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Research article

On n-Capable Groups

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Abstract

A group G is called n-capable if for a suitable group H we have $G \cong H/Z_n(H)$. In this article, we impose some conditions to an n-capable group G and find a group H with the mentioned condition such that $G \cong H/Z_n(H)$.

Keywords: Capability, n-Capable group

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1 Introduction

In 1938, Baer [1] initiated a systematic investigation of the question when a group G can be isomorphic to the group of inner automorphisms of some group H. Also, in Philip Hall's 1940 paper [4], it is shown the way towards the classification of groups of prime power order. Here is what Hall himself had to say about it:
"The question of what conditions a group G must fulfill in order that it may be the central quotient group of another group H,

$$G \cong \frac{H}{Z(H)}$$

is an interesting one. But while it is easy to write down a number of necessary conditions it is not so easy to be sure that they are sufficient."

Calling a group which is a central factor group a capable group occurred much later and is due to M. Hall and Senior [5]. Of course there are groups that are not capable (non-trivial cyclic groups for example), and so the condition that a group is capable imposes certain restrictions on its structure. The notion of capable groups is already studied by many authors (see for instance [2, 3, 8]). A group G is said to be n-capable if there is a group H such that $G \cong H/Z_n(H)$. In the present paper, we impose some properties to n-capable group G and we find a group H with these properties such that $G \cong H/Z_n(H)$.

2 Main Results

Let G and H be two groups. Then an *n-isoclinism* ($n \geq 1$) between G and H is a pair of isomorphisms (α, β) with $\alpha : G/Z_n(G) \longrightarrow H/Z_n(H)$ and $\beta : \gamma_{n+1}(G) \longrightarrow \gamma_{n+1}(H)$ such that the following diagram commutes:

$$\begin{array}{ccc} G/Z_n(G) \times \cdots \times G/Z_n(G) & \longrightarrow & \gamma_{n+1}(G) \\ \downarrow \alpha^{n+1} & & \downarrow \beta \\ H/Z_n(H) \times \cdots \times H/Z_n(H) & \longrightarrow & \gamma_{n+1}(H) \end{array}$$

where horizontal maps are defined by $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}) \longmapsto [[x_1, x_2], \ldots, x_{n+1}]$ such that $\bar{x}_i = x_i Z_n(G)$ and $\bar{x}_i = x_i Z_n(H)$ in the top and bottom horizontal maps, respectively (see [6] for more details). If there exists such an n-isoclinism, we say that G is n-isoclinic to H.

Lemma 1. *([6, Theorem 7.7]) Let G be a group. The following properties are equivalent.*

(a) G is n-isoclinic to a finite group.

(a) $G/Z_n(G)$ is finite.

(a) G is n-isoclinic to a finite section of itself.

Lemma 2. *([7]) Let G be a finite capable group. Then there is a finite group H such that $G \cong H/Z(H)$.*

The following proposition generalizes the above result which is one of the main lemmas of [7]. The notion of n-isoclinism helped us to provide a shorter proof than that presented in [7].

Proposition 1. *Let G be an n-capable finite group. Then there is a finite group H such that $G \cong H/Z_n(H)$.*

Proof. Since G is n-capable, there exists a group K such that $G \cong K/Z_n(K)$. As $K/Z_n(K)$ is finite, by part (b)⇒(a) of Lemma 1, K is n-isoclinic to a finite group H, that is $K/Z_n(K) \cong H/Z_n(H)$ and hence $G \cong H/Z_n(H)$. □

In the next results, we discuss the nilpotency and solvability conditions on H.

Proposition 2. *Let G be a nilpotent group of class m and there exists a group K such that $G \cong K/Z_n(K)$ $(m, n \geq 1)$. Then there is a nilpotent group H such that $G \cong H/Z_n(H)$.*

Proof. By hypothesis $K/Z_n(K)$ is nilpotent of class m. Thus

$$\frac{K}{Z_n(K)} = Z_m\left(\frac{K}{Z_n(K)}\right) = \frac{Z_{m+n}(K)}{Z_n(K)}.$$

Therefore $Z_{m+n}(K) = K$ and K is nilpotent of class at most $m+n$. Now, if we put $H := K$, then the proof will be completed. □

Proposition 3. *Let G be an n-capable solvable group. Then there is a solvable group H such that $G \cong H/Z_n(H)$.*

Proof. Clearly, for an arbitrary group K and for every $n \geq 0$, $Z_n(K)$ is solvable. Now, n-capability of G implies that for a group K we have $G \cong K/Z_n(K)$. Since $K/Z_n(K)$ and $Z_n(K)$ are solvable, K is also solvable. Therefore we can take $H := K$. □

A group G is called *polynilpotent* if it has a subnormal series

$$\{1\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G,$$

which the quotient groups G_{i+1}/G_i are nilpotent, for all $1 \leq i \leq n$.

Theorem 1. *Let G be an n-capable polynilpotent group. Then there is a polynilpotent group H such that $G \cong H/Z_n(H)$.*

Proof. Suppose that $G \cong K/Z_n(K)$ and consider the following subnormal series of $G \cong K/Z_n(K)$

$$\{1\} = G_0 \cong \frac{K_0}{Z_n(K)} \subseteq G_1 \cong \frac{K_1}{Z_n(K)} \subseteq \cdots \subseteq G_n = G \cong \frac{K_n}{Z_n(K)}.$$

Now, since for every group K and $n \geq 0$, $Z_n(K)$ is nilpotent, it is sufficient to show that K_{i+1}/K_i is nilpotent for all $1 \leq i \leq n$. The latter assertion is trivial as

$$\frac{K_{i+1}}{K_i} \cong \frac{K_{i+1}/Z_n(K)}{K_i/z_n(K)} \cong \frac{G_{i+1}}{G_i},$$

is nilpotent. In fact, K has the following subnormal series

$$\{1\} \subseteq Z_n(K) = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = K.$$

Therefore we can choose $H := K$. □

Theorem 2. *Let G be a finitely generated n-capable group with r generators. Then there exists a finitely generated group H with r generators such that $G \cong H/Z_n(H)$.*

Proof. Assume that $G \cong K/Z_n(K)$ and

$$\frac{K}{Z_n(K)} = \langle x_1Z_n(K), \ldots, x_rZ_n(K)\rangle.$$

Define $H = \langle x_1, \ldots, x_r\rangle \leq K$. First, we show that

$$Z_n(H) = Z_n(K) \cap H.$$

Let $x \in Z_n(H)$ and $k_1, \ldots, k_n$ be arbitrary elements of K. we can take $k_i = x_j z_j$ for some $z_j \in Z_n(K)$, ($1 \leq i \leq n$ and $1 \leq j \leq r$). Now, since we may consider $Z_n(K)$ as marginal subgroup of K

$$\begin{aligned} [k_1, k_2, \ldots, k_n, x] &= [x_{j_1} z_{j_1}, x_{j_2} z_{j_2}, \ldots, x_{j_n} z_{j_n}, x] \\ &= [x_{j_1}, x_{j_2}, \ldots, x_{j_n}, x] \\ &= 1. \end{aligned}$$

Therefore $x \in Z_n(K) \cap H$ and hence $Z_n(H) \subseteq Z_n(K) \cap H$. The converse of latter inclusion is obvious. Now, as $HZ_n(K) = K$ we have

$$\frac{H}{Z_n(H)} = \frac{H}{Z_n(K) \cap H} \cong \frac{HZ_n(K)}{Z_n(K)} = \frac{K}{Z_n(K)} \cong G,$$

and this completes the proof. □

Let π is a non-empty set of primes, a π-*number* is a positive integer whose prime divisors belong to π. An element of a group is called a π-*element*, if its order is a π-number and finally a group is called π-*group* if all of its elements are π-element.

Lemma 3. *([6, Lemma 7.8]) Let G be a finite group such that $G/Z_n(G)$ is a π-group. Then there exists a subgroup H of G such that H is a π-group which is n-isoclinic to G.*

Theorem 3. *Let G be an n-capable finite π-group. Then there is a finite π-group H such that $G \cong H/Z_n(H)$.*

Proof. Assume that $G \cong K/Z_n(K)$. Since $K/Z_n(K)$ is finite by Proposition 1, there is a finite group M such that $\frac{K}{Z_n(K)} \cong \frac{M}{Z_n(M)}$. As M is finite and $M/Z_n(M)$ is π-group, then by Lemma 3, there exists a subgroup H of M such that H is a π-group and M is n-isoclinic to H, that is $\frac{M}{Z_n(M)} \cong \frac{H}{Z_n(H)}$, which completes the proof. □

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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