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On n-Capable Groups

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Abstract A group G is called n-capable if for a suitable group H we have $G \cong H/Z_n(H)$. In this article, we impose some conditions to an n-capable group G and find a group H with the mentioned condition such that $G \cong H/Z_n(H)$.

Keywords Capability · n-Capable group

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1 Introduction

In 1938, Baer [1] initiated a systematic investigation of the question when a group G can be isomorphic to the group of inner automorphisms of some group H. Also, in Philip Hall's 1940 paper [4], it is shown the way towards the classification of groups of prime power order. Here is what Hall himself had to say about it:
"The question of what conditions a group G must fulfill in order that it may be the central quotient group of another group H,

$$G \cong \frac{H}{Z(H)}$$

is an interesting one. But while it is easy to write down a number of necessary conditions it is not so easy to be sure that they are sufficient."

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Calling a group which is a central factor group a capable group occurred much later and is due to M. Hall and Senior [5]. Of course there are groups that are not capable (non-trivial cyclic groups for example), and so the condition that a group is capable imposes certain restrictions on its structure. The notion of capable groups is already studied by many authors (see for instance [2, 3,8]). A group G is said to be n-capable if there is a group H such that $G \cong H/Z_n(H)$. In the present paper, we impose some properties to n-capable group G and we find a group H with these properties such that $G \cong H/Z_n(H)$.

2 Main results

Let G and H be two groups. Then an *n-isoclinism* ($n \geq 1$) between G and H is a pair of isomorphisms (α, β) with $\alpha : G/Z_n(G) \longrightarrow H/Z_n(H)$ and $\beta : \gamma_{n+1}(G) \longrightarrow \gamma_{n+1}(H)$ such that the following diagram commutes:

$$\begin{array}{ccc} G/Z_n(G) \times \cdots \times G/Z_n(G) & \longrightarrow & \gamma_{n+1}(G) \\ \downarrow{\scriptstyle \alpha^{n+1}} & & \downarrow{\scriptstyle \beta} \\ H/Z_n(H) \times \cdots \times H/Z_n(H) & \longrightarrow & \gamma_{n+1}(H) \end{array}$$

where horizontal maps are defined by $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}) \longmapsto [[x_1, x_2], \ldots, x_{n+1}]$ such that $\bar{x}_i = x_i Z_n(G)$ and $\bar{x}_i = x_i Z_n(H)$ in the top and bottom horizontal maps, respectively (see [6] for more details). If there exists such an n-isoclinism, we say that G is n-isoclinic to H.

Lemma 1 *([6, Theorem 7.7]) Let G be a group. The following properties are equivalent.*

(a) G is n-isoclinic to a finite group.
(b) $G/Z_n(G)$ is finite.
(c) G is n-isoclinic to a finite section of itself.

Lemma 2 *([7]) Let G be a finite capable group. Then there is a finite group H such that $G \cong H/Z(H)$.*

The following proposition generalizes the above result which is one of the main lemmas of [7]. The notion of n-isoclinism helped us to provide a shorter proof than that presented in [7].

Proposition 1 *Let G be an n-capable finite group. Then there is a finite group H such that $G \cong H/Z_n(H)$.*

Proof Since G is n-capable, there exists a group K such that $G \cong K/Z_n(K)$. As $K/Z_n(K)$ is finite, by part (b)$\Rightarrow$(a) of Lemma 1, K is n-isoclinic to a finite group H, that is $K/Z_n(K) \cong H/Z_n(H)$ and hence $G \cong H/Z_n(H)$.

In the next results, we discuss the nilpotency and solvability conditions on H.

Proposition 2 *Let G be a nilpotent group of class m and there exists a group K such that $G \cong K/Z_n(K)$ $(m, n \geq 1)$. Then there is a nilpotent group H such that $G \cong H/Z_n(H)$.*

Proof By hypothesis $K/Z_n(K)$ is nilpotent of class m. Thus

$$\frac{K}{Z_n(K)} = Z_m(\frac{K}{Z_n(K)}) = \frac{Z_{m+n}(K)}{Z_n(K)}.$$

Therefore $Z_{m+n}(K) = K$ and K is nilpotent of class at most $m+n$. Now, if we put $H := K$, then the proof will be completed.

Proposition 3 *Let G be an n-capable solvable group. Then there is a solvable group H such that $G \cong H/Z_n(H)$.*

Proof Clearly, for an arbitrary group K and for every $n \geq 0$, $Z_n(K)$ is solvable. Now, n-capability of G implies that for a group K we have $G \cong K/Z_n(K)$. Since $K/Z_n(K)$ and $Z_n(K)$ are solvable, K is also solvable. Therefore we can take $H := K$.

A group G is called *polynilpotent* if it has a subnormal series

$$\{1\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G,$$

which the quotient groups G_{i+1}/G_i are nilpotent, for all $1 \leq i \leq n$.

Theorem 1 *Let G be an n-capable polynilpotent group. Then there is a polynilpotent group H such that $G \cong H/Z_n(H)$.*

Proof Suppose that $G \cong K/Z_n(K)$ and consider the following subnormal series of $G \cong K/Z_n(K)$

$$\{1\} = G_0 \cong \frac{K_0}{Z_n(K)} \subseteq G_1 \cong \frac{K_1}{Z_n(K)} \subseteq \cdots \subseteq G_n = G \cong \frac{K_n}{Z_n(K)}.$$

Now, since for every group K and $n \geq 0$, $Z_n(K)$ is nilpotent, it is sufficient to show that K_{i+1}/K_i is nilpotent for all $1 \leq i \leq n$. The latter assertion is trivial as

$$\frac{K_{i+1}}{K_i} \cong \frac{K_{i+1}/Z_n(K)}{K_i/z_n(K)} \cong \frac{G_{i+1}}{G_i},$$

is nilpotent. In fact, K has the following subnormal series

$$\{1\} \subseteq Z_n(K) = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = K.$$

Therefore we can choose $H := K$.

Theorem 2 *Let G be a finitely generated n-capable group with r generators. Then there exists a finitely generated group H with r generators such that $G \cong H/Z_n(H)$.*

Proof Assume that $G \cong K/Z_n(K)$ and

$$\frac{K}{Z_n(K)} = \langle x_1 Z_n(K), \dots, x_r Z_n(K)\rangle.$$

Define $H = \langle x_1, \dots, x_r\rangle \leq K$. First, we show that

$$Z_n(H) = Z_n(K) \cap H.$$

Let $x \in Z_n(H)$ and $k_1, \dots, k_n$ be arbitrary elements of K. we can take $k_i = x_j z_j$ for some $z_j \in Z_n(K)$, $(1 \leq i \leq n$ and $1 \leq j \leq r)$. Now, since we may consider $Z_n(K)$ as marginal subgroup of K

$$\begin{aligned}[k_1, k_2, \dots, k_n, x] &= [x_{j_1} z_{j_1}, x_{j_2} z_{j_2}, \dots, x_{j_n} z_{j_n}, x] \\ &= [x_{j_1}, x_{j_2}, \dots, x_{j_n}, x] \\ &= 1.\end{aligned}$$

Therefore $x \in Z_n(K) \cap H$ and hence $Z_n(H) \subseteq Z_n(K) \cap H$. The converse of latter inclusion is obvious. Now, as $HZ_n(K) = K$ we have

$$\frac{H}{Z_n(H)} = \frac{H}{Z_n(K) \cap H} \cong \frac{HZ_n(K)}{Z_n(K)} = \frac{K}{Z_n(K)} \cong G,$$

and this completes the proof.

Let π is a non-empty set of primes, a π-*number* is a positive integer whose prime divisors belong to π. An element of a group is called a π-*element*, if its order is a π-number and finally a group is called π-*group* if all of its elements are π-element.

Lemma 3 *([6, Lemma 7.8]) Let G be a finite group such that $G/Z_n(G)$ is a π-group. Then there exists a subgroup H of G such that H is a π-group which is n-isoclinic to G.*

Theorem 3 *Let G be an n-capable finite π-group. Then there is a finite π-group H such that $G \cong H/Z_n(H)$.*

Proof Assume that $G \cong K/Z_n(K)$. Since $K/Z_n(K)$ is finite by Proposition 1, there is a finite group M such that $\frac{K}{Z_n(K)} \cong \frac{M}{Z_n(M)}$. As M is finite and $M/Z_n(M)$ is π-group, then by Lemma 3, there exists a subgroup H of M such that H is a π-group and M is n-isoclinic to H, that is $\frac{M}{Z_n(M)} \cong \frac{H}{Z_n(H)}$, which completes the proof.

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