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Research article

Unbounded Order-To-Order Continuous Operators on Riesz Spaces

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Abstract

Let E and F be two Riesz spaces. An operator $T: E \to F$ between two Riesz spaces is said to be unbounded order-to-order continuous whenever $x_\alpha \xrightarrow{uo} 0$ in E implies $Tx_\alpha \xrightarrow{o} 0$ in F for each net $(x_\alpha) \subseteq E$. This paper aims to investigate several properties of a novel class of operators and their connections to established operator classifications. Furthermore, we introduce a new class of operators, which we refer to as order-to-unbounded order continuous operators. An operator $T: E \to F$ between two Riesz spaces is said to be order-to-unbounded order continuous (for short, *ouo*-continuous), if $x_\alpha \xrightarrow{o} 0$ in E implies $Tx_\alpha \xrightarrow{uo} 0$ in F for each net $(x_\alpha) \subseteq E$. In this manuscript, we investigate the lattice properties of a certain class of objects and demonstrate that, under certain conditions, order continuity is equivalent to unbounded order-to-order continuity of operators on Riesz spaces. Additionally, we establish that the set of all unbounded order-to-order continuous linear functionals on a Riesz space E forms a band of $E^\sim$.

Keywords: Riesz space, Order convergence, Unbounded order convergence

Mathematics Subject Classification (2020): 47B60, 46A40

1 Introduction

The notion of unbounded order convergence, also known as *uo*-convergence, was initially introduced in [5] and further developed in [11]. In recent years, this concept has received significant attention and has been the subject of investigation in several papers, including [4, 6, 7]. An area of particular interest is the study of geometric properties of Banach lattices using uo-convergence. Wickstead provided a characterization of spaces in which weak convergence of nets is equivalent to *uo*-convergence, see [13]. It was followed in [6], Gao characterized the space E such that in its dual space E^*, *uo*-convergence implies w^*-convergence and vice versa. He also characterized the spaces in whose dual space simultaneous *uo*- and w^*-convergence imply weak/norm convergence. Bahramnezhad and Haghnejad Azar have introduced unbounded order continuous operators on Riesz spaces and investigated on the lattices properties of this classification of operators, see [3]. Also in another article, Hanghnejad Azar, Jalili, and Moghimi introduced a new classification of operators as order-to-norm topology continuous operators and order-to-weak topology continuous operators in [9]. They investigated the properties of these operators, and left as an open problem whether every order-to-norm continuous operator from a Riesz space to a normed Riesz space has a modulus. This manuscript introduces a new classification of operators, namely strongly order continuous operators, and investigates their lattice properties. Specifically, we demonstrate that if an order bounded linear functional f on a Riesz space E is strongly order continuous, then its modulus exists and is also strongly order continuous.

Recall that a net $(x_\alpha)_{\alpha\in\mathscr{A}}$ in a Riesz space E is *order convergent* (or, *o*-convergent for short) to $x \in E$, denoted by $x_\alpha \xrightarrow{o} x$ whenever there exists another net $(y_\beta)_{\beta\in\mathscr{B}}$ in E such that $y_\beta \downarrow 0$ and for every $\beta \in \mathscr{B}$, there exists $\alpha_0 \in \mathscr{A}$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_0$. A net (x_α) in a Riesz space E is *unbounded order convergent* (or, *uo*-convergent for short) to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in E^+$. We denote this convergence by $x_\alpha \xrightarrow{uo} x$ and write that (x_α) *uo*-convergent to x. This is an analogue of pointwise convergence in function spaces. Let $\mathbb{R}^A$ be the Riesz space of all real-valued functions on a non-empty set A, equipped with the pointwise order. It is easily seen that a net (x_α) in $\mathbb{R}^A$ *uo*-converges to $x \in \mathbb{R}^A$ if and only if it converges pointwise to x. For instance in c_0 and $\ell_p(1 \leq p \leq \infty)$, *uo*-convergence of nets is the same as coordinate-wise convergence. Assume that (Ω, Σ, μ) is a measure space and let $E = L_p(\mu)$ for some $1 \leq p < \infty$. Then *uo*-convergence of sequences in $L_p(\mu)$ is the same as almost everywhere convergence. Note that the *uo*-convergence in a Riesz space E does not necessarily correspond to a topology on E. For example, let $E = c$, the Banach lattice of real valued convergent sequences. Put $x_n = \Sigma_{k=1}^n e_k$, where (e_n) is the standard basis. Then (x_n) is *uo*-convergent to $x = (1, 1, 1, \ldots)$, but it is not norm convergent.

We show that the collection of all order bounded strongly order continuous linear functionals on a Riesz space E is a band of $E^\sim$ where $E^\sim$ is order dual of E [Theorem 2.7]. For unexplained terminology and facts on Banach lattices and positive operators, we refer the reader to [1,2]. Let us start with the definition. Recall that an operator $T: E \to F$ between two Riesz spaces is said to be order continuous (resp. σ-order continuous) if $x_\alpha \xrightarrow{o} 0$ (resp. $x_n \xrightarrow{o} 0$) in E implies $Tx_\alpha \xrightarrow{o} 0$ (resp. $Tx_n \xrightarrow{o} 0$) in F. The collection of all order continuous operators of $L_b(E,F)$ (the vector space of all order bounded operators from E to F) will be denoted by $L_n(E,F)$, that is

$$L_n(E,F) := \{T \in L_b(E,F) : \text{T is order continuous}\}.$$

Similarly, $L_c(E,F)$ will denote the collection of all order bounded operators from E to F that are σ-order continuous. That is,

$$L_c(E,F) := \{T \in L_b(E,F) : \text{T is } \sigma\text{-order continuous}\}.$$

Let E, F be two Riesz spaces. Recall from [3], an operator $T: E \to F$ between two Riesz spaces is said to be *uo*-continuous, if $x_\alpha \xrightarrow{uo} 0$ in E implies $T(x_\alpha) \xrightarrow{uo} 0$ in F. The collection of all *uo*-continuous operators will be denoted by $L_{uo}(E,F)$. Recall that from [8], a continuous operator $T: E \to F$ between two normed Riesz spaces is said to be σ-*uon*-continuous, if for each norm bounded *uo*-null sequence $(x_n) \subseteq E$ implies $T(x_n) \xrightarrow{\|\cdot\|} 0$ in F. When $T: E \to F$ is an order bounded, its order adjoint $T': F^\sim \to E^\sim$ satisfies

$$T'(f(x)) = f(T(x)),$$

for all $f \in F^\sim$ and $x \in E$. A Riesz space is said to be laterally complete (resp. σ-*laterally complete*) whenever every subset of pairwise disjoint positive vectors (if every disjoint sequence) has a supremum. For a set A, $\mathbb{R}^A$ is an example of σ-laterally complete Riesz space. A positive non-zero vector a in a Riesz space E is an atom if the ideal I_a generated by a coincides with span a. We say that E is non-atomic if it has no atoms. We say that E is atomic if E is the band generated by all the atoms in it.

Consider an order bounded operator $T: E \to F$ between two Riesz spaces with F Dedekind complete. Then the null ideal N_T of T is defined by $N_T = \{x \in E : |T|(|x|) = 0\}$.

2 Unbounded Order-To-Order Continuous Operators

Definition 1. *An operator $T: E \to F$ between two Riesz spaces is said to be:*

i. *unbounded order-to-order continuous or strongly order continuous (so-continuous for short), if $x_\alpha \xrightarrow{uo} 0$ in E implies $Tx_\alpha \xrightarrow{o} 0$ in F for each net $(x_\alpha) \subseteq E$.*

ii. *σ-unbounded order-to-order continuous or σ-strongly order continuous (σ-so-continuous for short), if $x_n \xrightarrow{uo} 0$ in E implies $Tx_n \xrightarrow{o} 0$ in F for each sequence $(x_n) \subseteq E$.*

The collection of all *so*-continuous operators of $L_b(E,F)$ will be denoted by $L_{so}(E,F)$, that is

$$L_{so}(E,F) := \{T \in L_b(E,F) : \text{T is } so\text{-continuous}\}.$$

Similarly, $L_{\sigma-so}(E,F)$ will denote the collection of all order bounded operators from E to F that are σ-*so*-continuous. That is,

$$L_{\sigma-so}(E,F) := \{T \in L_b(E,F) : \text{T is } \sigma\text{-}so\text{-continuous}\}.$$

Example 1. *Let E be a Riesz space, $e \in E^+$ and B_e be a band generated by e in E. The operator $T: E \to B_e$ that defined by $T(x) = |x| \wedge e$ is a so-continuous operator.*

Remark 1.
1. *The class of so-continuous operators differ from the calss of uo-continuous operators. For example the identity operator $I: c_0 \to c_0$ is a uo-continuous operator, while it is not so-continuous.*
2. *Let F has order continuous norm. If $T: E \to F$ is so-continuous, then it is a weakly compact operator. Let $(x_n) \subseteq E$ be norm bounded and uo-null sequence. By assumption $T(x_n) \xrightarrow{o} 0$ in F. Because F has order continuous norm, $(T(x_n))$ is norm-null in F. So T is a σ-uon-continuous operator. By Remark 2.9 of [8], T is M-weakly compact and therefore is weakly compact.*

If $T: E \to F$ is a *so*-continuous operator, then it is also order continuous. In the following example we show that the converse is not true in general.

Example 2. *The identity $I: \ell^1 \to \ell^1$ is order continuous, while it is not so-continuous. Because $(e_n) \subseteq \ell^1$ is uo-null while it is not o-null.*

As we said, if $T: E \to F$ is a *so*-continuous operator, it is an order continuous and so it is an order bounded operator. In the following example we show that the converse is not true in general.

Example 3.
1. *The identity operator $I: c_0 \to c_0$ is order continuous and therefore is order bounded but is not so-continuous. Indeed, the standard basis sequence of c_0 is uo-converges to 0 but is not order convergent.*
2. *The operator $T: \ell^1 \to \ell^\infty$ defined by*
$$T(x_1, x_2, \ldots) = (\sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \ldots),$$
is order bounded. Now if $(e_n)_n$ is the standard basis of ℓ^1, then $e_n \xrightarrow{uo} 0$ in ℓ^1 and $T(e_n) = (1,1,1,\ldots)$. Therefore T is not so-continuous.

Proposition 1.
1. *Let E, F be two Riesz spaces such that E is finite-dimensional. Then $L_{so}(E,F) = L_n(E,F)$ and $L_{\sigma-so}(E,F) = L_c(E,F)$.*
2. *Let E, F be two Riesz spaces such that F is finite-dimensional. Then $L_{so}(E,F) = L_{uo}(E,F)$ and $L_{\sigma-so}(E,F) = L_{\sigma-uo}(E,F)$.*
3. *Let G be a sublattice of E. If $T \in L_{so}(E,F)$, then $T \in L_{so}(G,F)$.*

Proof.
1. (and 2.) Follows immediately if we observe that in a finite-dimensional Riesz space order convergence is equivalent to *uo*-convergence.
3. Let $(x_\alpha) \subseteq G$ be a *uo*-null net. It is obvious that (x_α) is *uo*-null in E. By assumption, we have $T(x_\alpha) \xrightarrow{o} 0$ in F.

□

Problem 1. *Let E and F be two Riesz spaces. Under what conditions can it be said $L_{so}(E,F) = L_n(E,F) \cap L_{uo}(E,F)$?*

Proposition 2. *Let E, F, G be Riesz spaces. Then we have the following assertions.*

1. *If $T \in L_{so}(E,F)$ and $S \in L_n(F,G)$, then $ST \in L_{so}(E,G)$. As a consequence, $L_{so}(E)$ is a left ideal for $L_n(E)$. Similarly, $L_{\sigma-so}(E)$ is a left ideal for $L_c(E)$.*
2. *If $T \in L_{uo}(E,F)$ and $S \in L_{so}(F,G)$, then $ST \in L_{so}(E,G)$.*
3. *If $T \in L_{so}(E,E)$, then $T^n \in L_{so}(E,E)$ for all $n \in \mathbb{N}$.*
4. *If E is σ-Dedekind complete and σ-laterally complete and $S \in L_c(E,F)$ and $T \in L_{\sigma-so}(F,G)$, then $TS \in L_{\sigma-so}(E,G)$. In this case, $L_{\sigma-so}(E,F) = L_c(E,F)$.*

Proof.
1. Let (x_α) be a net in E such that $x_\alpha \xrightarrow{uo} 0$. By assumption, $Tx_\alpha \xrightarrow{o} 0$. So, $STx_\alpha \xrightarrow{o} 0$. Hence, $ST \in L_{so}(E,G)$.

2. Let (x_α) be a net in E such that $x_\alpha \xrightarrow{uo} 0$. By assumption, $Tx_\alpha \xrightarrow{uo} 0$. So, $STx_\alpha \xrightarrow{o} 0$. Therefore, $ST \in L_{so}(E,G)$.

3. Let (x_α) be a net in E such that $x_\alpha \xrightarrow{uo} 0$. By assumption, $Tx_\alpha \xrightarrow{o} 0$ and so $Tx_\alpha \xrightarrow{uo} 0$. Therefore, $T^2x_\alpha \xrightarrow{o} 0$. Hence, $T^2 \in L_{so}(E,E)$. By induction, $T^n \in L_{so}(E,E)$ for all $n \in \mathbb{N}$.

4. Let E be a σ-Dedekind complete and σ-laterally complete Riesz space. By Theorem 3.9 of [7], we see that a sequence (x_n) in E is uo-null if and only if it is order null. So, if (x_n) be a sequence in E such that $x_n \xrightarrow{uo} 0$, then $x_n \xrightarrow{o} 0$. Thus, $Sx_n \xrightarrow{o} 0$ and then $TSx_n \xrightarrow{o} 0$. Hence, $TS \in L_{\sigma-so}(E,G)$. Clearly, we have $L_{\sigma-so}(E,F) = L_c(E,F)$. This ends the proof.

□

Let $T\colon E \to F$ be a positive operator between two Riesz spaces. We say that an operator $S\colon E \to F$ is dominated by T (or that T dominates S) whenever $|Sx| \leq T|x|$ holds for each $x \in E$.

Theorem 1. *The following assertions are true.*

1. *If a positive so-continuous operator $T: E \to F$ dominates S, then S is so-continuous.*

2. *If E and F are Archimedean laterally complete Riesz spaces, G is order dense in Dedekind complete Riesz space E and $T: G \to F$ is order continuous lattice homomorphism, then T is σ-so-continuous.*

Proof. 1. Let $T\colon E \to F$ be a positive *so*-continuous operator between two Riesz spaces such that T dominates $S\colon E \to F$ and let $x_\alpha \xrightarrow{uo} 0$ in E. It is obvious that $|x_\alpha| \xrightarrow{uo} 0$. So, by assumption, $T|x_\alpha| \xrightarrow{o} 0$ and from the inequality $|Sx| \leq T|x|$, we have $Sx_\alpha \xrightarrow{o} 0$. Hence, S is *so*-continuous.

2. By Theorem 2.32 of [2], the formula

$$S(x) = \sup\{T(y) : y \in G \text{ and } 0 \leq y \leq x\}, \quad x \in E^+,$$

defines an extension of T from E to F, which is an order continuous lattice homomorphism. Let $(x_n) \subseteq G$ is a *uo*-null sequence. By Theorem 3.10 of [7], (x_n) is order-null. Since S is order continuous, therefore $T(x_n) = S(x_n) \xrightarrow{o} 0$ in F.

□

Recall from [4] that $f \in E^*$ is said to be *un*-continuous, if for each *un*-null net $(x_\alpha) \subseteq E$, we have $f(x_\alpha) \to 0$ in $\mathbb{R}$.
It is clear that if E is an atomice Banach lattice with order continuous norm, then by Theorem 5.3 of [4], each $f \in E^*$ is σ-*so*-continuous iff it is a σ-*un*-continuous.

Remark 2. *1. Let E be a atomic Banach lattice with order continuous norm. If $f\colon E \to \mathbb{R}$ is a positive σ-so-continuous, then $f = \lambda_1 f_{a_1} + \lambda_2 f_{a_2} + ... + \lambda_n f_{a_n}$, where $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$ and $a_1, a_2, ..., a_n$ are atoms. Let $(x_n) \subseteq E$ be a un-null net. Because E is atomic with order continuous norm, by Theorem 5.3 of [4], (x_n) is uo-null. by assumption we have $f(x_n) \xrightarrow{o} 0$ and therefore it is norm-null. By Corollary 5.4 of [10], the proof is complete.*

2. *If E is non-atomice and $f\colon E \to \mathbb{R}$ is continuous and so-continuous, then by Corollary 5.4 of [10], $f = 0$.*

3. *By Corollary 2.6 of [12], $E^\sim_{uo}$ is an ideal of $E^\sim_n$ (or $E^\sim$) and so $E^\sim_{so}$ is an ideal of $E^\sim_n$.*

Remark 3. *Let E be a Banach lattice and such that $E^\sim_{uo}$ separates the points of E. By Proposition 2.13 of [12], the following conditions are equivalent.*

1. *E is finite dimension space.*

2. *$E^\sim_{so} = E^\sim_n$.*

3. *$E^\sim_{so}$ is an band of $E^\sim$.*

Theorem 2. *For an order bounded linear functional f on a Riesz space E the following statements are equivalent.*

1. *f is so-continuous.*

2. *f^+ and f^- are both so-continuous.*

3. *$|f|$ is so-continuous.*

Proof. (1) $\Rightarrow$ (2) Let $(x_\alpha) \subseteq E^+$ and $x_\alpha \xrightarrow{uo} 0$. Let (r_α) be a net in $\mathbb{R}$ such that $r_\alpha \downarrow 0$. According to Proposition 3.1 of [7], in view of $f^+x = \sup\{fy : 0 \le y \le x\}$, there exists a net (y_α) in E with $0 \le y_\alpha \le x_\alpha$ for each α and $f^+x_\alpha - r_\alpha \le fy_\alpha$. So, $f^+x_\alpha \le fy_\alpha + r_\alpha$. Since $x_\alpha \xrightarrow{uo} 0$, we have $y_\alpha \xrightarrow{uo} 0$. Thus, by assumption, $fy_\alpha \xrightarrow{o} 0$. It follows from $f^+x_\alpha \le (fy_\alpha + r_\alpha) \xrightarrow{o} 0$ that $f^+x_\alpha \xrightarrow{o} 0$. Hence, f^+ is *so*-continuous. Now, as $f^- = (-f)^+$, we conclude that f^- is also *so*-continuous.
(2) $\Rightarrow$ (3) Follows from the identity $|f| = f^+ + f^-$.
(3) $\Rightarrow$ (1) Follows immediately from Theorem 1 by observing that $|f|$ dominates f. □

Remark 4. *One can easily formulate by himself the analogue of Theorem 2 for σ-so-continuous operators.*

Recall that a subset A of a Riesz space is said to be order closed whenever $(x_\alpha) \subseteq A$ and $x_\alpha \xrightarrow{o} x$ imply $x \in A$. An order closed ideal is referred to as a band. Thus, an ideal A is a band if and only if $(x_\alpha) \subseteq A$ and $0 \le x_\alpha \uparrow x$ imply $x \in A$. In the next theorem we show that $E^\sim_{so}$ and $E^\sim_{\sigma-so}$ are both bands of $E^\sim$. The details follow.

Theorem 3. *If E is a Riesz space, then $E^\sim_{so}$ and $E^\sim_{\sigma-so}$ are both bands of $E^\sim$.*

Proof. We only show that $E^\sim_{so}$ is a band of $E^\sim$. That $E^\sim_{\sigma-so}$ is a band can be proven in a similar manner. Note first that if $|g| \le |f|$ holds in $E^\sim$ with $f \in E^\sim_{so}$, then from Theorems 3.1 and 3.2 it follows that $g \in E^\sim_{so}$). That is $E^\sim_{so}$ is an ideal of $E^\sim$. To see that the ideal $E^\sim_{so}$ is a band, let $0 \le f_\lambda \uparrow f$ in $E^\sim$ with $(f_\lambda) \subset E^\sim_{so}$, and let $0 \le x_\alpha \xrightarrow{uo} 0$ in E. Then for each fixed λ we have

$$0 \le f(x_\alpha) = ((f - f_\lambda)(x_\alpha) + f_\lambda(x_\alpha)) \xrightarrow{o} 0.$$

So, $f(x_\alpha) \xrightarrow{o} 0$. Thus, $f \in E^\sim_{so}$, and the proof is finished. □

3 Order-To-Unbounded Order Continuous Operators

Definition 2. *An operator $T : E \to F$ between two Riesz spaces is said to be:*

1. *order-to-unbounded order continuous (for short, ouo-continuous), if $x_\alpha \xrightarrow{o} 0$ in E implies $Tx_\alpha \xrightarrow{uo} 0$ in F for each net $(x_\alpha) \subseteq E$.*

2. *σ-order-to-unbounded order continuous (for short, σ-ouo-continuous), if $x_n \xrightarrow{o} 0$ in E implies $Tx_n \xrightarrow{uo} 0$ in F for each sequence $(x_n) \subseteq E$.*

The collection of all *ouo*-continuous (resp. σ-*ouo*-continuous) operators from E into F will be denoted by $L_{ouo}(E,F)$, (resp. $L_{\sigma-ouo}(E,F)$).

It is obvious that each identity operator on Riesz space E is an *ouo*-continuous operator and also we have $f \in E^\sim$ if and only if $f \in E^\sim_{ouo}$.

Theorem 4. *Let E be a normed Riesz space with order continuous norm and F be an atomic Banach lattice with order continuous norm, then each continuous operator $T : E \to F$ is σ-ouo-continuous.*

Proof. Let $(x_n) \subseteq E$ be an order-null net. Since E has order continuous norm, (x_n) is a norm-null net. By continuity of T, we have, $(T(x_n))$ is norm-null and hence it is *un*-null. Because F is atomic with order continuous norm, by Theorem 5.3 of [4], $(T(x_n))$ is *uo*-null. □

On the other hand, if $T : E \to F$ is *ouo*-continuous, it follows that T is a σ-*ouo*-continuous operator. However, the converse is not necessarily true. An example illustrating this point is given in Example 1.55 on page 46 of [2], where a σ-*ouo*-continuous operator is presented that is not *ouo*-continuous. It should be noted that the operator T in Example 1.55 of [2] is σ-order continuous, and therefore σ-*ouo*-continuous. However, it is not *ouo*-continuous, as can be easily verified.
The following example shows that in general each *ouo*-continuous operator is not *uo*or *so*-continuous.

Example 4.

The functional $f\colon \ell^1 \to \mathbb{R}$ defined by

$$f((x_1,x_2,...)) = \sum_{i=1}^{\infty} x_i,$$

is ouo-continuous. Let $(x_\alpha) \subseteq E$ be an order-null net. Since ℓ^1 has order continuous norm, therefore (x_α) is norm-null and so $f(x_\alpha) \to 0$ in $\mathbb{R}$. On the other hands, $(e_n) \subseteq \ell^1$ is uo-null. But $(f(e_n))$ is not uo-null in $\mathbb{R}$. Hence f is not a uo-continuous operator.

The identity operator $I\colon \ell^1 \to \ell^1$ is ouo-continuous. Consider $(e_n) \subseteq \ell^1$ is uo-null, but it is not order-null in ℓ^1. Therefore $I\colon \ell^1 \to \ell^1$ is not so-continuous.

Theorem 5. *Every continuous operator from $C[0,1]$ to ℓ^1 is σ-ouo-continuous.*

Proof. Let $T\colon C[0,1] \to \ell^1$ is a continuous operator. By Exercise 3 of page 313 of [2], T is a compact operator. Since $C[0,1]^*$ has order continuous norm, by Theorem 5.44 of [2], there exist a reflexive Banach lattice F, the lattice homomorphism Q and compact operator S that $T = S \circ Q$. Let $(x_n) \subseteq C[0,1]$ be an o-null sequence. Because Q is lattice homomorphism and therefore is order continuous, so $(Q(x_n))$ is o-null in F. F is a reflexive, so it has order continuous norm. Therefore $(Q(x_n))$ is norm-null in F. By continiuty of S, we have $(S(Q(x_n)))$ is norm-null and therefore is un-null in ℓ^1. Since ℓ^1 is atomic with order continuous norm, by Theorem 5.3 of [4], $T(x_n) = (S(Q(x_n)) \xrightarrow{uo} 0$ in ℓ^1. □

In the following, we provide examples of new classifications of operators.

Example 5.

1. *Since, $L_1[0,1]$ has order continuous norm and c_0 is an atomic Banach lattice with order continuous norm, the operator $T\colon L_1[0,1] \to c_0$, given by*
$$T(f) = (\int_0^1 f(x) sinx dx, \int_0^1 f(x) sin2x dx, \dots),$$
is a σ-ouo-continuous operator.

2. *The operator $T\colon C[0,1] \to \ell^1$, given by*
$$T(f) = (\frac{\int_0^1 f(x)\sin x dx}{n^2}, \frac{\int_0^1 f(x)\sin 2x dx}{n^2}, \dots)$$
is a σ-ouo-continuous operator.

3. *Let B be a projection band of Riesz space E and P_B the corresponding band projection. It follows easily from $0 \le P_B \le I$ (see Theorem 1.44 of [2]) that if $x_\alpha \xrightarrow{o} 0$ in E then $P_B x_\alpha \xrightarrow{o} 0$ in B and therefore $P_B x_\alpha \xrightarrow{uo} 0$ in B. So P_B is an ouo-continuous operator.*

4. *Let $E^\sim$ be the order dual of Riesz space E. It is obvious that each $f \in E^\sim$ is a ouo-continuous operator.*

Remark 5.

1. *Let E, F be two Riesz spaces such that E is finite-dimensional. Then $L_{uo}(E,F) = L_{ouo}(E,F)$ and $L_{\sigma-uo}(E,F) = L_{\sigma-ouo}(E,F)$.*

2. *If $T: E \to F$ is an so-continuous operator and $S\colon F \to G$ is ouo-continuous, it is obvious that $S \circ T: E \to G$ is an uo-continuous operator.*

3. *If $T: E \to F$ is an ouo-continuous operator and $S\colon F \to G$ is so-continuous, it is obvious that $S \circ T: E \to G$ is an o-continuous operator.*

4. *If $T: E \to F$ is an ouo-continuous operator and $S\colon F \to G$ is uo-continuous, it is obvious that $S \circ T: E \to G$ is an ouo-continuous operator.*

5. *Let G be a sublattice of Dedekind complete Riesz space E. Then $T: E \to F$ is ouo-continuous if and only if $T|_G$ is ouo-continuous.*

6. *Let $T, S\colon E \to F$ be two operators and $0 \le T \le S$. If S is ouo-continuous, then T is an ouo-continuous operator.*

Since, the proofs of the three following theorems are straightforward, we will not provide them here.

Theorem 6. *Let E and F be two Riesz spaces that F is order continuous and atomic. An operator $T: E \to F$ is σ-ouo-continuous if and only if σ-oun-continuous operator.*

Theorem 7. *Let E and F be two Banach lattices. Then, by one of the following assertions, $T: E \to F$ is an ouo-continuous operator.*

1. *T is order continuous,*
2. *T is uo-continuous,*
3. *T is so-continuous.*

Theorem 8. 1. *Let $T: E \to F$ be an order bounded operator between two Riesz spaces with F Dedekind complete. If T is an uo-continuous operator, then T, T^+, T^- and $|T|$ are ouo-continuous operators.*

2. *If $T \in L_{uo}(E,E)$, then $T^n \in L_{ouo}(E,E)$ for all $n \in \mathbb{N}$.*

Theorem 9. *Let E and F be two Riesz spaces that F is a Dedekind complete. An operator $0 \le T: E \to F$ is ouo-continuous if and only if $x_\alpha \downarrow 0$ in E implies $T(x_\alpha) \downarrow 0$.*

Proof. Let T be an *ouo*-continuous operator and $(x_\alpha) \subseteq E$ with $x_\alpha \downarrow 0$ in E. Because $x_\alpha \xrightarrow{o} 0$ by assumption we have $T(x_\alpha) \xrightarrow{uo} 0$. On the other hand $T(x_\alpha) \downarrow z$ and therefore $T(x_\alpha) \xrightarrow{uo} z$. Since *uo*-convergence are unique, we have $z = 0$.

Conversely, now let $(x_\alpha) \subseteq E$ be an *o*-null net. there exists another net (y_β) in E such that $y_\beta \downarrow 0$ and that for every β, there exists α_0 such that $|x_\alpha| \le y_\beta$ for all $\alpha \ge \alpha_0$. By assumption, we have $T(y_\beta) \downarrow 0$. So $|T(x_\alpha)| \le T|x_\alpha| \le T(y_\beta)$. It means that $T(x_\alpha) \xrightarrow{o} 0$ and hence $T(x_\alpha) \xrightarrow{uo} 0$ in F. □

Corollary 1. *If F is Dedekind complete Riesz space and $T: E \to F$ is a positive operator, then T is order continuous if and only if it is ouo-continuous.*

Corollary 2. *Let E and F be two Archimedean Riesz spaces that F is a Dedekind complete. An operator $0 \le T: E \to F$ is ouo-continuous if and only if there is an order dense and topologically majorizing sublattice H such that $T|_H$ is ouo-continuous.*

Proposition 3. *If $T: E \to F$ is a so-continuous operator, then its order adjoint $T': F^\sim \to E^\sim$ is ouo-continuous.*

Proof. Let $T: E \to F$ be a *so*-continuous operator. It is obvious that it is an order continuous operator. By Lemma 1.54 of [2], T is an order bounded operator. Now by Theorem 1.73 of [2], its order adjoint $T': F^\sim \to E^\sim$ is order continuous. Therefore by Remark 7, T' is an *ouo*-continuous operator. □

Remark 6. *The converse of Proposition 3, is not true in general. Consider the identity operator $I: c_0 \to c_0$. Its order adjoint $I: \ell^1 \to \ell^1$ is ouo-continuous, while $I: c_0 \to c_0$ is not so-continuous.*

Theorem 10. *Let $T: E \to F$ be an operator between to Riesz spaces.Then there exist a vector lattice G, an operator $T_1: E \to G$ and an operator $T_2: G \to F$ that $T = T_2 \circ T_1$. Such that*

1. *T_1 is ouo-continuous.*
2. *T is so-continuous if T_2 is so-continuous.*
3. *T is ouo-continuous if T_2 is ouo-continuous.*

Proof. Let $T: E \to F$ be an operator and $(x_\alpha) \subseteq E$ be a *uo*-null net. We have for all $u \in E^+$, $(|x_\alpha| \wedge u)$ is *o*-null. Let $u \in E^+$ is an arbitrary vector and B_u be a band generated by u in E. We put $G = B_u$ and $T_1: E \to G$ by $T_1(x) = P_G(x)$, where P_G is band projection from E to G. It is clear that T_1 is well define and it is an *ouo*-continuous operator.

We put $T_2: G \to F$ by $T_2(z) = T_2(P_G x) = T(x)$ that $z \in G$. T_2 is well define and we have $T = T_2 \circ T_1$. Let $(x_\alpha) \subseteq E$ be a *uo*-null. Therefore $(P_G(x_\alpha))$ is *uo*-null. Now if T_2 is *so*-continuous, we have $T(x_\alpha) = T_2(P_G(x_\alpha)) \xrightarrow{o} 0$. So T is *so*-continuous. The same way, if T_2 is *ouo*-continuous, then T is an *ouo*-continuous operator. □

Proposition 4. *Let $T: E \to E$ be an operator. The following assertions are equivalent.*

1. E has finite dimensional.

2. T is *so*-continuous if and only if is *ouo*-continuous.

Proof. $1 \Rightarrow 2$ Let E has finite dimentional, it is clear that $T: E \to E$ is a *so*-continuous operator if and only if it is an *ouo*-continuous operator.

$2 \Rightarrow 1$ Conversely, let $T: E \to E$ is *so*-continuous if and only if it is an *ouo*-continuous operator. Suppose E has infinite dimensional. Therefore there exists a net $(x_\alpha) \subseteq E$ that it is *uo*-null while it is not *o*-null. It is a contradiction by assumption. □

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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