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D-Graphs, Graphs, that Arise from Some Linear Equations

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Abstract Many interesting structures are arising from the Tower of Hanoi puzzle. Some of them increase the number of pegs and some of the others relax the Divine Rule. But all of them accept discs of different diameters. In this paper, we increased the number of available pegs and changed the Divine Rule by considering similar discs, that is, all discs have the same size diameter. From this point of view, the Tower of Hanoi puzzle becomes the distributing of n identical discs (objects) into k distinct labeled pegs (boxes). We modify Lucas's legend to justify these variations. Each distribution of n discs on k pegs is a regular state. In a Diophantine Graph, every possible regular state is represented by a vertex. Two vertices are adjacent in a Diophantine Graph if their corresponding states differ by one move. The Diophantine Graphs have shown to possess attractive structures. Since it can be embedded as a subgraph of a Hamming Graph, the Diophantine Graph may find applications in fault-tolerant computing.

Keywords Linear equations · Diophantine graphs · Connectivity · Distance · Tower of Hanoi

Mathematics Subject Classification (2010) 80A20 · 65Z05 · 65L06

1 Introduction

Towers of Hanoi problem was introduced in 1883 by the French number theorist Edouard Lucas (1842-1891). The traditional puzzle consists of three vertical pegs and n discs, each of mutually different diameters. The puzzle starts with all discs on the first peg arranged in such a way that no larger disc lies on a smaller one (*divine rule*). A state obeying this divine rule is called regular

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state [1,6]. The goal of the puzzle is to transfer all discs to the third peg under the following stipulations:

1. Only one disc may be moved at a time;
2. Only the top disc on each peg can be moved;
3. The Divine Rule - A larger disc can never be placed on top of a smaller one.

As shown in the 2013 text by Heinz et al. [8], there are many variations on the Tower of Hanoi puzzle. One variation involves increasing the number of available pegs as in the Reve's puzzle [3]. Many variations relax the Divine Rule. These variations include the Bottleneck Tower of Hanoi [2], the Santa Claus Tower, and the Sinners' Tower [4,5]. In this paper, we consider two changes in the Tower of Hanoi puzzle, by increasing the number of vertical pegs and changing the Divine Rule by considering all discs had the same size.

With these changes, the Tower of Hanoi problem, changed to the Distribution Problem, i.e. distributing n identical objects into k distinct labeled boxes (pegs). On the other hand, the Distribution Problem consists of k pegs numbered $1, 2, \ldots, k$, and n identical discs. A (legal) *$n-$configuration or regular state* is a distribution of n discs among the pegs by stacking them on the pegs. A (legal) move changes one $n-$configuration into another by moving one topmost disc on one peg to the top of another peg. In the original setting, all discs lie on the peg numbered '1' (this is a perfect state of the Problem) Fig. 1, and the task is to transfer them to the last peg numbered 'k'.

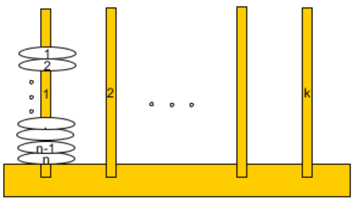


Fig. 1: Initial state, all n discs lie on the peg numbered '1'.

In this paper, we will show each regular state by a nondecreasing $n-$bit string $a_1a_2\ldots a_n$, where $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq k$. Here, we present a new interconnection structure, called the *Diophantine Graph*, which is inspired by the famous Diophantine Linear equations. One of the famous problems in elementary combinatorics is counting the number of ways of distributing n identical objects into k distinct labeled boxes. There are many interesting solutions in the literature.

One of these genius ways is representing each distribution by a binary string of length $n+k-1$. Suppose all boxes (pegs) are arranged side by side in a line by increasing labels. That is, they are arranged from left to right, numbered 1 through k. If we use a '0' to denote an object and a '1' to denote a separator vertical stroke between two adjacent boxes, then every way of distributing n identical objects in k distinct boxes, can be represented by a unique binary string of length $n+k-1$ with n 0's and $(k-1)$ 1's. This correspondence is indeed a bijection between the family of all distribution of n identical objects into k distinct boxes and the family of all such binary strings.

For example, in Fig. 2, the distribution of $n = 11$ objects into $k = 4$ boxes is represented by the binary string 00001001100000.

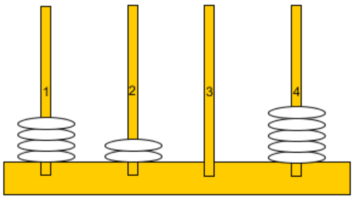


Fig. 2: A distribution and it's corresponds binary string 00001001100000

But, we can represent this binary string by a 4−ary string '11112244444' of length $n = 11$, in which we write the label of each box instead of its objects, that is, we labeled by 'i' for each object in the box of the label 'i'. It is easy to see that, there is a bijection between the family of all binary strings of length $n+k-1$ with n 0's and $(k-1)$1's and the family of all increasing $n-$bit string $a_1a_2\dots a_n$, where $1 \le a_1 \le a_2 \le \dots \le a_n \le k$.

Definition 1 Let $n, k \ge 1$ are given positive integers. Let $\alpha = a_1a_2\dots a_n$, be a string of length n over the set $\{1, 2, \dots, k\}$, then we say that the α is an increasing $n-$*string* over the set $\{1, 2, \dots, k\}$, if $1 \le a_1 \le a_2 \le \dots \le a_n \le k$.

Now, turn our attention to consider the following important and typical problem in combinatorics namely, finding the number of integer solutions to the linear equation:

$$x_1 + x_2 + \cdots + x_k = n, \qquad (1)$$

in k unknowns $x_1, x_2, \dots, x_k$, where $n \ge 1$ and $k \ge 1$ are nonnegative integers. An integer solution to the equation (1) is a $k-$tuple $(e_1, e_2, \dots, e_k)$ of integers satisfying (1) when x_i substituted by e_i, for each $i = 1, 2, \dots, k$. Now, every nonnegative integer solution $(e_1, e_2, \dots, e_k)$ to (1) corresponds to a way of distributing n identical objects to k distinct boxes as shown below:

$$\underbrace{\overbrace{o\cdots o}^{e_1}}_{\text{peg1}} + \underbrace{\overbrace{o\cdots o}^{e_2}}_{\text{peg2}} + \cdots + \underbrace{\overbrace{o\cdots o}^{e_k}}_{\text{pegk}} = n.$$

Clearly, different solutions to 1 correspond to different ways of distributing. On the other hand, every such way of distribution corresponds to a nonnegative

integer solution to (1). Since the correspondence is a one-to-one correspondence, thus, the number of increasing $k-$ary $n-$string is equal to the number of nonnegative solutions of the equation (1), is $\binom{n+k-1}{k-1}$.
From now on, we called each increasing $k-$ary $n-$string $\alpha = a_1 a_2 \dots a_n$ as a *Diophantine code* of the kind (n, k) or simply $(n, k)-$*Diophantine code*.

2 Diophantine Graph

A convenient and direct representation of the Distribution Problem is graph representation. In a Distribution Problem, every possible state of the Problem is represented by a vertex. Two vertices are adjacent in the Diophantine Graph if their corresponding states differ by one move. In this section, we define the *Diophantine graph* and investigate some basic parameter of it. But, before this we need the following definitions. Recall that the Hamming distance between two binary strings α and β is the number $H(\alpha, \beta)$ of bits, where α and β differ [9]. Now, we generalize this concept to Diophantine codes.

Definition 2 Let $n, k \geq 1$ be positive integers. The Diophantine graph D_k^n of kind (n, k) is the graph (V_k^n, E_k^n), where

$$V_k^n = \{a_1 a_2 \dots a_n : 1 \leq a_1 \leq a_2 \leq \dots \leq a_n \leq k\},$$

that is the set of all Diophantine codes of the kind (n, k) and $(\alpha, \beta) \in E_k^n$ if and only if $H(\alpha, \beta) = 1$.

Example 1 Let k=1 and n=3, then, D_1^3 is a graph with only one vertex $A = 111$ (Fig. 3(a)). For each $n \geq 1$ one can show that, $D_1^n \simeq K_1$.

Example 2 Let $k = 2$ and $n = 3$, then,

$$V_2^3 = \{111, 112, 122, 222\},$$

and $(3, 2)-$Diophantine Graph D_2^3 is isomorphic to P_4 (Fig. 3(b)).

For each $n \geq 1$, one can simply show that, $D_2^n \simeq P_{n+1}$.

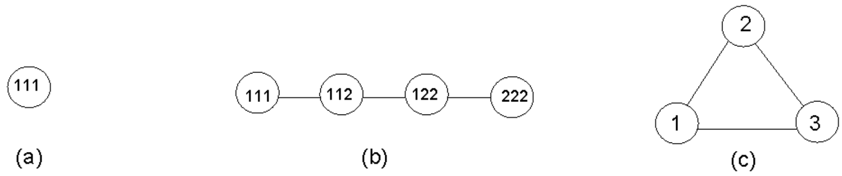


Fig. 3: Diophantine Graphs

Now, we want to draw the Diophantine graph corresponding to the non-negative integer solutions of the equation $x_1 + x_2 + x_3 = 1$. We want to show each solution of the equation by 1-string over the set $\{1, 2, 3\}$.The solutions of this equation are $x_1 = 1, x_2 = x_3 = 0$, which corresponds to the 1-string $a_1 = 1$; or the solution $x_2 = 1, x_1 = x_3 = 0$ where it corresponds to the 1-string $a_1 = 2$; and the last solution $x_3 = 1, x_1 = x_2 = 0$, which correspond to the 1-string $a_1 = 3$ (Fig. 3(c)).
Therefore, we have three distinct sequences of length 1, each pair is different in one component and their corresponding vertices in the Diophantine graph are adjacent. Hence, $D_3^1 \simeq K_3$. In general, we have the following lemma:

Lemma 1 *For each positive integer $k \geq 1$, we have $D_k^1 \simeq K_k$.*

Proof It is easy to prove by distributing of '1' object into k boxes. Then, there are n different $(1, k)-$Diophantine codes, in which, all pairs have one different component.

3 Some preliminary properties of D-Graphs

Our next task is to count $|V_k^n|$. It is a simple problem in introductory combinatorics (cf. [10]) to see that, each distribution of n identical objects in k labeled distinct boxes corresponds to a solution of the Diophantine equation $x_1 + x_2 + \cdots + x_k = n$, and hence, corresponding to an increasing $n-$string over the set $\{1, 2, \ldots, k\}$. So, we have the following lemma.

Lemma 2 $|V_k^n| = \binom{n+k-1}{k-1}$.

Now, we find some structural properties of the Diophantine graphs. Since each vertex of the Diophantine graph corresponds to a solution of the Diophantine linear equation $x_1 + x_2 + \cdots + x_k = n$.

Proposition 1 *Let $n \geq 1$ and $k \geq 1$ be positive integers. Let $\alpha = a_1 a_2 \ldots a_n$ is an arbitrary vertex of the Diophantine graph D_k^n, then*

$$deg_{D_k^n}(\alpha) = (a_n - a_1) + (k - 1).$$

Proof If $k = 1$, then for each positive integer $n \geq 1$ there is only one way to distribute n identical objects into one box. That is, in 1'st move, and 2nd move, ..., and n'th move, in each time only one object lies in the only box. Thus, $a_1 = a_2 = \cdots = a_n = 1$. Hence, the only one $(n, 1)-$Diophantine code is $\alpha = \overbrace{11\ldots1}^{n}$. Thus, $\deg_{D_1^n}(\alpha) = (1-1)+(1-1) = 0$. Now, suppose that, $k \geq 2$, and $\alpha = a_1 a_2 \ldots a_n$, $\beta = b_1 b_2 \ldots b_n$ are vertices of $(n, k)-$Diophantine graph D_k^n, we know α and β are adjacent if they are different in only one component. Thus, $(\alpha, \beta) \in E_k^n$ iff

- $a_1 \neq b_1$ and for each $i \neq 1$, $a_i = b_i$ and $1 \leq b_1 \leq a_2$, there is $a_2 - 1$ choice for b_1
  or

- $a_2 \neq b_2$ and for each $i \neq 2$, $a_i = b_i$ and $a_1 \leq b_2 \leq a_3$, there is $a_3 - a_1$ choice for b_2
  or
  $\vdots$
  or

- $a_{n-1} \neq b_{n-1}$ and for each $i \neq n-1$, $a_i = b_i$ and $a_{n-2} \leq b_n \leq a_n$, there is $a_n - a_{n-2}$ choice for b_{n-1}
  or

- $a_n \neq b_n$ and for each $i \neq n$, $a_i = b_i$ and $a_{n-1} \leq b_n \leq k$, there is $k - a_{n-1}$ choice for b_n.

Hence, by the Addition Principle, the desired number of adjacent vertices in the neighborhood of the vertex α is $(a_n - a_1) + k - 1$.

Immediately, from the above theorem the minimum and maximum degrees in the $(n, k)-$Diophantine graphs D_k^n are $k - 1$ and $2k - 2$, respectively. By Theorem 1, each of the vertices where $a_n = a_1$, has minimum degree $\delta = k - 1$ and those vertices whose labels include $a_n = k$ and $a_1 = 1$ have maximum degree $\Delta = 2k - 2$. Each vertex in $(n, k)-$Diophantine graph D_k^n labeled by a constant Diophantine code is called a *corner vertex*. Our main result is computed the size of the $(n, k)-$Diophantine graphs D_k^n, but first, we find the number of vertices, those Diophantine codes which are started with a given integers a_1 and terminated by an a_n.

Lemma 3 *Let,* $1 \leq a_1 \leq a_n \leq k$ *are given. Then, the number of vertices whose* $(n, k)-$*Diophantine Codes are started by* a_1 *and terminated by* a_n*, is*

$$\binom{n-2+(a_n-a_1+1)-1}{(a_n-a_1+1)-1} = \binom{n-2+(a_n-a_1)}{a_n-a_1}.$$

Proof Our main task is to find Diophantine Codes of kind (n, k), whose first and the last components are a_1 and a_n, respectively.
Since, $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq k$, and for each $1 \leq i \leq k$, we have $a_1 \leq a_i \leq a_n$, thus, the number of selections of a_i is $a_n - a_1 + 1$. On the other hand, for given a_1 and a_n, the number of distributing of n objects into $a_n - a_1 + 1$ boxes where, the first and the last boxes have at least one object analogous to distribution of $n - 2$ identical objects into, $a_n - a_1 + 1$ distinct labeled boxes and is

$$\binom{n-2+(a_n-a_1+1)-1}{(a_n-a_1+1)-1} = \binom{n-2+(a_n-a_1)}{a_n-a_1}.$$

In Theorem 1, we show that the degree of each vertex $\alpha = a_1a_2\dots a_n$ of (n,k)-Diophantine graphs, depends only on a_1 and a_n, by this fact and the hand shaking theorem [11], we can find the number of edges of the $(n,k)-$Diophantine graphs D_k^n.

Proposition 2 *Let $|E_k^n|$ be the number of edges of the Diophantine graph $D_k^n = (V_k^n, E_k^n)$, then*

$$|E_k^n| = q_k^n = \begin{cases} \binom{k}{2}, & \text{if } n=1,\\ \frac{(k-1)k(k+1)}{3}, & \text{if } n=2,\\ \frac{1}{2}\sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\binom{n+j-2}{j}j + \frac{1}{2}\binom{n+k-1}{k-1}(k-1), & \text{if } n\geq 3. \end{cases}$$

Proof If $n=1$, then for each $k\geq 1$, by lemma 1, $D_k^1 \simeq K_k$. Thus $q_k^1 = \binom{k}{2}$. Now, let $n=2$. Thus, we need all $(2,k)-$Diophantine Code ab, where we sorted them by lexicographic order in the following array:

$$\begin{array}{ccccc} 12 & 13 & 14 & \dots 1(k-1) & 1k \\ & 23 & 24 & \dots 2(k-1) & 2k \\ & & \ddots & \vdots & \vdots \\ & & & (k-2)(k-1) & (k-2)k \\ & & & & (k-1)k \end{array} \qquad (2)$$

This is a $(k-1)\times(k-1)$ array, we divided it along the main diagonal into $k-1$ disjoint subsets as follows:

$$L_i = \{ab \mid 1\leq a\leq b\leq k,\ b-a=i\},$$

where $i=1,2,\dots,k-1$. It is easy to check that each subset L_i, $i=1,2,\dots,k-1$, has $k-i$ elements. Thus,

$$\begin{aligned} 2q_k^2 &= \sum_{1\leq a\leq b\leq k} \deg(ab) \\ &= \sum_{1\leq a\leq b\leq k} [(b-a)+(k-1)] \qquad \text{(Theorem 1)} \\ &= \sum_{1\leq a\leq b\leq k}(b-a) + \sum_{1\leq a\leq b\leq k}(k-1) \\ &= \sum_{j=0}^{k-1}(k-j)j + \binom{2+k-1}{k-1}(k-1) \qquad \text{set} \quad b-a=j \\ &= \sum_{j=1}^{k-1}kj - \sum_{j=1}^{k-1}j^2 + \binom{k+1}{k-1}(k-1) \\ &= k(\frac{(k-1)k}{2}) - \frac{(k-1)k(2k+1)}{6} + \frac{(k-1)k(k+1)}{2} \end{aligned}$$

$$= \frac{k(k-1)}{2}\frac{6k-2k+1+3}{3} = 2\frac{(k-1)k(k+1)}{3}.$$

Thus,

$$q_k^2 = \frac{(k-1)k(k+1)}{3}.$$

Let $n \geq 3$ and $k \geq 3$. Since the degree of each vertex of D_k^n is dependent only on a_1 and a_n, it suffices to partition the vertex set V_k^n by $(2,k)-$ Diophantine Codes $a_1 a_n$, and hence, find the other components of the sequences $a_2, a_3, \ldots, a_{n-1}$. In other words, we want to find all Diophantine Codes of kind (n,k), $aa_2a_3\ldots a_{n-1}b$ for given a and b where $1 \leq a \leq b \leq k$.Therefore, for each component ab in array (2), we must compute the number of non-negative integer solutions of the following linear equation:

$$x_a + x_{a+1} + \cdots + x_b = n-2,$$

which equals to

$$\binom{n-2+((b-a)+1)-1}{((b-a)+1)-1} = \binom{n-2+(b-a)}{b-a}.$$

Let $b-a=j$. Since, $1 \leq a < b \leq k$ then, $j = 1,2,\ldots,k-1$. Thus,

$$\begin{aligned} 2q_k^n &= \sum_{v\in V_k^n} \deg(v) \\ &= \sum_{1\leq a<b\leq k} \deg(aa_2\ldots a_{n-1}b) \\ &= \sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\binom{n-2+j}{j}j + \sum_{v\in V_k^n}(k-1). \end{aligned}$$

Hence,

$$2q_k^n = \sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\binom{n+j-2}{j}j + \binom{n+k-1}{k-1}(k-1). \qquad (3)$$

In Proposition 2, by a little computing, one can reduce the double sigma into one sigma. Finally, we obtain a closed formula for the number of edges of this graph.

Proposition 3 *For each $n \geq 3$ and $k \geq 1$ we have,*

$$2q_k^n = \sum_{j=1}^{k-1}(k-j)j\binom{n+j-2}{j} + \binom{n+k-1}{k-1}(k-1). \qquad (4)$$

For this purpose, it is sufficient to show the following lemma.

Lemma 4 *For any positive integers $k \geq 2$ and $n \geq 1$ we have,*

$$\sum_{i=1}^{k-1}\sum_{j=1}^{k-i} j\binom{n+j-2}{j} = \sum_{j=1}^{k-1}(k-j)j\binom{n+j-2}{j}. \qquad (5)$$

Proof We will to show that both sides of the equality have the same recurrence relation with the same initial value. For this purpose, we set the left and right-hand-sides of (5) by $f(k)$ and $g(k)$, respectively. That is,

$$f(k) = \sum_{i=1}^{k-1}\sum_{j=1}^{k-i} j\binom{n+j-2}{j},$$

$$g(k) = \sum_{j=1}^{k-1}(k-j)j\binom{n+j-2}{j}.$$

It is easy to show that, $f(1) = g(1) = 0$, and $f(2) = g(2) = \binom{n-1}{1}$ and so on. But, for $k \geq 2$ we have,

$$f(k) = \sum_{j=1}^{k-1} j\binom{n+j-2}{j} + \sum_{i=2}^{k-1}\sum_{j=1}^{k-i} j\binom{n+j-2}{j}.$$

If we set $i - 1 = t$ then,

$$f(k) = \sum_{j=1}^{k-1} j\binom{n+j-2}{j} + \sum_{t=1}^{k-2}\sum_{j=1}^{(k-(1+t))} j\binom{n+j-2}{j}$$

That is,

$$f(k) = f(k-1) + \sum_{j=1}^{k-1} j\binom{n+j-2}{j}.$$

On the other hand,

$$g(k+1) = \sum_{j=1}^{k}((k-j)+1)j\binom{n+j-2}{j}$$

$$= \sum_{j=1}^{k}(k-j)j\binom{n+j-2}{j} + \sum_{j=1}^{k} j\binom{n+j-2}{j}.$$

So,

$$g(k+1) = g(k) + \sum_{j=1}^{k} j\binom{n+j-2}{j}.$$

Thus,

$$f(k+1) - f(k) = g(k+1) - g(k).$$

Hence, for each $n, k \geq 1$, we have,

$$f(k) = g(k).$$

Now, we write a closed formula for the sigma on the right side of equation (4).

Lemma 5 *For each $n \geq 3$ and $k \geq 3$ we have,*

$$\sum_{j=1}^{k-1}(k-j)j\binom{n+j-2}{j} = (k-1)(n-1)\binom{n+k-2}{k-2} - n(n-1)\binom{n+k-2}{k-3}. \quad (6)$$

Proof We consider the ordinary generating function

$$f(x) = \frac{1}{(1-x)^{n-1}} = \sum_{j=0}^{\infty}\binom{n+j-2}{j}x^j.$$

Now, by the derivative of the function $f(x)$, we have

$$f^{'}(x) = \frac{n-1}{(1-x)^n} = \sum_{j=1}^{\infty} j\binom{n+j-2}{j}x^{j-1}.$$

Now, define a new generating function $g(x)$ as

$$g(x) = xf^{'}(x) = \frac{(n-1)x}{(1-x)^n} = \sum_{j=1}^{\infty} j\binom{n+j-2}{j}x^j.$$

So,

$$g^{'}(x) = \frac{n-1}{(1-x)^n} + \frac{n(n-1)x}{(1-x)^{n+1}},$$

and so,

$$xg^{'}(x) = \frac{(n-1)x}{(1-x)^n} + \frac{n(n-1)x^2}{(1-x)^{n+1}} = \sum_{j=1}^{\infty} j^2\binom{n+j-2}{j}x^{j+1}.$$

Let,

$$\begin{aligned} h(x) = kg(x) - xg^{'}(x) &= \sum_{j=1}^{\infty}(kj-j^2)\binom{n+j-2}{j}x^{j+1} \\ &= \frac{k(n-1)x}{(1-x)^n} - \frac{(n-1)x}{(1-x)^n} - \frac{n(n-1)x^2}{(1-x)^{n+1}}. \end{aligned}$$

Suppose, $F(x) = \frac{h(x)}{1-x}$. Thus we have,

$$\begin{aligned} F(x) &= \frac{h(x)}{1-x} \\ &= \frac{k(n-1)x}{(1-x)^{n+1}} - \frac{(n-1)x}{(1-x)^{n+1}} - \frac{n(n-1)x^2}{(1-x)^{n+2}} \\ &= \sum_{j=1}^{\infty}(kj-j^2)\binom{n+j-2}{j}x^j\sum_{i=0}^{\infty}x^j \end{aligned}$$

$$= \sum_{t=1}^{\infty} (\sum_{j=1}^{t} (kj - j^2) \binom{n+j-2}{j}) x^t$$
$$= (k-1)(n-1)x + \sum_{j=0}^{\infty} \left\{ k(n-1)\binom{n+j+1}{j+1} - (n-1)\binom{n+j+1}{j+1} - n(n-1)\binom{n+j+1}{j} \right\} x^{j+2}.$$

Thus,

$$\sum_{j=1}^{k-1}(k-j)j\binom{n+j-2}{j} = (k-1)(n-1)\binom{n+k-2}{k-2} - n(n-1)\binom{n+k-2}{k-3}.$$

Therefore, we can rewrite Proposition 3 as follows.

Proposition 4 *For each $n \geq 3$ and $k \geq 3$ we have,*

$$2q_k^n = \sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\binom{n+j-2}{j}j + \binom{n+k-1}{k-1}(k-1)$$
$$= \sum_{j=1}^{k-1}(k-j)j\binom{n+j-2}{j} + \binom{n+k-1}{k-1}(k-1).$$

So,

$$2q_k^n = (k-1)(n-1)\binom{n+k-2}{k-2} - n(n-1)\binom{n+k-2}{k-3} + \binom{n+k-1}{k-1}(k-1).$$

Example 3 In Fig. (4a), for $n = 2$ and $k = 5$, We can easily find out with a few simple calculations

$$2q_5^2 = \sum_{i=1}^{5-1}\sum_{j=1}^{5-i}\binom{5+j-2}{j}j + \binom{2+5-1}{5-1}(5-1)$$
$$= (2-1)(5-1)\binom{2+5-2}{5-2} - 2(2-1)\binom{5+2-2}{5-3} + \binom{5+2-1}{5-1}(5-1)$$
$$= 4\binom{5}{3} - \binom{5}{2} + \binom{6}{4}4$$
$$= 80.$$

Hence, $q_5^2 = 40$.

Example 4 In graph Fig.(4b), with a few calculations one can find that

$$|E_4^4| = q_4^4 = 84.$$

4 Coloring of the D-Graph D_k^n

In this section, we use this labeling, which is key to coloring the vertices. It is customary to number the boxes $0, 1, \ldots, k-1$. Thus, far we have looked at known properties of the Diophantine graphs. we are now ready to proved a new result. The Diophantine graphs are complicated, but thanks to their symmetry and our convenient labeling, they can easily be colored.

For a positive integer c, a graph can be *c-colored* if there is a way to label the vertices with the colors $0, 1, \ldots, c-1$ such that adjacent vertices are different colors. The *chromatic number* of a graph G is the smallest number of colors needed and is denoted $\chi(G)$. For example, $\chi(D_k^1) = \chi(K_k) = k$.
As, induced subgraph $< \{11\ldots 1a_n | 1 \leq a_n \leq k\} >$ is isomorphic to K_k thus, $\chi(D_k^n) \geq k$. To see that k colors suffice, color the vertex labeled $a_1a_2\ldots a_n$ by the sum of its box numbers modulo k. That is,

$$f(a_1a_2\ldots a_n) = a_1 + a_2 + \cdots + a_n \quad mod(k).$$

To check that f is a k-coloring, observe that two vertices of D_k^n are adjacent if and only if they differ in exactly one place. Hence, we have the following theorem.

Proposition 5 *Let $k \geq 2$ and $n \geq 1$, then $\chi(D_k^n) = k$.*

5 Connectivity

In this section, we want to show that for each $(n,k)-$ Diophanteen graph D_k^n are connected graphs where $n, k \geq 1$ are positive integers.

Proposition 6 *Let $n, k \geq 1$ be positive integer. Thus, the Diophantine graph D_k^n is connected.*

Proof It is sufficient to show that there is a path between any two vertices of the Diophantine graph D_k^n. Let $A = a_1a_2\ldots a_n$ and $B = b_1b_2\ldots b_n$ are two vertices of D_k^n. Define $n-$string $C = A - B = c_1c_2\ldots c_n$ and call it difference vector, where $c_i = a_i - b_i$ for each $i = 1, 2, \ldots, n$. There are some coordinates c_i's are positive, negative, or equal to zero, for each $i = 1, 2, \ldots, n$.
Suppose for each $i \in \{i_1, i_2, \ldots, i_t\}$, the coordinate $c_i < 0$, where

$$1 \leq i_t < i_{t-1} < \ldots < i_1 \leq n,$$

and for each $j \in \{j_1, j_2, \ldots, j_s\}$, $c_j > 0$ where $1 \leq j_1 < j_2 < \ldots < j_s \leq n$ and for other indices i, c_i's are equal to 0. Let $A_0 = A$. Suppose i_1 be the index of the rightmost bit of C, such that $c_{i_1} < 0$, then, in A_0 we converse a_{i_1} into b_{i_1} and rest the remaining coordinates. Call the new sequence by A_1. Since i_1 is the greatest index such that $a_{i_1} < b_{i_1}$ thus, $b_{i_1+1} \leq a_{i_1+1}$. So $a_{i_1-1} \leq a_{i_1} < b_{i_1} \leq b_{i_1+1} \leq a_{i_1+1}$. Hence, $A_1 = a_1\ldots a_{i_1-1}b_{i_1}a_{i_1+1}\ldots a_n$ is an admissible string. Since A_0 and A_1 differ by exactly one coordinate, then, they are adjacent vertices of the Diophantine graph D_k^n. So, in the same way as above, one can show that for each l, $1 \leq l \leq t$ we can find the vertex A_l from A_{l-1}, by conversing the coordinate a_{i_l} into b_{i_l}. Hence, A_{i_l-1} and A_{i_l} are differ on exactly one digit, thus, they are adjacent in D_k^n. So, we have a sequence of consecutive adjacent vertices. Hence, $P_1 = A_0 A_1 \dots A_t$ is a path from A_0 through A_t. Now, set $B_0 = A_t$. If we compute the difference vector $C = B_0 - B$ then, all its coordinates are non negative, that is; $c_j \geq 0$ for each $j = 1, 2, \dots, n$. For each $j \in \{j_1, j_2, \dots, j_s\}$ where $1 \leq j_1 < j_2 < \dots < j_s \leq n$, the coordinates c_j of difference vector are positive and all remaining coordinates of the vector are zero. Let j_1 be the least indices of $n-$string B_0 such that c_{j_1} is positive. Replace the j_1'th coordinate B_0 by j_1'th coordinate b_{j_1}. Hence, we obtain the $n-$string $B_1 = a_1 \dots a_{j_1-1} b_{j_1} a_{j_1+1} \dots a_n$. Since, j_1 is the least indices of B_0 in which $a_{j_1} > b_{j_1}$. Thus, $b_{j_1-1} \geq a_{j_1-1}$. Hence,

$$a_{j_1-1} \leq b_{j_1-1} \leq b_{j_1} < a_{j_1} \leq a_{j_1+1}.$$

So, the $n-$string B_1 is an admissible string. Since the vertices B_0 and B_1 differ on exactly one position then, they are adjacent in the Diophantine graph D_k^n. So, in the same way as above, one can show that for each l, $1 \leq l \leq s$ one can find the vertex B_l from the vertex B_{l-1}, by replacing the coordinate a_{j_l} by b_{j_l}. So, we obtain the consecutive adjacent vertices $B_0, B_1, \dots, B_s$ and hence, the path $A_0 A_1 \dots A_t B_1 \dots B_s$ from A through B. Thus, the graph D_k^n is connected.

Since the minimum degree of vertices, D_k^n is $k-1$. Then, $\kappa(D_k^n) \leq k-1$. On the other hand the Diophantine graph D_k^n is a subdivision graph of the complete graph K_k. So, $\kappa(D_k^n) \geq k-1$. Hence, we have the following theorem.

Proposition 7 *For each $n, k \geq 1$ connectivity of the Diophantine graph D_k^n is $(k-1)$. That is, $\kappa(D_k^n) = k-1$.*

Example 5 Suppose in Fig. 4b, A and B are 2222 and 1234, respectively. Then, the diference vector C is $A - B = +0--$. So, by the above algorithm we have the consecutive adjacent vertices $A_0 = 2222$, $A_1 = 2224$, $A_2 = 2234$ and $B_1 = 1234$. Therefore, $P = A_0 A_1 A_2 B_1$ is a (shortest) path between the vertices A and B.

6 Distance properties of a D-Graphs

One of the most basic parameters of a graph G is the notion of distance. The distance between two vertices in a graph is a simple but surprisingly useful notion. It has led to the definition of several graph parameters such as the diameter, the radius, the average distance, and the metric dimension. In this section, we find the radius and diameter of the Diophantine graphs.

Definition 3 Let G be a connected graph. The distance $d_G(u, v)$ between two vertices u, v of a graph G is defined as the length of the shortest path between u and v in G.

Informally and naturally, the distance between u and v equals the least possible number of edges traversed from u to v. Specially $d_G(u,u)=0$.

Definition 4 Let G be a graph. We define, concerning G, the following notions:

- The eccentricity of a vertex $ecc(v)$ is the largest distance from v to another vertex;
$$ecc(v)=\max_{x\in V(G)} d_G(v,x).$$
- The diameter $diam(G)$ of G is the largest eccentricity over its vertices, and the radius $rad(G)$ of G is the smallest eccentricity over its vertices.
- The vertex u is a central vertex if $ecc(u)=rad(G)$. The center of G, $Z(G)$ is defined as
$$Z(G)=\{u\in V(G)\mid ecc(u)=rad(G)\}.$$

As can be seen, the definition of a Diophantine graph is analogous to that of the Boolean cube. In the following, we show that the shortest path between two vertices of the Diophantine graph D_k^n is the Hamming distance between them. The Hamming distance between two strings of equal length is the number of positions at which the corresponding symbols are different. In another way, it measures the minimum number of substitutions required to change one string into the other. On the other hand, let $A=a_1a_2\dots a_n$ and $B=b_1b_2\dots b_n$ are two arbitrary vertices of the Diophantine graph D_k^n. The Hamming distance between A and B is defined as the following function,

$$H(A,B)=\Big|\{i: a_i\neq b_i,\ \text{where } A=a_1a_2\dots a_n \text{ and } B=b_1b_2\dots b_n\}\Big|.$$

For example, $H(11223345,11123455)=3$. Since, the $(n,k)-$Diophantine graph D_k^n, contains all increasing $n-$string over the alphabet $\{1,2,\dots,k\}$, then, two vertices A and B are adjacent if and only if $H(A,B)=1$. One can simply show that $d_{D_k^n}(A,B)=H(A,B)$, where $d_{D_k^n}(A,B)$ is the shortest path between A and B. Indeed, by the proof of the theorem 6, there is a path of length $H(A,B)$ between two vretices A and B. So, $H(A,B)\geq d_{D_k^n}(A,B)$. Now, suppose $P=A_0A_1\dots A_m$ is a shortest path between $A=A_0$ and $B=A_m$. Two consecutive vertices of the path are adjacent iff $H(A,B)=1$. So, A and B are different at most in $d_{D_k^n}(A,B)$ positions. Hence, $H(A,B)=d_{D_k^n}$.
Now, we want to compute the eccentricity of each vertex of the Diophantine graph. Suppose $A=a_1a_2\dots a_n$ be a given vertex of the graph D_k^n, set $M_i(A)=\{j|a_j=i\}$ where, $1\leq j\leq n$ and $i=1,2,\dots,k$.

Lemma 6 *Let A and B be two arbitrary vertices of the graph D_k^n. Thus, there is a corner vertex $\bar{i}$ such that, $d(A,\bar{i})\geq d(A,B)$.*

Proof Set $m=\min\{|M_i(A)\cap M_i(B)|: i=1,2,\dots,k\}$, thus, there is a j where $1\leq j\leq k$ such that, $m=|M_j(A)\cap M_j(B)|$. By the definition of Hamming distance, we have

$$d(A,B)=n-\sum_{i=1}^{k}|M_i(A)\cap M_i(B)|\leq n-m=d(A,\bar{j}).$$

Thus, for calculating the eccentricity of vertex$A$, it is sufficient to only calculate its distance from the Corner vertices. So,

$$ecc(A) = \max\{d(A, \bar{i}) : i = 1, 2, \ldots, k\}.$$

Since, the corner vertices $\bar{1} = 11\ldots1$ and $\bar{k} = kk\ldots k$ are different on n coordinates, then, $diam(D_k^n) = n$. Thus we have the following lemma.

Lemma 7 *For each positive integer $n \geq 1$and $k \geq 1$, the diameter of the Diophantine graph D_k^n is n.*

By Lemma 6, for calculating the eccentricity of a vertex A, it is sufficient to only calculate its distance from the Corner vertices. Indeed, for each vertex $A \in D_k^n$, we have

$$\begin{aligned} ecc(A) &= \max\{d(A, X) : X \in V(D_k^n)\} \\ &= \max\{d(A, \bar{i}) : i \in \{1, 2, \ldots, k\}\} \\ &= \max\{n - n_i : i = 1, 2, \ldots, k\}, \end{aligned}$$

where $n_i = |M_i(A)|$. Let A be a vertex of the graph D_k^n, then, by Lemma 6 there is a corner vertex $\bar{i}$, such that $ecc(A) = n - n_i$. If $n_i < \lfloor \frac{n}{k} \rfloor$, then $n - n_i > n - \lfloor \frac{n}{k} \rfloor$. But, the eccentricity of the vertex

$$B = \overbrace{1\ldots1}^{n_1}\overbrace{2\ldots2}^{n_2}\ldots\overbrace{k\ldots k}^{n_k},$$

is $n - \lfloor \frac{n}{k} \rfloor$ where, $n_i \geq \lfloor \frac{n}{k} \rfloor$ for $i = 1, 2, \ldots, n$. Hence, the vertex A can not be a central vertex. Thus, we have the following lemma.

Lemma 8 $rad(D_k^n) = n - \lfloor \frac{n}{k} \rfloor$.

Now, by Lemma 6 and Lemma 8, one can construct many graphs with given radius.

Proposition 8 *Let $n \geq 1$ and $k \geq 1$ be positive integers then, the center $Z(D_k^n)$ is an induced subgraph of the Diophantine graph D_k^n of order $\binom{r+k-1}{k-1}$ where $r = n - k\lfloor \frac{n}{k} \rfloor$.*

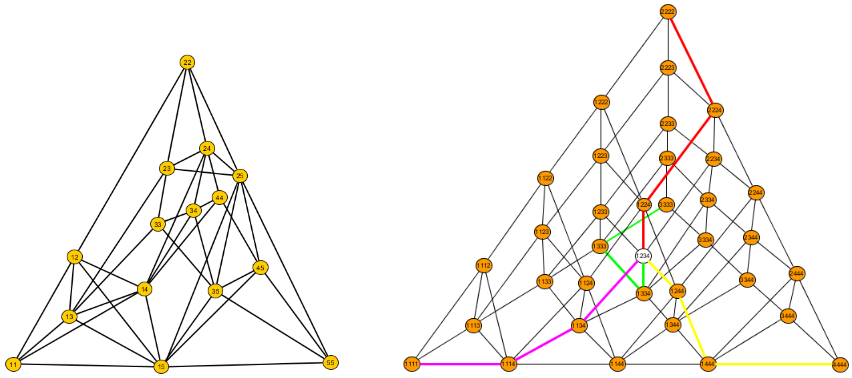
Corollary 1 *Let $n \geq 1$ and $k \geq 1$ be positive integers. then,*

a) If $n < k$ then $Z(D_k^n) \cong D_k^n$ that is, the Diophantine graph is self-center.
b) If $k|n$ then, the Diophantine graph D_k^n is a mono center that is $Z(D_k^n) \cong K_1$ with center

$$V(Z(D_k^n)) = \{\overbrace{1\ldots1}^{m}\overbrace{2\ldots2}^{m}\ldots\overbrace{k\ldots k}^{m}\},$$

where $m = \frac{n}{k}$.

Example 6 The following figures show some D-graphs with various centers and radii.



(a) Self center graph $Z(D_5^2) = D_5^2$ with radius $rad(D_5^2) = 2$

(b) The center and radius of the D-graph D_4^4.

Fig. 4: The radius and center of some D-graphs

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