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Research article

Generalized k-Rainbow and Generalized 2-Rainbow Domination in Graphs

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Abstract

Assume we have a set of k colors and to each vertex of a graph G we assign an arbitray of these colors. If we require that each vertex to set is assigned has in its closed neighborhood all k colors, then this is called the generalized k-rainbow dominating function of a graph G. The corresponding γ_{gkr}, which is the minimum sum of numbers of assigned colors over all vertices of G, is called the gk-rainbow domination number of G. In this paper, we present a linear algorithms for determining a minimum generalized 2-rainbow dominating set of a tree and on $GP(n,2)$.

Keywords: Generalized 2-rainbow, Domination, Honeycomb network

Mathematics Subject Classification (2020): 05C38, 90B10, 05C07

1 Introduction

Domination and its variations in graphs have been extensively studied, c.[1,2]. For a graph $G=(V,E)$, a set S is a domination set if every vertex in V\S is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. We call a dominating set of cardinality $\gamma(G)$ a $\gamma(G)$-set. For subsets $S,T \subseteq V$, the set S is said to dominate T if every vertex of T is adjacent to a vertex of S.

Domination represents situation in which each vertex location that a guard does not occupy needs to have a guard in a closed neighboring vertex location. In these situations, only one type of guard is considered. Assume a more complex situation where there are different types of guards (let there be k such types), and we require that each vertex location that is not occupied with a guard has in its closed neighborhood all types of guards. [10] From a practical point of view, suppose 5 mechanics in a workshop are doing the same job and each of them needs 7 different tools. Why only a mechanic who has no tools should have access to all the tools in his neighborhood?! Rather, this condition should be considered for all of them. That is, a mechanic who only has one, two, three, ... or six tools should have access to 7 required tools around him. Therefore, I with change the definition in the following article. "All graph vertices in their neighborhood see all labels." This relaxation leads to the following definitions.

Let G be a graph and let f be a function that assigns to each vertex a set of colors chosen from the set $\{1,2\}$; that is, f:V(G)→ $P\{(1,2)\}$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have $\bigcup_{u \in V(G)} f(u) = \{1,2\}$. Type domination in graphs Assume we have a set of 2 colors and to each vertex of a graph G we assign an arbitary of these colors. If we require that each vertex to which an empty set is assigned has in its neighborhood all 2 colors, then this is called the 2-rainbow dominating function of a graph G. The corresponding γ_{2r}, which is the minimum sum of numbers of assigned colores over all vertices of G, is called the 2-rainbow domination number of G.

Definition 1. *Let G be a graph and let f be a function that assigns to each vertex a set of colors chosen from the set $\{1,\ldots,k\}$; that is, $f : V(G) \to p\{(1,\ldots,k)\}$. If for each vertex $v \in V(G)$, we have $\bigcup_{u \in V(G)} f(u) = \{1,\ldots,k\}$; then f is called generalized k-rainbow dominating function(gkrdf) of G. The weight, $\omega(f)$, of a function f is defined as $\omega(f) = \sum_{v \in V(G)} |f(v)|$. Given a graph G, the minimum weight of a GKRDF is called the generalized k-rainbow dominating number of G, which we denote by $a\gamma_{gkr}(G)$.*

Definition 2. *Let G be a graph and let f be a function that assigned to each vertex a set of colors chosen from the set $\{1,2\}$; that is, $f : V(G) \to P(\{1,2\})$, of a function f is defined, if for each vertex $v \in V(G)$ we have $\bigcup_{u \in N[v]} f(u) = \{1,2\}$. then f is called generalized 2-rainbow dominating function($G2RDF$) of G. The weight, $\omega(f)$, of a function f is defined as $\omega(f) = \sum_{v \in V(G)} |f(v)|$. Given a graph G, the minimum weight of a G2RDF is called the generalized 2-rainbow dominating number of G, which we denote by a $\gamma_{g2r}(G)$.*

Theorem 1. *[9] Let G be a graph. Then for any $k \geq 2$*

$$\min\{|G|, \gamma(G) + k - 2\} \leq \gamma_{grk} \leq k\gamma(G). \tag{1}$$

The attempt in [9] to characterize graphs with $\gamma = \gamma_{r2}$ was inspired by the following famous problem.

2 Generalized 2-Rainbow Domination Function Graphs

2.1 $\gamma_g 2r$ for Graphs k_n

Generally, for graphs k_n that $V(k_n) = v_1,\ldots,v_n$, we labeled $f(v_1) = \{1,2\}$ and the rest of the vertices are labeled $\emptyset$, s0 $\gamma_{g2r} = 2$ and $\gamma(G) = 1$, then we observe relationship (1) is established.

2.2 $\gamma_g 2r$ for Graphs k_m, n

Generally, for graphs $k_{m,n}$, V($k_{m,n}$)=$\{u_1,\ldots,u_m,v_1,\ldots,v_n\}$ and u_i not adjacent to v_j, v_i not adjacent to u_j and any u_i are adjacent all v_j, and any v_i are adjacent all u_j, then if $f(v_1) = \emptyset$ then for example, we label $f(u_1) = \{1\}$ and $f(u_2) = \{2\}$ because $f(u_1) = \{1\}$ so, at least f should be $\{2\}$ for one of v. We assume $f(u_2) = \{2\}$ then should for example $f(v_3) = \{1\}$, therefore, it is sufficient that the rest of the vertex have enough $\emptyset$ then $w(f) = 4$. We observe relationship (1) is established.

2.3 $\gamma_g 2r$ for Graphs k_1, n

γ_{g2r} for graphs $k_{1,n}$ or star graphs, are $w(f) = 2$, then establishing relationship (1) for these graphs are easily visible.

Definition 3. *A tree graph that has n vertices with k hanging vertices that has degree $k+2$ and also the beginning and end vertices of the graph have degree $k+1$ and is represented by F_k.*

Theorem 2. *For graphs F_k, $\gamma_{g2r} = 2n$.*

Proof. The Proof is readily available. □

2.4 $\gamma_g 2r$ for Paths

γ_{g2r} for paths are as follows:

- For p_i if $i = 2,3$ then $\gamma_{g2r} = 2$.
- For p_i if $i,n \in N$ and $3n+1 \leq i \leq 3(n+1)$ then $\gamma_{g2r} = 2n+2$.

2.5 $\gamma_g 2r$ for Graphs C_n

Generally, γ_{g2r} for these graphs are as follows:

- For C_i if $i = 2,3$ then $\gamma_{g2r} = 2$.
- For C_i if $i,n \in N$ and $i = 3n$ then $\gamma_{g2r} = 2n$.
- For C_i if $i,n \in N$ and $3n+1 \leq i \leq 3(n+1)$ then $\gamma_{g2r} = 2n+2$.

Definition 4. *Let $n \geq 3$ and k be relatively prime natural numbers $k < n$. The generalized Petersen graph $GP(n,k)$ is defined as follows. Let C_n, C'_n be two disjoint cycles of length n. Let the vertices of C_n be $u_1,\ldots,u_n$ and edges u_iu_{i+1} for $i = 1,\ldots,n-1$ and u_nu_1. Let the vertices of C'_n be $v_1,\ldots,v_n$ and edges v_iv_{i+k} for $i = 1,\ldots,n$, the sum $i+k$ being taken modulo n (throughout this section). The graph $GP(n,k)$ is obtained from the union of C_n and C'_n by adding the edges u_iv_i for $i = 1,\ldots,n$. Its obvious that $GP(n,k) = GP(n,n-k)$. The graph $GP(5,2)$ or $GP(5,3)$ is the well-known Petersen graph.*

2.6 $\gamma_g 2r$ for Graphs GP(N,2)

Theorem 3. *For graphs $GP(n,2)$ that $n \geq 3$ and n and 2 are prime to each other ((n,2)=1), then γ_{g2r} for these graphs are as follows:*

$$\gamma_{g2r} \leq \begin{cases} 4[\frac{n}{3}], & \text{if } n \equiv 0(mod3), \\ 4([\frac{n}{3}]+1), & \text{if } n \equiv 1(mod3) \text{ and } n \equiv 2(mod3). \end{cases} \tag{2}$$

Proof. (a) If $n \equiv 0(mod3)$, we use the following algorithm and define the function f on $GP(n,2)$ such that $(n,2) = 1$.

Step 1: $f(u_i) = \emptyset$ if $i \not\equiv 0(mod3)$ and $f(u_i) = \{1,2\}$ if $i \equiv 0(mod3)$.

Step 2: $f(v_{3t}) = f(v_{3t-1}) = \emptyset$ for $t = 1,2,\ldots$ and $f(v_{3t+1}) = \{1,2\}$ for $t = 0,1,2,\ldots$

In the graphs $GP(n,2)$, in the outer circle of the graph, all the vertices with a multiplier of 3 have labeled with $\{1,2\}$ then $\omega(f)$ for the outer circle of the graph is equal to $2[\frac{n}{3}]$ and in the inner round of the graph, all the vertices with a multiplier of $3k+1$ have labeled with $\{1,2\}$ then $\omega(f)$ for the inner circle of the graph is equal to $2[\frac{n}{3}]$. Therefore $\gamma_{g2r}(G) \leq 2[\frac{n}{3}]+2[\frac{n}{3}] = 4[\frac{n}{3}]$.

(a) If $n \equiv 2(mod3)$, we use the following algorithm and define the function f on $GP(n,2)$ such that $(n,2) = 1$.

Step 1: If $i \neq n$ and $i \not\equiv 0(mod3)$, then $f(u_i) = \emptyset$ and $f(u_i) = \{1,2\}$ if $i \equiv 0(mod3)$ and $f(u_n) = \{1,2\}$.

Step 2: $f(v_{3t}) = \emptyset = f(v_{3t-1})$ for $t = 0,1,2,\ldots$ and $f(v_{3t+1}) = \{1,2\}$ for $t = 1,2,\ldots$. In the graphs, $GP(n,2)$, in the outer circle of the graph, all vertices with a multiplier of 3 and vertices u_n have the labe of $\{1,2\}$, then $\omega(f)$ for the outer circle of the graph is equal to $2[\frac{n}{3}]+2$ and in the inner round of the graph, all vertices with a multiplier of $3k+1$ have labeled with $\{1,2\}$ then $\omega(f)$ for the inner circle of the graph is equal to $2[\frac{n}{3}]+2$. So,

$$\gamma_{g2r}(GP(n,2)) \leq 2[\frac{n}{3}]+2+2[\frac{n}{3}]+2 = 4[\frac{n}{3}]+4.$$

(a) If $n \equiv 1(mod3)$, we use the following algorithm and define the function f on $GP(n,2)$ such that $(n,2) = 1$.

Step 1: If $i \neq n$ and $i \not\equiv 0(mod3)$ then $f(u_i) = \emptyset$ and if $i \equiv 0(mod3)$ then $f(u_i) = \{1,2\}$.

Step 2: $f(v_{3t}) = f(v_{3t-1}) = \emptyset$ for $t = 1,2,\ldots$ and $f(v_{3t+1}) = \{1,2\} = f(v_n)$ for $t = 0,1,2,\ldots$.

So, In the graphs $GP(n,2)$, in the outer circle of the graph, all vertices with a multiplier of 3 and u_n are labeled with $\{1,2\}$. Therefore $\omega(f)$ for the outer circle of the graph is equal to $2[\frac{n}{3}]+2$ and in the inner round of the graph, all vertices with a multiplier of $3k+1$ and vertices v_n have labeled with $\{1,2\}$. So, $\omega(f)$ for the inner circle of the graph is equal $2[\frac{n}{3}]+2$. Finally, $\gamma_{g2r}(GP(n,2)) \leq 2[\frac{n}{3}]+2+2[\frac{n}{3}]+2 = 4([\frac{n}{3}]+1)$.

□

Theorem 4. *For graphs GP(n,3) that, $n \geq 4$ and n and 3 are prime to each other((n,3)=1), then γ_{g2r} for these graphs are as follows:*

$$\gamma_{g2r}GP(n,3) \leq 4([\frac{n}{3}]+1).$$

Proof. We use the following partition of V(GP(n,3));
In the graphs GP(n,3), in the outer circle of the graph the vertices of u_{3k+1} that $k = 0,1,2,\dots$ are labeled of $\{1,2\}$ and the rest of the vertices are labeled of $\emptyset$. Then, γ_{g2r} the outer circle of the graph is equal $2[\frac{n}{3}]$. But in the inner round of the graph

- the vertices of v_{3k+2} that $k = 0,2,4,\dots$ are labeled of $\{1,2\}$ then γ_{g2r} this vertices is equal (for more caution, we label a vertex more with $\{1,2\}$) $[\frac{n}{3}]+2$.
- the vertices of v_{3k+2} that $k = 1,3,5,\dots$ are labeled of $\{1,2\}$ and γ_{g2r} this vertices is equal (for more caution, we label a vertex more with $\{1,2\}$) $[\frac{n}{3}]+2$, and the rest of the vertices are labeled of $\emptyset$.

As a result, γ_{g2r} the inner round of the graph is $2[\frac{n}{3}]+4$. Eventually,

$$\gamma_{g2r}GP(n,3) \leq 2[\frac{n}{3}]+[\frac{n}{3}]+2+[\frac{n}{3}]+2 = 4([\frac{n}{3}]+1).$$

□

Definition 5. *The honeycomb network $HC(1)$ is a hexagon. The honeycomb network $HC(2)$ is obtained adding six hexagon to the boundary edges of $HC(1)$. Inductively, honeycomb network $HC(n)$ is obtained from $HC(n-1)$ by adding a large of hexagons around the boundary of $HC(n-1)$. The number of vertices and edges of $HC(n)$ are $6n^2$ and $9n^2-3n$ respectively. The application of Honeycomb network is very vest, it is applied in different networking such as all-to-all broadcasting, in cellular services, in computer networking. It is also used in chemistry to represent the structures of different compounds. The following results are required.*

Lemma 1. *([3]) The boundary of $HC(n)$ is the cycle $C_{6(2n-1)}$.*

Lemma 2. *([3]) For $n \geq 2$, $|V(HC(n))| - |V(HC(n-1))| = 6(2n-1)$.*

2.7 $G2RDF$ for Honeycomb Network $HC(N)$

Theorem 5. *For Honeycomb network $HC(n)$,*

$$\gamma_{g2r}(HC(n)) \leq 4\sum_{k=2}^{n}(2k-1).$$

Proof. According the lemma 1, the boundary of $HC(1)$, $HC(2)$ and $HC(n)$ is C_6, C_{18} and $C_{6(2n-1)}$ respectively. For to get the generalized 2-rainbow dominating number Honeycomb network $HC(n)$, at first we label each circle separately to reach the nth circle, secondly with calculate the $\omega(f)$ for each round and finally we find sum of them. It is done in following way. The vertices of the first round (C_6) of the Honeycomb network $HC(n)$ with $\emptyset$ is labeled. For second round, we consider an arbitrary vertex with degree 3 and call that with w and its label is $\{1,2\}$. Then, the label of the other vertices of this round is like label of a cycle graph (the label of first vertex is $\{1,2\}$, the labels of second and third vertices are $\emptyset$ and it continues in the same way until to end). Since, the $\gamma_{g2r}(C_{3n}) = 2n$ and $f : HC(2) \to \{1,2\}$, we have

$$\omega(f) = \gamma_{g2r}(HC(2)) = \gamma_{g2r}(C_{6(2(2)-1)}) = \gamma_{g2r}(C_{18}) = \gamma_{g2r}(C_{3\times 6}) = 2(4k-2),$$

that k is the number of rounds.
Then, with the our method, for $f : HC(3) \to \{1,2\}$, we have

$$\begin{aligned}\omega(f) &= \gamma_{g2r}(C_{6(2(3)-1)}) + \gamma_{g2r}(C_{6(2(2)-1)})\\ &= \gamma_{g2r}(C_{30}) + \gamma_{g2r}(C_{18})\\ &= 4\sum_{k=2}^{3}(2k-1).\end{aligned}$$

In the same way until, for $f : HC(n) \to \{1,2\}$, we have

$$\omega(f) = \sum_{k=2}^{n}\gamma_{g2r}(C_{3(4k-2)}) = 4\sum_{k=2}^{n}(2k-1),$$

then,

$$\gamma_{g2r}(HC(n)) \leq 4\sum_{k=2}^{n}(2k-1).$$

□

3 Conclusion and Future Works

In this paper Based on the concept of usability k-rainbow domination applicability, we generalized it to be more evident in the field of application and at the same time reduce costs. For this purpose, we remove one of the conditions of k-rainbow (vertex with empty label) and instead we did added the condition of having k neighbors for each vertex. we did this generalized k-rainbow domination for simple graphs and so did this generalized 2-rainbow domination for simple graphs and $GP(n,2)$ and $GP(n,3)$.
According to the above process, For future works, we can expand generalized k-rainbow domination for $GP(n,k)$ and present different algorithms.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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