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Research article

On NeutroEngelGroups

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Abstract

In this paper We introduce the notion of NeutroEngelGroups and we show some of it's results. Also, we show that the intersection of two NeutroEngelGroups and the quotient of a NeutroEngelGroups are NeutroEngelGroups too. Moreover, we prove that NeutroEngel is closed with respect to homomorphic image. Also, by several examples we show the diferences between Engel groups and NeutroEngelGroups.

Keywords: NeutroGroup, NeutroEngelGroup, NeutroGroup homomorphism

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1 Introduction

Your text comes heLotfi A. Zadeh in 1965, introduced fuzzy sets as an extension of the classical notion of set. Then in 2003 Smarandache [14], generalized the concept of fuzzy logic/set to neutrosophic logic/set. It has many applications in sciences, engineering, technology and social sciences, medical diagnosis and multiple decision-making. Also, Neutrosophic set has been used in several areas of mathematics. In 2019, Smarandache [12] introduced NeutroStructures and AntiStructures. Also, in [13] he introduced the notion of NeutroAlgebras and AntiAlgebras. In [9], he calculated the number of NeutroAlgebras and AntiAlgebras in a classical algebra.

Moreover, NeutroAlgebras as a generalization of Partial Algebras are studied by Smarandache (see [11]). Agboola et al [3], examined NeutroAlgebras and AntiAlgebras viz a viz the classical number systems **Z, Q, R** and **C**. Smarandache [13], introduced NeutroGroup.

Kandasamy and Smarandache studied several neutrosophic algebraic structures in [5–7]. Some of them are neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups,neutrosophic loops, neutrosophic groupoids, and so on. Also, in [2], Agboola studied NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition). Moreover, some autors introduced the new notions such as NutroRings, NeutroIdeals, NeutroQuotientRings and NeutroHomomorphism by the concept of NeutroAxioms and NeutroLow (see [10], [2]).

Now, we introduce the notion of NeutroEngelGroups and provied a few examples of NeutroEngelGroups and also we present some elementary results of them. Moreover, we prove that the intersection of two NeutroEngelGroups is a NeutroEngelGroup and the homomorphic image of a NeutroEngelGroup is a NeutroEngelGroup too.

2 Preliminaries

In this section we sumerize some basic definitions and results which will be used throughout the paper.

Definition 1 ([4]). *A classical group is a nonempty set G endowed with a binary operation $*$, denoted by $(G,*)$, satisfying the following axioms:*

(1) $x*y \in G$ for all $x,y \in G$.

(2) $x*(y*z) = (x*y)*z$ for all $x,y,z \in G$.

(3) There exists $e \in G$ such that $x*e = e*x = x$ for all $x \in G$.

(4) For all $x \in G$ there exists $y \in G$ such that $x*y = y*x = e$, where e is the neutral element of G.

If $(G,*)$ satisfies the following condition, then we call it an abelian group:
(5) $x*y = y*x$ for all $x,y \in G$.

Definition 2 ([6]). *A group $(G,*)$ is called nilpotent if for all i the factor G_{i+1}/G_i is contained in the center of G/G_i in a normal series $e = G_0 \leq G_1 \leq \cdots \leq G_n = G$. The smallest such n is called the nilpotent class of G.*

Definition 3 ([6]). *If $(G,*)$ is a group and $x_1,\ldots,x_n \in G$, then*

$$c(x_1,x_2) = {x_1}^{-1}{x_2}^{-1}x_1x_2,$$

is the commutator of x_1 and x_2. For $n > 2$ the commutator $c(x_1,\ldots,x_n)$ is $c(c(x_1,\ldots,x_{n1}),x_n)$, where $c(x_1) = x_1$.

Definition 4. *An algebra $(T,*)$, where $*$ is a binary operation is a NeutroGroup if it satisfying the following axioms:*
(T1) There are (x,y,z) and $(u,v,w) \in T$ such that $x*(y*z) = (x*y)*z$ and $u*(v*w) \neq (u*v)*w$ or$u*(v*w =$indeterminate or $(u*v)*w =$indeterminate (NeutroAssociativity).
(T2) There are exists at least an element $a \in T$ that has a single neutral element i.e., we have $e \in T$ such that $a*e = e*a = a$ and for $b \in T$ there does not exist $e \in T$ such that $b*e = e*b \neq b$ or there exists $e_1,e_2 \in T$ such that $b*e_1 = e_1*b = b$ or $b*e_2 = e_2*b = b$ with $e_1 \neq e_2$ or there exists at least an element $c \in T$ that there is $d \in T$ such that $c*d = d*c =$indeterminate (NeutroNeutralElement).
(T3) There exists an element $a \in T$ that has an inverse $b \in T$ w.r.t. a unit element $e \in T$ i.e., $a*b = b*a = e$, or there exists at least one element $b \in T$ that has two or more inverses $c,d \in T$ w.r.t. some unit element $u \in T$ i.e., $b*c = c*b = u$, $b*d = d*b = u$ or there exists at least one element $r \in T$ that has one element $s \in T$ such that $r*s = s*r =$indeterminate (NeutroInverseElement).
(T4) the structure $(T,*)$ is said a NeutroAbelianGroup (NeutroAbelianGroup) if there are (a,b) and $(c,d) \in T$ such that $c*d \neq d*c$, or $c*d =$indeterminate or $d*c =$indeterminate.

Definition 5. *Let $(T,*)$ be a NeutroGroup and $\Phi \neq H \subseteq G$. Then H is called a NeutroSubgroup of T if $(H,*)$ is also a NeutroGroup of the same type as T and we denote by $H \sqsubseteq T$.*

Example 1. *Let $U = \{a,b,c,d,e,f\}$ and $T = \{a,b,c,d\}$ be a subset of U. Consider table 1. Then $*_1$ is a NeutroAbelian.*

From now on, in this paper T is a NeutroGroup unless otherwise state. For all $x,y \in T$ we use xy instead of xy and η_x, ε_x represent the NeutroNeutral element and the NeutroInverse elements, respectively.

3 Some Results On NeutroEngelGroups

In this section we define NeutroEngelGroup on a NeutroGroup T with three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral and NeutroInverse elements). Then we study some results on NeutroEngelGroups.
The commutator of $x,y \in T$, is $c(x,y) = (\varepsilon_x\varepsilon_y)(xy)$, where for any $x \in T$, we take $\varepsilon_x = x$ if there is not ε_x. Also, for any $x,y_1,\ldots,y_n \in T$, the commutator of weigh $n \in \mathbf{N}$ is $c(x,y_1,\ldots,y_n) = c(c(x,y_1,\ldots,y_{n-1}),y_n)$ and we use $c(x,_n y)$ instead of $c(x,\underbrace{y,\ldots,y}_{n})$.

Table 1. The table of NeutoEngelGroup $(T, *_1)$

$*_1$	**a**	**b**	**c**	**d**
a	b	c	d	a
b	c	d	a	c
c	d	a	b	d
d	a	b	c	a

Definition 6. *Let for any $x \in T$ there exists at least one $g, z \in T$ such that $c(x,_n g) = \eta_z$. Then T is saied a NeutroEngelGroup. Let E_n be the class of all n-NeutroEngelGroups.*

In the next Example we show that $\mathbf{Z}_{10}$ is a NeutroAbelianGroup which is in E_1.

Example 2. *We define a binary operation "$*$" on $G = \mathbf{Z}_{10}$ by $x * y = x + 2y$ for any $x, y \in G$, where "$+$" is addition modula 10. Then $(G, *)$ is a NeutroAbelianGroup (for more detailes see [1]). We have $\eta_0 = 0, \eta_5 = 5$ and $\eta_1, \eta_2, \eta_3, \eta_4, \eta_6, \eta_7, \eta_8, \eta_9$ do not exist. Now, we can see that $G \in E_1$.*

Example 3. *In Example 1, T is Neutro associative, because*

$$a *_1 (b *_1 c) = (a *_1 b) *_1 c \text{ and } b *_1 (d *_1 c) \neq (b *_1 d) *_1 c.$$

Also, $\eta_a = d$, but η_b, η_c, η_d do not exist. Moreover, $\varepsilon_a = c$ and $\varepsilon_b, \varepsilon_c, \varepsilon_d$ do not exist (see [1]). Since $c(a,b) = (\varepsilon_a *_1 \varepsilon_b) *_1 (a *_1 b) = c *_1 b *_1 a *_1 b = d$, $c(c,b) = b$, $c(d,b) = d$ and $c(b,b) = a$, so we have $c(a,_3 b) = d = \eta_a$, $c(c,_3 b) = c(a,b) = \eta_a$, $c(d,_3 b) = \eta_a$ and $c(b,_3 b) = \eta_a$. Therefore, $T \in E_3$.

Definition 7. *[8] Let there is $g_1, \ldots, g_n, z \in T$ such that $c(x, g_1, \ldots, g_n) = \eta_z$ for any $x \in T$. Then T is called a NeutroNilpotentGroup of class n. Let N_n be the class of all NeutroNilpotentGroup of class n.*

In group theory every n-Engel group is $n+1$-Engel group. But we show that it is not valid in NeutroGroups.

Example 4. *As Example 3, we have $T \in E_3$. Now, for any $x \in T$ there exist $b \in T$ such that $c(x,_3 b) = \eta_a$, so $c(x,_4 b) = c(c(x,_3 b), b) = c(d,b) = d = \eta_a$. Therefore, $T \in E_4$.*

The following example shows a NeutroGroup H such that $H \in E_1$ but $H \notin E_2$.

Example 5. *Let $U = \{a,b,c,d\}$ and $T = \{a,b,c\}$ be a NeutroGroup by the following tabel:*

$*_3$	a	b	c
a	a	c	b
b	c	a	c
c	a	c	d

$*_4$	a	b
a	a	c
b	c	a

Let $H = \{a,b\}$. Then $(H, *_4) \sqsubseteq T$. We have $\eta_a = a$, $\varepsilon_a = a$, η_b and ε_b dont exist. So $c(a,a) = c(b,b) = a = \eta_a$. Therefore, $H \in E_1$ and is a NeutroNilpotent. Moreover, $H \notin E_2$. Since $c(b,_2 a)$ and $c(b,_2 b)$ dont exist.

Now, by Example 4 and 5, we have the following theorem.

Theorem 1. *Every n-NeutroEngelGroup is not $n+1$-NeutroEngelGroup in general.*

Theorem 2. *Assume that T_1 and T_2 are two NeutroGroups. Then $T_1 \times T_2 \in E_n$ if and only if T_1 and $T_2 \in E_n$.*

Proof. Since $T_1 \times T_2 \in E_n$ so, for any $(x,y) \in T_1 \times T_2$, there exist

$$(x_1, y_1) \in T_1 \times T_2, z \in T_1,$$

and $t \in T_2$ such that $(\eta_z, \eta_t) = c((x,y),_n (x_1,y_1)) = (c(x,_n x_1), c(y,_n y_1))$. So $c(x,_n x_1) = \eta_z$ and $c(y,_n y_1) = \eta_t$. Therefore, $T_1, T_2 \in E_2$. The converse of theorem is similar. □

Table 2. The table of NeutoSubgroup $(R, *_5)$

$*_3$	**a**	**c**	**d**
a	b	d	a
c	d	b	d
d	a	c	a

By the following example we have a NeutroSubgroup which is not NeutroNilpotent and is not a NeutroEngelGroup.

Example 6. *Assume the NeutroGroup $(T, *_1)$ as table 3 and $R = \{a, c, d\}$. Then, $(R, *_5)$ is not a NeutroNilotentSubgroup of T (see [3]). Also, R is not NeutroEngel, since $c(a, d) = a$, $c(a, a)$ and $c(a, c)$ do not exist, so $c(a,_n g)$ does not exist for any $g \in R$.*

Example 7. *Let $(R, *_5)$ and $(H, *_4)$ be as Example 6 and 5 respectively. Then, R is not a NeutroEngelGroup and H is a NeutroEngelGroup. But $R \times H$ is not a NeutroEngelGroup.*

We know that every nilpotent group is an Engel group and the converse is not true in general. Now, we check this result in NeutroGroups.

Theorem 3. *Every n-NeutroEngelGroup is NeutroNilpotent. Moreover, if T is not a NeutroNilpotentGroup, then it is not NeutroEngel.*

Proof. Suppose that $T \in E_n$. Then, for any $x \in T$ there exist $g, z \in T$ such that $c(x,_n g) = \eta_z$ and so $c(x, \underbrace{g, \ldots, g}_{n}) = \eta_z$. Therefore, $T \in N_n$. □

Theorem 4. *If $T \in N_1$, then $T \in E_1$.*

Proof. We get the result by definitions of N_1 and E_1. □

Theorem 5 ([1]). *Let $R \sqsubseteq T$. The sets*

$$(T/R)_l = \{xR : x \in T\},$$

and

$$(T/R)_r = \{Rx : x \in T\},$$

are two NeutroGroups with operations $\circ_l$ and $\circ_r$ where for any

$$xR, yR \in (T/R)_l, \;\; Rx, Ry \in (T/R)_r, \;\; x, y \in T,$$

we have

$$xR \circ_l yR = xyR \text{ and } Rx \circ_r Ry = Rxy.$$

Theorem 6. *If $T \in E_n$, then $(T/R)_l, (T/R)_r \in E_n$.*

Proof. Suppose that $R \sqsubseteq T$ and $gR \in (T/R)_l$. Since $T \in E_n$ there exist $g_1, z \in T$ such that $c(g,_n g_1) = \eta_z$ and so $c(gR,_n g_1R) = c(g,_n g_1)R = \eta_z R$. On the otherhand, $\eta_z R$ is a NeutroNatural element of $(T/R)_l$, since $(zR) \circ_l (\eta_z R) = (z\eta_z)R = zR = \eta_z R \circ_l zR$. Therefore, $(T/R)_l \in E_n$. In a similar way $(T/R)_r \in E_n$. □

Example 8. *Let T and H be as Example 3.6 and consider $(T/H)_l = \{aH\}$. We can see that $(T/H)_l$ is NeutroEngelGroup. Since $c(aH, aH) = aH = \eta_{aH}$.*

We know that the intersection of two NeutroGroup is a NeutroGroup (see [1]). Now we have the following theorem:

Theorem 7. *The intersection of two NeutroEngelGroups is also a NeutroEngelGroup.*

Definition 8 ([1]). *Let $(T_1, *)$ and $(T_2, \circ)$ be two NeutroGroups. NeutroGroup homomorphism $\varphi : T \to R$ is a map such that for any $x, y \in G$, we have $\varphi(x * y) = \varphi(x) \circ \varphi(y)$.*
A NeutroGroup isomorphism is a NeutroGroup homomorphism which is a Neutrobijection. If φ from G to H is a NeutroGroup isomorphism, then we denoted it by $G \approx H$. NeutroGroup epimorphism, NeutroGroup monomorphism, NeutroGroup endomorphism are defined similarly.

Theorem 8. *[1] Let $\varphi : (T_1, *) \to (T_2, \circ)$ be a NeutroGroup homomorphism with NeutroNeutralElements e_1 and e_2, respectively. Then $\varphi(e_1) = e_2$.*

Theorem 9. *The homomorphic image of a n-NeytroEngelGroup is also a n-NeutroEngelGroup.*

Proof. Assume $R \sqsubseteq T$, where T is a n-NeutroEngelGroup, and e_1, e_2 be NeutroNeutralElements in T and R, respectively. Suppose that $\psi : T \to R$ is a NeutroGroup epimorphism. Then for any $h \in R$ there exist $x \in T$ such that $h = \psi(x)$. Since $T \in E_n$ for $x \in T$ there exist $g_1 \in T$ such that $c(x,_n g_1) = e_1$. Put $k_1 = \psi(g_1)$. Therefore, $c(h,_n k_1) = \psi(c(x,_n g_1)) = \psi(e_1) = e_2$, and so $R \in E_n$. □

In the following theorem, we define a NeutroEngelGroup by an n-Engel group.

Theorem 10. *Let $(G, \cdot)$ be a n-Engel group and $g_0 \in G$. Let $a \notin G$ and $P_G = G \cup \{a\}$. Then $(P_G, \circ)$ is a NeutroEngelGroup where, $\circ$ is defined as follows:*
(1) $a \circ a = g_0$,
(2) $a \circ g = g \circ a = a$, for all $g \in G - \{a\}$,
(3) $x \circ y = x.y$, for all $(x,y) \in G^2$.

Proof. By (2), it is clear that $\eta_a = g$, for any $g \in G$. So $\eta_a = g_0$. Now, using (1), we conclude that $\varepsilon_a = a$. Therefore, $(P_G, \circ)$ is a NeutroGroup. We have,

$$c(a,g) = (\varepsilon_a \circ \varepsilon_g) \circ (a \circ g) = (a \circ g^{-1}) \circ (a \circ g) = a \circ a = \eta_a = g_0,$$

and

$$c(g,a) = (\varepsilon_g \circ \varepsilon_a) \circ (g \circ a) = (g^{-1} \circ a) \circ a = a \circ a = \eta_a = g_0.$$

Since G is n-Engel, we have $c(a,_{n+1} g) = c(c(a,g),_n g) = c(g_0,_n g) = e$. Also, $c(g,_{n+1} a) = c(c(g,a),_n a) = c(g_0,_n a) = \ldots = c(g_0, a) = g_0 = \eta_a$. Therefore, $(P_G, \circ)$ is a NeutroEngelGroup. □

In the above theorem if $a \circ a = a$, then $(P_G, \circ)$ is not a NeutroEngelGroup. By the proof of Theorem 3.20, for all $g \in G$ we have $\eta_a = g$. For all $z \in G$,

$$c(a,g) = (\varepsilon_a \circ \varepsilon_g) \circ (a \circ g) = (a \circ g^{-1}) \circ (a \circ g) = a \circ a = a \neq \eta_z,$$

and $c(a,a) = a \neq \eta_z$. Then $c(a,_n g) \neq \eta_z$ and $c(a,_n a) \neq \eta_z$. Consequently, $(P_G, \circ)$ is not a NeutroEngelGroup.

Example 9. *We know D_8 and Q_8 are 2-Engel groups. Then by Theorem 3.20, P_{D_8} and P_{Q_8} are neutroEngel groups.*

Theorem 11. *For the NeutroGroup isomorphism $\varphi : T \to R$, if $T \in E_n$, then R is too.*

Proof. Let $x \in R$. By f is an isomorphism, there exists $x \in T$ such that $f(x') = x$. But x'/inE_n. So there exists $g', z' \in T$ such that $c(x',_n g') = z'$. Let $z = f(x'), g = f(g')$. Then

$$c(x,_n g) = c(f(x'),_n f(g')) = fc(x', g') = f(\eta') = \eta_z.$$

□

Theorem 12. *If K is a neutroGroup and $G \approx H$, then $G \times K \approx H \times K$. In particular, if K and G are NeutroEngel groups, then $H \times K$ is too.*

Theorem 13. *If $f : G \to H$ is a neutroGroup homomorphism and K is a neutroEngel subgroup of G, then $f(K)$ is a neutroEngel subgroups of H.*

Theorem 14. *If $f : G \to H$ is a neutroGroup homomorphism and K is a neutroEngel subgroup of H, then $f^{-1}(K)$ is a neutroEngel subgroups of G.*

Example 10. *Let $a, r \notin \mathbf{Z}$, $P = \mathbf{Z} \cup \{a\}$ and $P' = (\{0\} \times \mathbf{Z}) \cup \{(0,r)\}$. By the proof of Theorem 3.20, P and P' are two NeutroEngel groups. The function $f : P \Rightarrow P'$ by $f(x) = (0,x)$ for $x \in \mathbf{Z}$ and $f(x) = (0,r)$ for $x = a$, is a Neutro isomomorhism. Since $A = 2\mathbf{Z} \cup \{a\}$ is a neutrpEngel subgroup of P, so by 13, $f(A)$ is a neutrpEngel subgroup of P'*

Theorem 15. *If $G \approx H$, then $P_G \approx P_H$. Moreover, if G is an Engel group, then P_H is NeutroEngel group.*

4 Conclusion

In this paper, we introduce a subclass of NeutroNilpotentGroups, named NeutroEngelGroups, and their elementary properties were presented. We try to show the differences between classic Engel groups and NeutroEngelGroups throughout of several examples.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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