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Existence Results for a Dirichlet Quasilinear Elliptic Problem

Ghasem A. Afrouzi · Armin Hadjian*

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Abstract In this paper, existence results of positive classical solutions for a class of second-order differential equations with the nonlinearity dependent on the derivative are established. The approach is based on variational methods.

Keywords Dirichlet problem · Critical points · Variational methods

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1 Introduction

Dirichlet boundary value problems have been widely studied because of their applications in various fields of applied sciences, as mechanical engineering, control systems, computer science, economics, artificial or biological neural networks and many others.

The aim of this paper is to establish the existence of at least one nontrivial solution for the following Dirichlet quasilinear elliptic problem on a bounded

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G.A. Afrouzi
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
Tel.: +98-11-35302463
E-mail: afrouzi@umz.ac.ir

*Corresponding author
A. Hadjian
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail: a.hadjian@umz.ac.ir

interval $[a,b]$ in $\mathbf{R}$ $(a<b)$

$$\begin{cases} -(p-1)|u'(x)|^{p-2}u''(x)=\lambda f(x,u)h(x,u'), & x\in(a,b),\\ u(a)=u(b)=0, \end{cases} \tag{1}$$

where $p>1$, λ is a positive parameter, $h:[a,b]\times\mathbf{R}\to[0,+\infty)$ is a bounded and continuous function with $m:=\inf_{(x,t)\in[a,b]\times\mathbf{R}}h(x,t)>0$ and $f:[a,b]\times\mathbf{R}\to\mathbf{R}$ is an L^1-Carathéodory function.

Our main tool is the Ricceri variational principle [4, Theorem 2.5] as given in [1, Theorem 5.1] which is below recalled.

For a given non-empty set X, and two functionals $\Phi,\Psi:X\to\mathbf{R}$, we define the following functions

$$\beta(r_1,r_2):=\inf_{v\in\Phi^{-1}(]r_1,r_2[)}\frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)}\Psi(u)-\Psi(v)}{r_2-\Phi(v)},$$

$$\rho_2(r_1,r_2):=\sup_{v\in\Phi^{-1}(]r_1,r_2[)}\frac{\Psi(v)-\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{\Phi(v)-r_1},$$

for all $r_1,r_2\in\mathbf{R}$, with $r_1<r_2$.

Theorem 1 ([1, Theorem 5.1]) *Let X be a real Banach space; $\Phi:X\to\mathbf{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi:X\to\mathbf{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Put $I_\lambda:=\Phi-\lambda\Psi$ and assume that there are $r_1,r_2\in\mathbf{R}$, with $r_1<r_2$, such that*

$$\beta(r_1,r_2)<\rho_2(r_1,r_2). \tag{2}$$

Then, for each $\lambda\in\left]\frac{1}{\rho_2(r_1,r_2)},\frac{1}{\beta(r_1,r_2)}\right[$ there is $u_{0,\lambda}\in\Phi^{-1}(]r_1,r_2[)$ such that $I_\lambda(u_{0,\lambda})\le I_\lambda(u)$ for all $u\in\Phi^{-1}(]r_1,r_2[)$ and $I'_\lambda(u_{0,\lambda})=0$.

Let $h:[a,b]\times\mathbf{R}\to[0,+\infty)$ be a bounded and continuous function with

$$m:=\inf_{(x,t)\in[a,b]\times\mathbf{R}}h(x,t)>0,$$

and $f:[a,b]\times\mathbf{R}\to\mathbf{R}$ be an L^1-Carathéodory function.

We recall that $f:[a,b]\times\mathbf{R}\to\mathbf{R}$ is an L^1-Carathéodory function if

(a) $x\mapsto f(x,\xi)$ is measurable for every $\xi\in\mathbf{R}$;
(b) $\xi\mapsto f(x,\xi)$ is continuous for almost every $x\in[a,b]$;
(c) for every $\rho>0$ there is a function $l_\rho\in L^1([a,b])$ such that

$$\sup_{|\xi|\le\rho}|f(x,\xi)|\le l_\rho(x)$$

for almost every $x\in[a,b]$.

Corresponding to f and h we introduce the functions $F : [a, b] \times \mathbf{R} \to \mathbf{R}$ and $H : [a, b] \times \mathbf{R} \to [0, +\infty)$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)d\xi$$

and

$$H(x,t) := \int_0^t \left(\int_0^\tau \frac{(p-1)|\delta|^{p-2}}{h(x,\delta)}\, d\delta \right) d\tau$$

for all $x \in [a, b]$ and $t \in \mathbf{R}$. Also, we use the following notation:

$$M := \sup_{(x,t)\in[a,b]\times\mathbf{R}} h(x,t).$$

Here and in the following, let $X := W_0^{1,p}([a, b])$, equipped with the norm

$$\|u\| := \left(\int_a^b |u'(x)|^p dx \right)^{1/p}.$$

Then, X is a reflexive real Banach space. Since $p > 1$, X is compactly embedded in $C^0([a, b])$ and

$$\|u\|_\infty \le \frac{(b-a)^{(p-1)/p}}{2} \|u\|, \tag{3}$$

for all $u \in X$.

By a *classical solution* of problem (1), we mean a function u such that $u \in C^1([a, b])$, $u' \in AC([a, b])$, and $u(t)$ satisfies (1) a.e. on $[a, b]$. We say that a function $u \in X$ is a *weak solution* of the problem (1) if

$$\int_a^b \left(\int_0^{u'(x)} \frac{(p-1)|\tau|^{p-2}}{h(x,\tau)}\, d\tau \right) v'(x)dx - \lambda \int_a^b f(x, u(x))v(x)dx = 0$$

for all $v \in X$.

The following lemma is taken from [2, Lemma 2.2].

Lemma 1 *A weak solution to* (1) *in* X *coincides with a classical solution to* (1).

2 Main results

In this section we present our main results.

Throughout the sequel, α, β are two positive constants such that $\alpha + \beta < b - a$. Now, put

$$D := \frac{(p-1)^{p-2}}{p} \left(\alpha^{-p+1} + \beta^{-p+1} \right).$$

Given two nonnegative constants c, d, with

$$m(2c)^p \ne Dd^p (b-a)^{p-1} pM,$$

put

$$a_d(c) := \frac{\displaystyle\int_a^b \max_{|t|\le c} F(x,t)dx - \int_{a+\alpha}^{b-\beta} F(x,d)dx}{m(2c)^p - Dd^p(b-a)^{p-1}pM}.$$

Theorem 2 *Assume that there exist a nonnegative constant c_1 and two positive constants c_2, d, with*

$$c_1 < \frac{(b-a)^{(p-1)/p}(pD)^{1/p}}{2} d < \left(\frac{m}{M}\right)^{1/p} c_2, \tag{4}$$

such that

(A_1) $F(x,t) \ge 0$ *for all* $(x,t) \in ([a,a+\alpha] \cup [b-\beta,b]) \times [0,d]$*;*
(A_2) $a_d(c_2) < a_d(c_1)$.

Then, for each

$$\lambda \in \frac{1}{(b-a)^{p-1}pmM} \left]\frac{1}{a_d(c_1)}, \frac{1}{a_d(c_2)}\right[,$$

problem (1) *admits at least one nontrivial classical solution $\bar{u} \in X$, such that*

$$\frac{2m^{1/p}}{(b-a)^{(p-1)/p}M^{1/p}} c_1 < \|\bar{u}\| < \frac{2}{(b-a)^{(p-1)/p}} c_2.$$

Proof Our aim is to apply Theorem 1 to our problem. To this end, for each $u \in X$, let the functionals $\Phi, \Psi : X \to \mathbf{R}$ be defined by

$$\Phi(u) := \int_a^b H(x,u'(x))dx, \qquad \Psi(u) := \int_a^b F(x,u(x))dx,$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \qquad \forall\ u \in X.$$

Note that the weak solutions of (1) are exactly the critical points of I_λ. The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 1.

Since $m \le h(x,t) \le M$ for all $(x,t) \in [a,b] \times \mathbf{R}$, we see that

$$\frac{1}{pM}\|u\|^p \le \Phi(u) \le \frac{1}{pm}\|u\|^p \qquad \text{for all } u \in X. \tag{5}$$

Now, put

$$r_1 := \frac{2^p}{(b-a)^{p-1}pM} c_1^p, \qquad r_2 := \frac{2^p}{(b-a)^{p-1}pM} c_2^p,$$

and

$$w(x) := \begin{cases} \dfrac{1}{\alpha^{p-1}} d(x-a)^{p-1}, & \text{if } a \le x < a+\alpha, \\ d, & \text{if } a+\alpha \le x \le b-\beta, \\ \dfrac{1}{\beta^{p-1}} d(b-x)^{p-1}, & \text{if } b-\beta < x \le b. \end{cases}$$

It is easy to verify that $w \in X$ and, in particular, one has

$$\|w\|^p = d^p(p-1)^{p-2}\left(\alpha^{-p+1} + \beta^{-p+1}\right) = pDd^p.$$

So, from (5), we have

$$\frac{Dd^p}{M} \leq \Phi(w) \leq \frac{Dd^p}{m}.$$

From the condition (4), we obtain $r_1 < \Phi(w) < r_2$. For all $u \in X$ such that $\Phi(u) < r_2$, taking (3) into account, one has $|u(x)| < c_2$ for all $x \in [a,b]$, from which it follows

$$\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) = \sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \int_a^b F(x, u(x))dx \leq \int_a^b \max_{|t| \leq c_2} F(x,t)dx.$$

Arguing as before, we obtain

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \leq \int_a^b \max_{|t| \leq c_1} F(x,t)dx.$$

Since $0 \leq w(x) \leq d$ for each $x \in [a,b]$, the assumption (A_1) ensures that

$$\Psi(w) \geq \int_{a+\alpha}^{b-\beta} F(x,d)dx.$$

Therefore, one has

$$\begin{aligned}
\beta(r_1, r_2) &\leq \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\
&\leq (b-a)^{p-1} pmM \frac{\int_a^b \max_{|t| \leq c_2} F(x,t)dx - \int_{a+\alpha}^{b-\beta} F(x,d)dx}{m(2c_2)^p - Dd^p(b-a)^{p-1}pM} \\
&= \left[(b-a)^{p-1} pmM\right] a_d(c_2).
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
\rho_2(r_1, r_2) &\geq \frac{\Psi(w) - \sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(w) - r_1} \\
&\geq (b-a)^{p-1} pmM \frac{\int_{a+\alpha}^{b-\beta} F(x,d)dx - \int_a^b \max_{|t| \leq c_1} F(x,t)dx}{Dd^p(b-a)^{p-1}pM - m(2c_1)^p} \\
&= \left[(b-a)^{p-1} pmM\right] a_d(c_1).
\end{aligned}$$

Hence, from the assumption (A_2), one has $\beta(r_1, r_2) < \rho_2(r_1, r_2)$. Therefore, from Theorem 1, for each $\lambda \in \frac{1}{(b-a)^{p-1}pmM}\left]\frac{1}{a_d(c_1)}, \frac{1}{a_d(c_2)}\right[$, the functional I_λ admits at least one critical point $\bar{u}$ such that $r_1 < \Phi(\bar{u}) < r_2$, that is

$$\frac{2m^{1/p}}{(b-a)^{(p-1)/p} M^{1/p}} c_1 < \|\bar{u}\| < \frac{2}{(b-a)^{(p-1)/p}} c_2.$$

So, applying Lemma 1, the conclusion is achieved.

Now, we point out an immediate consequence of Theorem 2 by taking $c_1 = 0$ and $c_2 = c$.

Theorem 3 *Assume that there exist two positive constants c, d, with*

$$\frac{(b-a)^{(p-1)/p}(pD)^{1/p}}{2} d < \Big(\frac{m}{M}\Big)^{1/p} c,$$

such that the assumption (A$_1$) *in Theorem 2 holds. Furthermore, suppose that*

$$(\mathrm{A}_3) \quad \frac{\int_a^b \max_{|t|\le c} F(x,t)dx}{m(2c)^p} < \frac{\int_{a+\alpha}^{b-\beta} F(x,d)dx}{Dd^p(b-a)^{p-1}pM}.$$

Then, for each

$$\lambda \in \left] \frac{Dd^p}{m\int_{a+\alpha}^{b-\beta} F(x,d)dx}, \frac{(2c)^p}{(b-a)^{p-1}pM\int_a^b \max_{|t|\le c} F(x,t)dx} \right[,$$

problem (1) *admits at least one nontrivial classical solution $\bar{u} \in X$, such that $|\bar{u}(x)| < c$ for all $x \in [a,b]$.*

Let $\gamma \in L^1([a,b])$ such that $\gamma(x) \ge 0$ a.e. $x \in [a,b]$, $\gamma \not\equiv 0$, and let $g : \mathbf{R} \to \mathbf{R}$ be a nonnegative continuous function. Consider the following Dirichlet boundary value problem

$$\begin{cases} -(p-1)|u'(x)|^{p-2}u''(x) = \lambda\gamma(x)g(u)h(x,u'), & x \in (a,b), \\ u(a) = u(b) = 0. \end{cases} \tag{6}$$

Put $G(t) := \int_0^t g(\xi)d\xi$ for all $t \in \mathbf{R}$, and set $\|\gamma\|_1 := \int_a^b \gamma(x)dx$.

Theorem 4 *Assume that there exist two positive constants c, d, with*

$$\frac{(b-a)^{(p-1)/p}(pD)^{1/p}}{2} d < \Big(\frac{m}{M}\Big)^{1/p} c,$$

such that

$$(\mathrm{A}_4) \quad \frac{G(c)}{c^p} < \left(\frac{2^p m \int_{a+\alpha}^{b-\beta} \gamma(x)dx}{D(b-a)^{p-1}pM\|\gamma\|_1} \right) \frac{G(d)}{d^p}.$$

Then, for each

$$\lambda \in \left] \frac{D}{m\int_{a+\alpha}^{b-\beta} \gamma(x)dx} \frac{d^p}{G(d)}, \frac{2^p}{(b-a)^{p-1}pM\|\gamma\|_1} \frac{c^p}{G(c)} \right[,$$

problem (6) *admits at least one positive classical solution $\bar{u} \in X$, such that $\bar{u}(x) < c$ for all $x \in [a,b]$.*

Proof Put $f(x, \xi) := \gamma(x)g(\xi)$ for all $(x, \xi) \in [a, b] \times \mathbf{R}$. Clearly, one has $F(x, t) = \gamma(x)G(t)$ for all $(x, t) \in [a, b] \times \mathbf{R}$. Therefore, taking into account that G is a nondecreasing function, Theorem 3 ensures the existence of a non-zero classical solution $\bar{u}$. Now, it is straightforward to show that $\bar{u}$ is nonnegative. Hence, owing to the strong maximum principle (see, e.g., [3, Theorem 11.1]) the classical solution $\bar{u}$, being non-zero, is positive and the conclusion is achieved.

3 Conclusion

In this paper, employing a very recent local minimum theorem for differentiable functionals obtained by Bonanno [1], the existence of at least one nontrivial solution for problem (1) is established. It is worth noticing that, usually, to obtain the existence of one solution, asymptotic conditions both at zero and at infinity in the nonlinear term are requested, while, here, it is assumed only a unique algebraic condition (see (A_4) in Theorem 4).

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