain that*

$$M^d = \frac{1}{16} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 3 \\ 0 & 4 & 0 & 0 & 0 & 0 & 4 \\ 0 & -4 & 0 & 0 & 0 & 0 & -4 \\ 0 & 4 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Theorem 31 *Let* $A \in \mathcal{L}(X)^{\mathbf{D}}$, $D \in \mathcal{L}(Y)^{\mathbf{D}}$ *and* M *be given by* (1.1). *If* $A^2A^\pi BC = 0$, $BCA^\pi BC = 0$, $CAA^\pi BC = 0$ *and* $D = CA^dB$. *If*

$$AW = A^2A^d + AA^dBCA^d,$$

has Drazin inverse, then $M \in \mathcal{L}(X \oplus Y)^{\mathbf{D}}$ *and*

$$M^d = \left[(P^d)^2 + \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix} (P^d)^3 + \begin{pmatrix} 0 & AA^\pi B \\ 0 & CA^\pi B \end{pmatrix} (P^d)^4\right] \times \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$(P^d)^n = \sum_{i=0}^{k} (P_1^d)^{i+n} \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}^i,$$

$$(P_1^d)^i = \begin{pmatrix} I \\ CA^d \end{pmatrix} \left((AW)^d\right)^{i+1} A \begin{pmatrix} I & A^dB \end{pmatrix},$$

for every $i \in \mathbb{N}$ *and* $k = ind(A)$.

Proof We easily see that

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}, \text{ and } Q = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}.$$

By assumption, we verify that $P^2QP = 0$, $(QP)^2 = 0$ and $Q^2 = 0$. Clearly, Q has Drazin inverse, we get $Q^d = 0$. Moreover, according to Theorem 3, we have

$$M^d = \left[(P^d)^2 + Q(P^d)^3 + PQ(P^d)^4\right](P + Q). \tag{4.4}$$

Let

$$P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}, \text{ and } P_2 = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix},$$

then we have $P = P_1 + P_2$, P_2 is nilpotent and $P_2P_1 = 0$, so $P_2^d = 0$. Simillary to proof of Theorem 29, P_1 has Drazin inverse and for every $n \in \mathbb{N}$,

$$(P^d)^n = \sum_{i=0}^{k} (P_1^d)^{i+n} P_2^i. \tag{4.5}$$

and

$$\left((P_1)^d\right)^i = \begin{pmatrix} I \\ CA^d \end{pmatrix} \left((AW)^d\right)^{i+1} A \begin{pmatrix} I & A^dB \end{pmatrix}.$$

for every $i \in \mathbb{N}$. After substituting this expressions and equation (4.5) into equation (4.4), we complete the proof.

Example 32 *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where*

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

We have that $ind(A) = 2$ *and*

$$A^d = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D^d = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix},$$

$$A^{\pi} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ (AW)^d = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We easily check that $A^2A^{\pi}BC = 0$, $BCA^{\pi}BC = 0$, $CAA^{\pi}BC = 0$ *and* $D = CA^dB$. *Then by Theorem 31, we obtain that*

$$M^d = \frac{1}{243} \begin{pmatrix} 18 & -18 & 0 & 0 & 36 & 0 \\ -9 & 9 & 0 & 0 & -18 & 0 \\ 9 & 234 & 243 & 0 & 18 & 0 \\ 3 & -3 & 0 & 0 & 6 & 0 \\ 27 & -27 & 0 & 0 & 54 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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