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The Fastest Three-Step with Memory Method by Four Self-Accelerating Parameters

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Abstract In this paper, a new family of eighth-order iterative methods for solving simple roots of nonlinear equations is developed. Each member of the proposed family requires four functional evaluations in each iteration that it is optimal according to the sense of Kung-Traub's conjecture. They have four self-accelerating parameters that are calculated using the adaptive method. The R-order of convergence has increased from 8 to 16 (maximum improvement).

Keywords With-memory method · Accelerator parameter · Weight function · R-order of convergence · Nonlinear equations

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1 Introduction

1.1 Definition

Definition 1 Suppose an iterative method (IM) converges to some limit α and lets x_n be an arbitrary sequence in R^n which converges to α. Hence

$$R_m(x) = \begin{cases} \lim\limits_{n\to\infty} sup||x_n - \alpha||^{\frac{1}{n}}, & \text{for} \quad m = 1, \\ \lim\limits_{n\to\infty} sup||x_n - \alpha||^{\frac{1}{m^n}}, & \text{for} \quad m > 1, \end{cases}$$

is named the R-factor of the sequence x_n [28].

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Definition 2 The convergence R-order of an iterative method (IM) at the spot α is

$$O_R((IM),\alpha)=\begin{cases}+\infty, & \text{if } \ R_m((IM),\alpha)=0, \ \text{ for all } \ m\in[1,+\infty),\\ \inf\{m\in[1,+\infty):R_m((IM),\alpha)=1\}, \text{otherwise.}\end{cases}$$

Convergence R-order is very proper for determining the convergence rate of the with-memory methods [28].

1.2 Literature

One of the most studied problems in numerical analysis is the approximation of the roots of the nonlinear equations. A powerful tool is the use of iterative methods. We propose methods derivative-free. Steffensen-type methods are suitable. These methods use divided differences to compute the roots of non-differentiable problems. Among the repetitive methods, multi-step methods have a higher efficiency index. One can be studied the Newton-like single-step methods in [5,20,32]. Kung and Traub [45], Neta [26], and Bi et al.'s methods [4] are examples of three-step schemes without memory that converge to the root of the nonlinear equation with a suitable approximation. These methods have more efficiency to fourth-order iterative method. References [8,6,11, 17,39] provide information about two-step and three-step without memory methods. Jaiswal [16], Lotfi et al. [23,24], Petković et al. [29], Soleymani et al. [36,37], Torkashvand and Kazemi [42], and Wang- Zhang [46], have also provided efficiency index for the scheme method based on technique memorization.

1.3 Existing iterative method

In 2008, Chun proposed the idea of the weight function for improving the order convergence of a two-step method as follows and made the fourth-order method the conditions given by the weight function $g(x)$ [15]

$$\begin{cases}y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \ n=0,1,2,\ldots,\\ t_n=\frac{f(y_n)}{f(x_n)}, \ g(0)=1, \ g'(0)=-2, \ |g''(0)|<\infty,\\ x_{n+1}=y_n-\frac{f(y_n)}{f'(x_n)g(t_n)}.\end{cases} \tag{1}$$

In 2009, Bi et al. made the three-step method based on Chun's method. Their optimal method eight-order is as follows [3]

$$\begin{cases}y_n=x_n-\frac{f(x_n)}{f'(x_n)},\\ z_n=y_n-\frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f(x_n)},\\ \beta=-0.5, \ t_n=\frac{f(z_n)}{f(x_n)}, \ g(0)=1, \ g'(0)=2, \ |g''(0)|<\infty,\\ x_{n+1}=z_n-g(t_k)\frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}.\end{cases} \tag{2}$$

In 2020, Torkashvand et al. developed the first two-step method with the highest efficiency index. The conditions of the weight function used to be as follows [41]

$$\begin{cases} x_0, \gamma_0, \beta_0, \lambda_0 \text{ is given}, \\ \beta_k = -\frac{1}{N'_{3n}(x_n)}, \gamma_n = -\frac{N''_{3n+1}(w_n)}{2N'_{3n+1}(w_n)}, \lambda_n = \frac{N'''_{3n+2}(y_n)}{6}, \; n \geq 4, \\ w_n = x_n + \beta_n f(x_n), \; y_n = x_n - \frac{f(x_n)}{f[w_n, x_n] + \gamma_n f(w_n)}, \; n = 0, 1, 2, \ldots, \\ t_n = \frac{f(y_n)}{f(x_n)}, \; g(0) = 1, \; g'(0) = -1, \; |g''(0)| < \infty, \\ x_{n+1} = y_n - g(t_n) \frac{f(x_n)^2}{(f(x_n) - f(y_n))^2} \frac{f(y_n)}{f[y_n, w_n] + \gamma_n f(w_n) + \lambda_n (y_k - x_n)(y_n - w_n)}. \end{cases} \tag{3}$$

1.4 Motivation and organization

Our motivation from this article is the construction of the first-class three-step methods. We use the idea of the weight function to improve the convergence order. A new Steffensen-type method, in addition, to overcome the weakness of the Newton method (use of the derivative function), uses only a weight function. In other words, we have a 100% improvement in the convergence order from 8 to 16.

In this work, we construct a three-step the second step of Chun's weight function idea has been used and the eight order methods transform a family of the with-memory methods by convergence order of 16. The rest of this paper is organized as follows. In Section 2, the study of the without-memory method is presented. Section 3 is devoted to the following steps about the supreme efficiency index. Some numerical performances are presented in Section 4. We finalize the paper with some concluding remarks in Section 5.

2 Description of the methods

2.1 Derivation

We initially target constructing an optimal eighth-order method, which is used as the first three-step method in building our sixteenth-order with-memory family in section 3. To this end, notice the following three-point method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0, 1, 2, \ldots, \\ z_n = y_n - \frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \end{cases} \tag{4}$$

in which the Newton's iteration has just been repeated three times. The convergence order of the method (4) is 8 and the error equation for the method is given as

$$e_{n+1} = c_2^7 e_n^8 + O(e_n^9). \tag{5}$$

Two major drawbacks of the schemes (4) are that they involve derivative and are not optimal methods. For solving these problems, initially $f'(x_n),\ f'(y_n)$ and $f'(z_n)$ in order to maintain the order of convergence at the highest level with the least evaluation in each iteration. Now, $f'(x_n),\ f'(y_n)$ and $f'(z_n)$ must be approximated so that the order does not die down. In order to suggest a wide class of optimal eighth-order methods, we make use of weight function approach at the second and third step by using all known data.

$$\begin{cases} f'(x_n) \approx \frac{f(w_n)-f(x_n)}{w_n-x_n},\ w_n = x_n + \beta f(x_n),\ n = 0, 1, 2, \dots, \\ f'(y_n) \approx \frac{f[w_n,y_n]}{G(t_n)},\ t_n = \frac{f(y_n)}{f(x_n)}, \\ f'(z_n) \approx f[z_n, y_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n). \end{cases} \tag{6}$$

Consequently a novel three-step iteration by using weight function approach including one free parameter can be defined as follows (denoted by TM8)

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]},\ w_n = x_n + \beta f(x_n),\ n = 0, 1, 2, \dots, \\ z = y_n - G(t_n)\frac{f(y_n)}{f[w_n,y_n]},\ t_n = \frac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,y_n,x_n](z_n-y_n)+f[z_n,y_n,x_n,w_n](z_n-y_n)(z_n-x_n)}. \end{cases} \tag{7}$$

To recover the optimal eighth-order convergence, we find some suitable conditions on the introduced weight function

$$G(0) = 1,\ G'(0) = 1. \tag{8}$$

We show that the convergence order reaches the optimal case, i.e. 8, with only four evaluations. The proposed class of derivative-free methods has a high-efficiency index of 1.68. We prove the convergence of the iterative method (7) through the following theorem.

Theorem 1 *Suppose that f is a sufficiently differentiable real function and $\xi \in D$ is a simple zero of f. If the initial approximation x_0 is close adequate to ξ, then the sequence x_n generated by any method of the family (3) converges to ξ with eighth-order of convergence if G is real sufficiently differentiable functions satisfying $G(0) = G'(0) = 1$.*

Proof Let us introduce the following notations

$$\begin{aligned} &e_n = x_n - \xi, e_{n,w} = w_n - \xi, e_{n,y} = y_n - \xi, e_{n,z} = z_n - \xi, \\ &e_{n+1} = x_{n+1} - \xi, c_r = \frac{f^{(r)}(\xi)}{r! f'(\xi)},\ r = 2, 3, \dots \end{aligned} \tag{9}$$

Using Taylor's expansion and taking into account $f(\xi) = 0$, we have

$$f(x_n) = f'(\xi)(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8), \tag{10}$$

and expanding w_n at ξ we get

$$\begin{aligned} f(w_n) =&(1+\beta f'(\xi))e_n + f'(\xi)(1+\beta f'(\xi)(3+f'(\xi))c_2e_n^2 \\ &+ f'(\xi)(2+f'(\xi)\beta(1+\beta f'(\xi))c_2^2 + \beta f'(\xi)c_3 + (1+\beta f'(\xi))^3e_n^3 \\ &+ f'(\xi)(c_4+\beta f'(\xi)(\beta f'(\xi)c_2^3 + (1+\beta f'(\xi))(5+3\beta f'(\xi))c_2c_3 \\ &+ (5+\beta f'(\xi)(6+\beta f'(\xi)(4+\beta f'(\xi))))c_4))e_n^4 + \cdots + O(e_n^9). \end{aligned} \tag{11}$$

Hence, substituting (10) and (11) in the first step of (3), we get

$$\begin{aligned} e_{n,y} =&y_n - \xi = (1+\beta f'(\xi))c_2e_n^2 + (-(2+\beta f'(\xi)(2+\beta f'(\xi)))c_2^2 \\ &+ (1+\beta f'(\xi))(2+\beta f'(\xi))c_3)e_n^3 + ((4+\beta f'(\xi)(5+\beta f'(\xi)(3+\beta f'(\xi))) \\ &- (7+\beta f'(\xi)(10+\beta f'(\xi)(7+2\beta f'(\xi))))c_2c_3 + (1+\beta f'(\xi))(3+\beta f'(\xi) \\ &(3+\beta f'(\xi)))c_4e_n^4 + \cdots + O(e_n^8), \end{aligned} \tag{12}$$

and in the combination of Taylor series expansion of $f(y_n) = f(x_n - \frac{f(x_n)}{f[x_n,w_n]})$ about $x_n = \xi$, we have

$$\begin{aligned} f(y_n) =&f'(\xi)(1+\beta f'(\xi))c_2e_n^2 + (-(2+\beta f'(\xi)(2+\beta f'(\xi)))c_2^2 \\ &+ (1+\beta f'(\xi))(2+\beta f'(\xi))c_3)e_n^3 + f'(\xi)((5+\beta f'(\xi)(7+\beta f'(\xi) \\ &(4+\beta f'(\xi)))c_2^3 - (7+\beta f'(\xi)(10+\beta f'(\xi)(7+2\beta f'(\xi))))c_2c_3 \\ &+ (1+\beta f'(\xi))(3+\beta f'(\xi)(3+\beta f'(\xi)))c_4e_n^4 + \cdots + O(e_n^8). \end{aligned} \tag{13}$$

Now dividing (13) by (10) ends in

$$\begin{aligned} t_n =&\frac{f(y_n)}{f(x_n)} = (1+\beta f'(\xi))c_2e_n + (-(3+\beta f'(\xi)(3+\beta f'(\xi)))c_2^2 \\ &+ (1+\beta f'(\xi))(2+\beta f'(\xi))c_3)e_n^2 + ((2+\beta f'(\xi))(4+\beta f'(\xi)) \\ &(3+\beta f'(\xi))c_2^3 - 2(5+\beta f'(\xi)(7+\beta f'(\xi)(4+\beta f'(\xi))))c_2c_3 \\ &+ (1+\beta f'(\xi))(3+\beta f'(\xi)(3+\beta f'(\xi)))c_4)e_n^3 + (-(2+\beta f'(\xi)) \\ &(10+\beta f'(\xi)(10+\beta f'(\xi)(5+\beta f'(\xi))))c_2^4 + (37+\beta f'(\xi)(3+\beta f'(\xi)) \\ &(20+\beta f'(\xi)(8+3\beta f'(\xi))))c_2^2c_3 - (8+\beta f'(\xi)(15+\beta f'(\xi)(13+\beta f'(\xi) \\ &(6+\beta f'(\xi))))c_3^2 - (14+\beta f'(\xi)(5+2\beta f'(\xi))(5+\beta f'(\xi)(2+\beta f'(\xi))))c_2c_4 \\ &+ (1+\beta f'(\xi))(2+\beta f'(\xi))(2+\beta f'(\xi)(2+\beta f'(\xi)))c_5)e_n^4 + \cdots + O(e_n^8), \end{aligned} \tag{14}$$

and expanding G at 0 yields

$$G(t_n) = G(0) + G'(0)t_n + O(t_n^2). \tag{15}$$

By substituting (10)-(15) into (7), we obtain

$$\begin{aligned} e_{n,z} =& z_n - \xi = -(-1+G(0))(1+\beta f'(\xi))c_2 e_n^2 + (-(2-3G(0)+G'(0)+2\\ & f'(\xi)\beta(1-2G(0)+G'(0)) + f'(\xi)^2\beta^2(1-2G(0)+G'(0))c_2^2 -\\ & (-1+G(0))(1+\beta f'(\xi))(2+\beta f'(\xi))c_3)e_n^3 + ((4+\beta f'(\xi)(5+\beta f'(\xi))\\ & (3+\beta f'(\xi))) + G'(0)(1+\beta f'(\xi))(6+\beta f'(\xi)(7+\beta f'(\xi))) - G(0)\\ & ((7+\beta f'(\xi)(11+\beta f'(\xi)(8+3\beta f'(\xi)))))c_2^3 + (-7-2G'(0)(1+\beta f'(\xi))^2\\ & (2+\beta f'(\xi)) - \beta f'(\xi)(10+\beta f'(\xi)(7+2\beta f'(\xi))) + 2G(0)(5+\beta f'(\xi)\\ & (9+\beta f'(\xi)(7+2\beta f'(\xi)))))c_2^2 c_3 - (-1+G(0))(1+\beta f'(\xi))\\ & (3+\beta f'(\xi)(3+\beta f'(\xi)))(c_4)e_n^4 + \cdots + O(e_n^8). \end{aligned} \tag{16}$$

We now need to vanish coefficients e_n^2 and e_n^3 for making obtained the first both steps which simplifies future relations. It is sufficient to ask the weight function G to assure conditions $G(0)=1$ and $G'(0)=1$. Hence

$$\begin{aligned} e_{n,z} =& z_n - \xi = (1+\beta f'(\xi))^2 c_2((3+\beta f'(\xi))c_2^2 - c_3)e_n^4 + (1+\beta f'(\xi)) \\ & ((3+\beta f'(\xi))(6+\beta f'(\xi)(7+\beta f'(\xi))c_4^2 - (20+\beta f'(\xi)(34+\beta f'(\xi)\\ & (19+3\beta f'(\xi)))c_2^3 c_3 + (1+\beta f'(\xi))(2+\beta f'(\xi))c_3^2 + (1+\beta f'(\xi))\\ & (2+\beta f'(\xi))c_2 c_4 e_n^5 + \cdots + O(e_n^8). \end{aligned} \tag{17}$$

For the third step, we also require

$$\begin{aligned} f(z_n) =& f'(\xi)(1+\beta f'(\xi))^2 c_2((3+\beta f'(\xi))c_2^2 - c_3)e_n^4 + f'(\xi)\\ & (1+\beta f'(\xi))((3+\beta f'(\xi))(6+\beta f'(\xi)(7+3\beta f'(\xi))c_{2^4} - (20+\beta f'(\xi)\\ & (34+\beta f'(\xi)(19+3\beta f'(\xi))))c_2^2 c^3 + (1+\beta f'(\xi))(2+\beta f'(\xi))c_3^2\\ & + (1+\beta f'(\xi))(2+\beta f'(\xi))c_2 c_4 e_n^5 + \cdots + O(e_n^8). \end{aligned} \tag{18}$$

With (13) and(18), we have

$$\begin{aligned} f[z_n, y_n] =& f'(\xi) + f'(\xi)(1+\beta f'(\xi))c_2^2 e_n^2 + (-f'(\xi)(2+\beta f'(\xi)\\ & (2+\beta f'(\xi)))c_2^3 + f'(\xi)(1+\beta f'(\xi))(2+\beta f'(\xi))c_2 c_3)e_n^3 + f'(\xi)\\ & c_2(7+2\beta f'(\xi)(6+\beta f'(\xi)(4+\beta f'(\xi))))c_2^3 - (7+\beta f'(\xi)(10+\beta f'(\xi)\\ & (7+2\beta f'(\xi))))c_2 c_3 + (1+\beta f'(\xi))(3+\beta f'(\xi)(3+\beta f'(\xi)))c_4 e_n^4\\ & + \cdots + O(e_n^8). \end{aligned} \tag{19}$$

Furthermore, we can obtain that

$$\begin{aligned} f[z_n, y_n, x_n] =& f'(\xi)c_2 + f'(\xi)c_3 e_n + f'(\xi)((1+\beta f'(\xi))c_2 c_3 + c_4)e_n^2 + f'(\xi)\\ & (-(2+\beta f'(\xi)(2+\beta f'(\xi)))c_2^2 c_3 + (1+\beta f'(\xi))(2+\beta f'(\xi))c_3^2\\ & (1+\beta f'(\xi))c_2 c_4 + c_5)e_n^3 + f'(\xi)(7+2\beta f'(\xi)(6+\beta f'(\xi)\\ & (4+\beta f'(\xi))))c_2^3 c_3 - c_2^2 c_4 + (1+\beta f'(\xi))(5+\beta f'(\xi)(4+\beta f'(\xi)))\\ & c_3 c_4 + c_2(-2(2+\beta f'(\xi)(2+\beta f'(\xi)(2+\beta f'(\xi)))))c_3^2\\ & + (1+\beta f'(\xi))c_5 + c_6)e_n^4 + \cdots + O(e_n^8), \end{aligned} \tag{20}$$

and

$$
\begin{aligned}
f[z_n, y_n, x_n, w_n] =& f'(\xi)c_3 + f'(\xi)(2+\beta f'(\xi))c_4 e_n + f'(\xi)((1+2\beta f'(\xi))c_2c_4 \\
&+ (3+\beta f'(\xi)(3+\beta f'(\xi)))c_5)e_n^2 + f'(\xi)(-(2+\beta f'(\xi) \\
&(2+\beta f'(\xi)))c_2^2c_4 + (2+\beta f'(\xi))(4+\beta f'(\xi))c_3c_4 + \\
&(2+3\beta f'(\xi)(2+\beta f'(\xi)))c_2c_5 + (2+f'(\xi))(2+\beta f'(\xi) \\
&(2+\beta f'(\xi)))c_6)e_n^3 + f'(\xi)((7+2\beta f'(\xi)(6+\beta f'(\xi) \\
&(4+\beta f'(\xi))))c_2^3c_4 + (3+\beta f'(\xi)(7+\beta f'(\xi)(4+\beta f'(\xi))))c_4^3 \\
&-(1+\beta f'(\xi))(3+\beta^2 f'(\xi)^2)c_2^2c_5 + (4+\beta f'(\xi)(11+\beta f'(\xi) \\
&(7+\beta f'(\xi))))c_3c_5 + c_2(-2(2+\beta f'(\xi))(2+\beta f'(\xi) \\
&(2+\beta f'(\xi)))c_3c_4 + (3+\beta f'(\xi)(3+\beta f'(\xi)(3+\beta f'(\xi))))c_6) \\
&+(5+\beta f'(\xi)(10+\beta f'(\xi)(10+\beta f'(\xi)(5+\beta f'(\xi)))))c_7)e_n^4 \\
&+\cdots + O(e_n^8).
\end{aligned} \tag{21}
$$

By substituting (9)-(21) in (7), we obtain the error equation

$$
\begin{aligned}
e_{n+1} =& x_{n+1} - \xi \\
=& (1+\beta f'(\xi))^4 c_2^2((3+\beta f'(\xi))c_2^2 - c_3)((3+\beta f'(\xi))c_2^3 - c_2c_3 + c_4)e_n^8 \\
&+ O(e_n^9).
\end{aligned} \tag{22}
$$

We find that the order of convergence for the TM8 is 8.

Some simple weight functions satisfying conditions (8) are

$$
\begin{cases}
G_1(t) = 1+t, & G_2(t) = Arctan(t)+1, \quad G_3(t) = 1+\sin(t), \\
G_4(t) = \dfrac{\cos(t)}{1-\sin(t)}, & G_5(t) = \dfrac{1}{1-t}, \quad G_6(t) = \sin(t)+\cos(t), \\
G_7(t) = e^t, & G_8(t) = \dfrac{2+t}{2-t}, \quad G_9(t) = 1+te^t, \\
G_{10}(t) = \dfrac{11+2t}{11-9t}, & G_{11}(t) = 2 - \dfrac{1}{1+t}, \quad G_{12}(t) = 2t + \dfrac{1}{1+t}, \\
G_{13}(t) = e^t\cos(t), & G_{14}(t) = \dfrac{a+bt}{a-(a-b)t}, \quad G_{15}(t) = (1+t)^{\frac{m}{n}} + \dfrac{n-m}{n}t, \\
G_{16}(t) = \dfrac{5+2t}{5-3t}, & G_{17}(t) = \sqrt{1+t} + \dfrac{t}{2}, \quad G_{18}(t) = (1+t)^{\frac{1}{3}} + \dfrac{2}{3}t, \\
G_{19}(t) = \dfrac{1}{\cos(t)-\sin(t)}, & G_{20}(t) = \cos(t)+\tan(t), \quad G_{21}(t) = 2t - \sin(t) + 1, \\
G_{22}(t) = 2t + e^{-t},
\end{cases} \tag{23}
$$

where $a, b, m, n \in \mathbb{R}^+$, and $n, b \neq 0$.
By entering three new self-accelater parameters the method of (7) can be rewritten as follows

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]+\alpha f(w_n)}, \; w_n = x_n + \beta f(x_n), \; n = 0,1,2,\ldots, \\ z = y_n - G(t_n)\frac{f(y_n)}{f[w_n,y_n]+\alpha f(w_n)+\gamma(y_n-x_n)(y_n-w_n)}, \; t_n = \frac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,y_n,x_n](z_n-y_n)+f[z_n,y_n,x_n,w_n](z_n-y_n)(z_n-x_n)+\lambda(z_n-y_n)(z_n-x_n)(z_n-w_n)}. \end{cases} \tag{24}$$

Theorem 2 *Let $G(t_n)$ be differentiable function that satisfy the conditions $G(0) = G'(0) = 1$. If an initial approximation x_0 is sufficiently close to the root ξ of a function f, then the convergence order of the class of three-step methods (24) is at least eight and its error equation is given by*

$$\begin{aligned} e_{n+1} = & x_{n+1} - \xi = f'(\xi)^{-2}(1+\beta f'(\xi))^4(\alpha+c_2)^2(\gamma+f'(\xi)\alpha^2 \\ & (1+\beta f'(\xi)) + f'(\xi)c_2(2\alpha(2+\beta f'(\xi)) + (3+\beta f'(\xi))c_2) - f'(\xi)c_3) \\ & (-\lambda + c_2(\gamma + f'(\xi)\alpha^2(1+\beta f'(\xi)) + f'(\xi)c_2(2\alpha(2+\beta f'(\xi)) \\ & + (3+\beta f'(\xi))c_2) - f'(\xi)c_3) + f'(\xi)c_4)e_n^8 + O(e_n^9). \end{aligned} \tag{25}$$

Proof The proof is similar to the proof of Theorem 1, therefore it is omitted.

Remark 1 Family of multipoint methods (7), and (24) uses four evaluations in each iteration, and has eighth-order convergence which is consistent with the conjecture of Kung-Traub [19]. The multipoint iteration without memory based on n evaluations achieves optimal convergence order $2^{n-1}$for the case $n = 4$. The efficiency indices of methods both the families come out to be equal 1.68.

We can assure that the convergence order of (24) is still eight, with independence of parameters β, α, γ and λ. This order can be raised from 8 to 12, 14, 15,15.52,15.97, and 16, by obtaining $\beta = \frac{-1}{f'(\xi)}$, $\alpha = -c_2$, $\gamma = f'(\xi)c3$ and $\lambda = f'(\xi)c_4$, but root ξ is not known. To improve the convergence rate of (24), we re-calculate the value of parameters β, α, γ and λ in each iteration, by taking $\beta \approx \frac{-1}{f'(\xi)}$, $\alpha \approx -c_2$, $\gamma \approx f'(\xi)c_3$ and $\lambda \approx f'(\xi)c_4$, while $f'(\xi)$, $f''(\xi)$, $f'''(\xi)$, and $f^{(4)}(\xi)$ are not provided. We represent these estimations through β_n, α_n, γ_n and λ_n and they are computed by using current and previous iteration satisfying.

$$\begin{cases} \lim_{n\to\infty} \beta_n = \frac{-1}{f'(\xi)}, \\ \lim_{n\to\infty} \alpha_n = \frac{-f''(\xi)}{2f'(\xi)}, \\ \lim_{n\to\infty} \gamma_n = \frac{f'''(\xi)}{3!}, \\ \lim_{n\to\infty} \lambda_n = \frac{f^{(4)}(\xi)}{4!}. \end{cases} \tag{26}$$

We consider the following formulas for $\beta_n, \alpha_n, \gamma_n$ and λ_n using Newton interpolating polynomials

$$\begin{cases} \beta_n = -\frac{1}{N_4'(x_n)}, \quad \alpha_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}, \\ \gamma_n = \frac{N_6'''(y_n)}{3!}, \quad \lambda_k = \frac{N_7^{(4)}(z_n)}{4!}, \end{cases} \tag{27}$$

where

$$\begin{cases} N_4(t) = N_4(t; x_k, x_{k-1}, w_{k-1}, y_{k-1}, z_{k-1}),\ k \geqslant 1 \\ N_5(t) = N_5(t; w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}, z_{k-1}), \\ N_6(t) = N_6(t; y_k, w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}, z_{k-1}), \\ N_7(t) = N_7(t; z_k, y_k, w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}, z_{k-1}). \end{cases} \tag{28}$$

In next section, we are going to design new methods with memory by using one, two, three and four self-accelerating parameters from family (24).

3 Development of new methods with memry

In the remainder of this essay, we will present with memory methods with convergence rates of $12, 14, 15, 15.5$ and 16.

3.1 One-parametric methods

First, we present the with-memory method which has a self-accelerator parameter (denoted by TM12)

$$\begin{cases} \beta_n = -\frac{1}{N_4'(x_n)},\ n = 1, 2, 3, \ldots, \\ y_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + \alpha f(w_n)},\ w_n = x_n + \beta_n f(x_n),\ n = 0, 1, 2, \ldots, \\ z = y_n - G(t_n) \frac{f(y_n)}{f[w_n, y_n] + \alpha f(w_n) + \gamma (y_n - x_n)(y_n - w_n)},\ t_n = \frac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n) + \lambda (z_n - y_n)(z_n - x_n)(z_n - w_n)}, \end{cases} \tag{29}$$

where

$$N_4'(x_n) = N_4(t; x_n, x_{n-1}, w_{n-1}, y_{n-1}, z_{n-1}).$$

$N_4(x_n)$ is Newton's interpolatory polynomial of fourth degree, set through five available approximations $x_n, x_{n-1}, w_{n-1}, y_{n-1}, z_{n-1}$. To study the convergence analysis of the new with memory(29), we need following Lemma.

Lemma 1 *If* $\beta_n = -\frac{1}{N_4'(x_n)}$ *then* $(1 + \beta_n f'(\xi)) \sim e_{n-1} e_{n-1,w} e_{n-1,y} e_{n-1,z}$.

Theorem 3 *Let the varying parameter* β_n *in the iterative method (29) be calculated by (27). If an initial approximation* x_0 *is sufficiently close to a simple root* ξ *of* $f(x) = 0$*, then the R-order of convergence of the iterative methods (29) is at least 12.*

Proof First we assume that the $R-$order of convergence in sequence x_n, w_n, y_n and z_n is at least r, r_1, r_2 and r_3, respectively. Hence

$$\begin{cases} e_{n+1} \sim e_n^r \sim e_{n-1}^{r^2}, \\ e_{n,w} \sim e_n^{r_1} \sim e_{n-1}^{r r_1}, \\ e_{n,y} \sim e_n^{r_2} \sim e_{n-1}^{r r_2}, \\ e_{n,z} \sim e_n^{r_3} \sim e_{n-1}^{r r_3}. \end{cases} \tag{30}$$

By using Lemma 1 , we obtain

$$1+\beta_n f'(\xi) \sim e_{n-1}^{r_1+r_2+r_3+1}. \tag{31}$$

On the other hand, we get

$$\begin{cases} e_{n,w} \sim (1+\beta_n f'(\xi))e_n, \\ e_{n,y} \sim (1+\beta_n f'(\xi))e_n^2, \\ e_{n,z} \sim (1+\beta_n f'(\xi))^2 e_n^4, \\ e_{n+1} \sim (1+\beta_n f'(\xi))^4 e_n^8. \end{cases} \tag{32}$$

Combining (30), (31), and (32), we conclude

$$\begin{cases} e_{n,w} \sim e_{n-1}^{(1+r_1+r_2+r_3)+r}, \\ e_{n,y} \sim e_{n-1}^{(1+r_1+r_2+r_3)+2r}, \\ e_{n,z} \sim e_{n-1}^{2(1+r_1+r_2+r_3)+4r}, \\ e_{n+1} \sim e_{n-1}^{4(1+r_1+r_2+r_3)+8r}. \end{cases} \tag{33}$$

By considering, (30) and (33) we find

$$\begin{cases} (1+r_1+r_2+r_3)+r-rr_1=0, \\ (1+r_1+r_2+r_3)+2r-rr_2=0, \\ 2(1+r_1+r_2+r_3)+4r-rr_3=0, \\ 4(1+r_1+r_2+r_3)+8r-r^2=0. \end{cases} \tag{34}$$

This system has the solution $r_1 = 2$, $r_2 = 3$, $r_3 = 6$ and $r = 12$. Therefore, the R-order of the with memory methods(29) is at least 12.

3.2 Two-parametric methods

Now, we improve the convergence order of the proposed method (29). To apply two self-accelerators on our proposed scheme (29), we have 75% of improvement in the convergence R-order e.g. 14 according to (denoted by TM14)

$$\begin{cases} \beta_n = -\frac{1}{N_4'(x_n)}, \ \alpha_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}, \ n = 1, 2, 3, \dots, \\ y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]+\alpha_n f(w_n)}, \ w_n = x_n + \beta_n f(x_n), \ n = 0, 1, 2, \dots, \\ z = y_n - G(t_n)\frac{f(y_n)}{f[w_n, y_n]+\alpha_n f(w_n)+\gamma(y_n-x_n)(y_n-w_n)}, \ t_n = \frac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n]+f[z_n, y_n, x_n](z_n-y_n)+f[z_n, y_n, x_n, w_n](z_n-y_n)(z_n-x_n)+\lambda(z_n-y_n)(z_n-x_n)(z_n-w_n)}. \end{cases} \tag{35}$$

Lemma 2 *If* $\beta_n = -\frac{1}{N_4'(x_n)}$ *and* $\alpha_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}$ *then*

$$\begin{cases} (1+\beta_n f'(\xi)) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z}, \\ (\alpha_n + c_2) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z}. \end{cases} \tag{36}$$

Theorem 4 *Suppose that* x_0 *is an approximation to a simple zero* ξ *of* f, *then the R-order of convergence of the three-step method with memory (35) is at least 14.*

3.3 Three-parametric methods

Also, interpolating tri-parameter of the self-accelerator, the new method with memory with convergence rate of 15 can be constructed as follows (denoted by TM15)

$$\begin{cases} \beta_n = -\frac{1}{N_4'(x_n)},\ \alpha_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}, \gamma_n = \frac{N_6'''(y_n)}{3!},\ n = 1, 2, 3, \dots, \\ y_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + \alpha_n f(w_n)},\ w_n = x_n + \beta_n f(x_n),\ n = 0, 1, 2, \dots, \\ z = y_n - G(t_n)\frac{f(y_n)}{f[w_n, y_n] + \alpha_n f(w_n) + \gamma_n (y_n - x_n)(y_n - w_n)},\ t_n = \frac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n) + \lambda(z_n - y_n)(z_n - x_n)(z_n - w_n)}. \end{cases} \tag{37}$$

Lemma 3 *If* $\beta_n = -\frac{1}{N_4'(x_n)}$, $\alpha_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}$ *and* $\gamma_n = \frac{N_6'''(y_n)}{3!}$ *then*

$$\begin{cases} (1 + \beta_n f'(\xi)) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z}, \\ (\alpha_n + c_2) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z}, \\ (\gamma + f'(\xi)\alpha^2(1 + \beta f'(\xi)) + f'(\xi)c_2(2\alpha(2 + \beta f'(\xi)) \\ +(3 + \beta f'(\xi))c_2) - f'(\xi)c_3) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z}. \end{cases} \tag{38}$$

Theorem 5 *Suppose that x_0 is an approximation to a simple zero ξ of f, then the R-order of convergence of the three-step method with memory (37)is at least 15.*

3.4 Four-parametric methods

In this section, we derive the convergence order of the proposed method (29). To apply two self-accelerators on our proposed two-step scheme (29). In this way we have 93.75% of improvement in the convergence R-order e.g. 15.52 according to (denoted by TM15.5)

$$\begin{cases} \beta_n = -\frac{1}{N_4'(x_n)},\ \alpha_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}, \gamma_n = \frac{N_6'''(y_n)}{3!}, \lambda_k = \frac{N_7^{(4)}(z_n)}{4!},\ n = 1, 2, 3, \dots, \\ y_n = x_n - \frac{f(x_n)}{f[x_n, w_n] + \alpha_n f(w_n)},\ w_n = x_n + \beta_n f(x_n),\ n = 0, 1, 2, \dots, \\ z = y_n - G(t_n)\frac{f(y_n)}{f[w_n, y_n] + \alpha_n f(w_n) + \gamma_n (y_n - x_n)(y_n - w_n)},\ t_n = \frac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, y_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n) + \lambda_n(z_n - y_n)(z_n - x_n)(z_n - w_n)}. \end{cases} \tag{39}$$

Lemma 4 *If* $\beta_n = -\frac{1}{N_4'(x_n)}$, $\alpha_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}, \gamma_n = \dfrac{N_6'''(y_n)}{3!}$ *and* $\lambda_k = \frac{N_7^{(4)}(z_n)}{4!}$ *then*

$$\begin{cases}(1+\beta_n f'(\xi)) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z},\\ (\alpha_n + c_2) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z},\\ (\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n(2+\beta_n f'(\xi)) + (3+\beta_n f'(\xi))c_2)\\ -f'(\xi)c_3) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z},\\ (-\lambda_n + c_2(\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n(2+\beta_n f'(\xi))\\ +(3+\beta_n f'(\xi))c_2) - f'(\xi)c_3) + f'(\xi)c_4) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z}.\end{cases} \tag{40}$$

Theorem 6 *Suppose that* x_0 *is an approximation to a simple zero* ξ *of* f, *then the R-order of convergence of the three-step method with memory (39)is at least 15.52.*

Proof First we assume that the $R-$order of convergence of sequence x_n, w_n, y_n and z_n is at least r, r_1, r_2 and r_3, respectively. Hence

$$\begin{cases} e_{n+1} \sim e_n^r \sim e_{n-1}^{r^2},\\ e_{n,w} \sim e_n^{r_1} \sim e_{n-1}^{rr_1},\\ e_{n,y} \sim e_n^{r_2} \sim e_{n-1}^{rr_2},\\ e_{n,z} \sim e_n^{r_3} \sim e_{n-1}^{rr_3}.\end{cases} \tag{41}$$

By considering (41), and Lemma 4, we obtain

$$\begin{aligned} &1+\beta_n f'(\xi) \sim e_{n-1}^{r_1+r_2+r_3+1}, \qquad (42)\\ &\alpha_n + c_2 \sim e_{n-1}^{r_1+r_2+r_3+1},\\ &(\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n(2+\beta_n f'(\xi)) + (3+\beta_n f'(\xi))c_2)\\ &- f'(\xi)c_3) \sim e_{n-1}^{r_1+r_2+r_3+1},\\ &(-\lambda_n + c_2(\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n(2+\beta_n f'(\xi))\\ &+ (3+\beta_n f'(\xi))c_2) - f'(\xi)c_3) + f'(\xi)c_4) \sim e_{n-1}^{r_1+r_2+r_3+1}.\end{aligned}$$

On the other hand, we get

$$\begin{aligned} &e_{n,w} \sim (1+\beta_n f'(\xi))e_n, \qquad (43)\\ &e_{n,y} \sim (1+\beta_n f'(\xi))(\alpha_n + c_2)e_n^2,\\ &e_{n,z} \sim (1+\beta_n f'(\xi))^2(\alpha_n + c_2)(\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi)) + f'(\xi)c_2\\ &(2\alpha_n(2+\beta_n f'(\xi)) + (3+\beta f'(\xi))c_2)e_n^4,\\ &e_{n+1} \sim (1+\beta_n f'(\xi))^4(\alpha_n + c_2)^2(\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi)) + f'(\xi)c_2\\ &(2\alpha_n(2+\beta_n f'(\xi)) + (3+\beta_n f'(\xi))c_2)(-\lambda_n + c_2(\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi))\\ &+ f'(\xi)c_2(2\alpha_n(2+\beta_n f'(\xi)) + (3+\beta_n f'(\xi))c_2) - f'(\xi)c_3) + f'(\xi)c_4)e_n^8.\end{aligned}$$

Combining (41), (42) and (43) we conclude

$$\begin{cases} e_{n,w} \sim e_{n-1}^{(1+r_1+r_2+r_3)+r}, \\ e_{n,y} \sim e_{n-1}^{2(1+r_1+r_2+r_3)+2r}, \\ e_{n,z} \sim e_{n-1}^{4(1+r_1+r_2+r_3)+4r}, \\ e_{n+1} \sim e_{n-1}^{8(1+r_1+r_2+r_3)+8r}. \end{cases} \tag{44}$$

by using equations (41) and (44) we have

$$\begin{cases} (1+r_1+r_2+r_3)+r-rr_1=0, \\ 2(1+r_1+r_2+r_3)+2r-rr_2=0, \\ 4(1+r_1+r_2+r_3)+4r-rr_3=0, \\ 8(1+r_1+r_2+r_3)+8r-r^2=0. \end{cases} \tag{45}$$

This system of equations has the solution

$$r_1 = \frac{1}{16}(15+\sqrt{257}) \simeq 1.94, \qquad r_2 = \frac{1}{8}(15+\sqrt{257}) \simeq 3.88,$$
$$r_3 = \frac{1}{4}(15+\sqrt{257}) \simeq 7.76, \qquad r = \frac{1}{2}(15+\sqrt{257}) \simeq 15.52.$$

Now, to introduce convergence methods of 15.5 to 16, we will rewrite the following three methods

$$\begin{cases} \beta_n = -\frac{1}{N'_{4n}(x_n)},\ \alpha_n = -\frac{N''_{4n+1}(w_n)}{2N'_{4n+1}(w_n)}, \gamma_n = \frac{N'''_{4n+2}(y_n)}{3!}, \lambda_n = \frac{N^{(4)}_{4n+3}(z_n)}{4!},\ n=1,2,3,\ldots, \\ y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]+\alpha_n f(w_n)},\ w_n = x_n + \beta_n f(x_n),\ n=0,1,2,\ldots, \\ z = y_n - G(t_n)\frac{f(y_n)}{f[w_n,y_n]+\alpha_n f(w_n)+\gamma_n(y_n-x_n)(y_n-w_n)},\ t_n = \frac{f(y_n)}{f(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,y_n,x_n](z_n-y_n)+f[z_n,y_n,x_n,w_n](z_n-y_n)(z_n-x_n)+\lambda_n(z_n-y_n)(z_n-x_n)(z_n-w_n)}. \end{cases} \tag{46}$$

As an illustration, here we also can define

$$\begin{cases} N_8(t) = N_8(t;\, x_k, x_{k-1},\, w_{k-1},\, y_{k-1},\, z_{k-1},\, x_{k-2},\, w_{k-2},\, y_{k-2},\, z_{k-2}),\ k \geqslant 2, \\ N_9(t) = N_9(t;\, w_k,\, x_k, x_{k-1},\, w_{k-1},\, y_{k-1},\, z_{k-1},\, x_{k-2},\, w_{k-2},\, y_{k-2},\, z_{k-2}), \\ N_{10}(t) = N_{10}(t;\, y_k,\, w_k,\, x_k, x_{k-1},\, w_{k-1},\, y_{k-1},\, z_{k-1},\, x_{k-2},\, w_{k-2},\, y_{k-2},\, z_{k-2}), \\ N_{11}(t) = N_{11}(t;\, z_k,\, y_k,\, w_k,\, x_k, x_{k-1},\, w_{k-1},\, y_{k-1},\, z_{k-1},\, x_{k-2},\, w_{k-2},\, y_{k-2},\, z_{k-2}). \end{cases}$$

Lemma 5 *If* $\beta_n = -\frac{1}{N'_{4k}(x_n)}$, $\alpha_n = -\frac{N''_{4k+1}(w_n)}{2N'_{4k+1}(w_n)}, \gamma_n = \dfrac{N'''_{4k+2}(y_n)}{3!}$ *and* $\lambda_k = \frac{N^{(4)}_{4k+3}(z_n)}{4!}$ *then*

$$\begin{cases} (1+\beta_n f'(\xi)) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z}, \\ (\alpha_n + c_2) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z}, \\ (\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n(2+\beta_n f'(\xi)) + (3+\beta_n f'(\xi))c_2) \\ -f'(\xi)c_3) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z}, \\ (-\lambda_n + c_2(\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n(2+\beta_n f'(\xi)) \\ +(3+\beta_n f'(\xi))c_2) - f'(\xi)c_3) + f'(\xi)c_4) \sim e_{n-1}e_{n-1,w}e_{n-1,y}e_{n-1,z}. \end{cases} \tag{47}$$

Theorem 7 *If an initial approximation x_0 is sufficiently close to the zero of $f(x)$ and the parameters β_n, γ_n, α_n, λ_n in the iterative scheme (46) is recursively calculated then, the R-order of convergence of the method (46) is at least* 15.97 *and* 16*.*

Proof The convergence order of the methods is given separately.
Method TM15.97
Let $\{x_n\}, \{w_n\}, \{y_n\}$, and $\{z_n\}$, be convergent with orders r, r_1, r_2, and r_3, respectively. Then

$$\begin{cases} e_{n+1} \sim e_n^r \sim e_{n-1}^{r^2} \sim e_{n-2}^{r^3} \sim e_{n-3}^{r^4} \sim e_{n-4}^{r^5}, \\ e_{n,w} \sim e_n^{r_1} \sim e_{n-1}^{r_1 r} \sim e_{n-2}^{r^2 r_1} \sim e_{n-3}^{r^3 r_1} \sim e_{n-4}^{r^4 r_1}, \\ e_{n,y} \sim e_n^{r_2} \sim e_{n-1}^{r_2 r} \sim e_{n-2}^{r^2 r_2} \sim e_{n-3}^{r^3 r_2} \sim e_{n-4}^{r^4 r_2}, \\ e_{n,z} \sim e_n^{r_3} \sim e_{n-1}^{r r_3} \sim e_{n-2}^{r^2 r_3} \sim e_{n-3}^{r^3 r_3} \sim e_{n-4}^{r^4 r_3}. \end{cases} \tag{48}$$

Using Theorem 1 and error equation (11) obtain

$$\begin{aligned} &e_{n,w} \sim (1 + \beta_n f'(\xi))e_n, \qquad (49)\\ &e_{n,y} \sim (1 + \beta_n f'(\xi))(\alpha_n + c_2)e_n^2, \\ &e_{n,z} \sim (1 + \beta_n f'(\xi))^2(\alpha_n + c_2)(\gamma_n + f'(\xi)\alpha_n^2(1 + \beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n \\ &(2 + \beta_n f'(\xi)) + (3 + \beta f'(\xi))c_2)e_n^4, \\ &e_{n+1} \sim (1 + \beta_n f'(\xi))^4(\alpha_n + c_2)^2(\gamma_n + f'(\xi)\alpha_n^2(1 + \beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n \\ &(2 + \beta_n f'(\xi)) + (3 + \beta_n f'(\xi))c_2)(-\lambda_n + c_2(\gamma_n + f'(\xi)\alpha_n^2(1 + \beta_n f'(\xi)) + f'(\xi)c_2 \\ &(2\alpha_n(2 + \beta_n f'(\xi)) + (3 + \beta_n f'(\xi))c_2) - f'(\xi)c_3) + f'(\xi)c_4)e_n^8, \end{aligned}$$

and

$$\begin{aligned} &(1 + \beta_n f'(\xi)) \sim e_{n-2}e_{n-1}e_{n-2,w}e_{n-1,w}e_{n-2,y}e_{n-1,y}e_{n-2,z}e_{n-1,z}, \qquad (50)\\ &(\alpha_n + c_2) \sim e_{n-2}e_{n-1}e_{n-2,w}e_{n-1,w}e_{n-2,y}e_{n-1,y}e_{n-2,z}e_{n-1,z}, \\ &(\gamma_n + f'(\xi)\alpha_n^2(1 + \beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n(2 + \beta_n f'(\xi)) + (3 + \beta f'(\xi))c_2) \\ &\sim e_{n-2}e_{n-1}e_{n-2,w}e_{n-1,w}e_{n-2,y}e_{n-1,y}e_{n-2,z}e_{n-1,z}, \\ &(-\lambda_n + c_2(\gamma_n + f'(\xi)\alpha_n^2(1 + \beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n(2 + \beta_n f'(\xi)) + (3 + \beta_n \\ &f'(\xi))c_2) - f'(\xi)c_3) + f'(\xi)c_4) \sim e_{n-2}e_{n-1}e_{n-2,w}e_{n-1,w}e_{n-2,y}e_{n-1,y}e_{n-2,z}e_{n-1,z}. \end{aligned}$$

Combining relations (48), (49) and (50) we get

$$\begin{cases} e_{n,w} \sim e_{n-2}^{(1+r)(1+r_1+r_2+r_3)}, \\ e_{n,y} \sim e_{n-2}^{2(1+r)(1+r_1+r_2+r_3)}, \\ e_{n,z} \sim e_{k-2}^{4(1+r)(1+r_1+r_2+r_3)}, \\ e_{n+1} \sim e_{n-2}^{8(1+r)(1+r_1+r_2+r_3)}. \end{cases} \tag{51}$$

Now, by comparing (48) and (51) we have

$$\begin{cases} r^2 r_1 = (1+r)(1+r_1+r_2+r_3) + r^2, \\ r^2 r_2 = 2(1+r)(1+r_1+r_2+r_3) + 2r^2, \\ r^2 r_3 = 4(1+r)(1+r_1+r_2+r_3) + 4r^2, \\ r^3 = 8(1+r)(1+r_1+r_2+r_3) + 8r^2. \end{cases} \tag{52}$$

The positive real solution of this system is $r_1 \simeq 2, r_2 \simeq 3.99, r_3 \simeq 7.99$ and $r \simeq 15.97$ that displays the R-order of the methods (46) is at least 15.97.
Method TM16:
Analogy

$$\begin{aligned} &(1+\beta_n f'(\xi)) \\ &\sim e_{n-3}e_{n-2}e_{n-1}e_{n-3,w}e_{n-2,w}e_{n-1,w}e_{n-3,y}e_{n-2,y}e_{n-1,y}e_{n-3,z}e_{n-2,z}e_{n-1,z}, \\ &(\alpha_n + c_2) \\ &\sim e_{n-3}e_{n-2}e_{n-1}e_{n-3,w}e_{n-2,w}e_{n-1,w}e_{n-3,y}e_{n-2,y}e_{n-1,y}e_{n-3,z}e_{n-2,z}e_{n-1,z}, \\ &(\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n(2+\beta_n f'(\xi)) + (3+\beta f'(\xi))c_2) \\ &\sim e_{n-3}e_{n-2}e_{n-1}e_{n-3,w}e_{n-2,w}e_{n-1,w}e_{n-3,y}e_{n-2,y}e_{n-1,y}e_{n-3,z}e_{n-2,z}e_{n-1,z}, \\ &(-\lambda_n + c_2(\gamma_n + f'(\xi)\alpha_n^2(1+\beta_n f'(\xi)) + f'(\xi)c_2(2\alpha_n(2+\beta_n f'(\xi)) + (3+\beta_n \\ &f'(\xi))c_2) - f'(\xi)c_3) + f'(\xi)c_4) \\ &\sim e_{n-3}e_{n-2}e_{n-1}e_{n-3,w}e_{n-2,w}e_{n-1,w}e_{n-3,y}e_{n-2,y}e_{n-1,y}e_{n-3,z}e_{n-2,z}e_{n-1,z}. \end{aligned} \tag{53}$$

Combining (48), (49), and (53) we get

$$\begin{cases} e_{n,w} \sim e_{n-3}^{(1+r+r^2)(1+r_1+r_2+r_3)}, \\ e_{n,y} \sim e_{n-3}^{2(1+r+r^2)(1+r_1+r_2+r_3)}, \\ e_{n,z} \sim e_{n-3}^{4(1+r+r^2)(1+r_1+r_2+r_3)}, \\ e_{n+1} \sim e_{n-3}^{8(1+r+r^2)(1+r_1+r_2+r_3)}. \end{cases} \tag{54}$$

Comparing (48), and (54) we have

$$\begin{cases} r^3 r_1 = (1+r+r^2)(1+r_1+r_2+r_3) + r^3, \\ r^3 r_2 = 2(1+r+r^2)(1+r_1+r_2+r_3) + 2r^3, \\ r^3 r_3 = 4(1+r+r^2)(1+r_1+r_2+r_3) + 4r^3, \\ r^4 = 8(1+r+r^2)(1+r_1+r_2+r_3) + 8r^3. \end{cases} \tag{55}$$

The positive real solution of this system is $r_1 \simeq 2, r_2 \simeq 4, r_3 \simeq 8$ and $r \simeq 16$. We conclude that the R-order of the with memory methods (46) is at least 16. This completes the proof of the Theorem.

4 Numerical results and comparisons

In this section we now consider some numerical examples to demonstrate the performance of the newly developed iterative methods. In this study, methods (TM8) (3), (TM12)(24), (TM14)(27), (TM15)(29), (TM15.5)(35), (TM15.97) (37) $k = 2$, (TM15.99) (37) $k = 3$, and (TM16) (37) $k = 4$, are employed to solve some nonlinear equations and compared with Behl et al.'s method (BCMTM)[2], Bi et al.'s method (BRWM) [3,4], Chun-Neta(CNM) [11], Chun-Lee(CLM) [12], Cordero et al.'s method (CLTAMM) [6], Khattri-Steihaug's method (KSM) [17], Kim's method (KM) [18], Kung-Traub's methods (KTM) [19], Neta's method (NM) [26], Lotfi et al.'s method(LSSSM) [21], Lotfi et al.'s method(LSSSKM) [22], Sharma et al.'s method(SGGM) [30], Sharifi et al.'s method(SSSLM) [33], Soleymani's method (SM) [35], Soleymani et al's method (SLTKM) [36], Soleymani-Vanani's method [37], Thukral-Petkovic's method (TPM) [40], and Zheng et al's method (ZLHM) [50]. Furthermore, we used a programming package Mathematica 10 with sufficiently large number of digits and precision. Before proceeding with the discussion of convergence and convergence of the methods proposed with other iterative methods for solving nonlinear equations, it is essential to select the initial suitable approximation for the root in each nonlinear equation to be convergence [42]. Meanwile in Tables 1 and 2 denote $a \times 10^b$.
The computational order of convergence (COC) introduced in [49]

$$\rho \approx COC = \frac{\ln|x_{k+1} - \xi| / \ln|x_k - \xi|}{\ln|x_k - \xi| / \ln|x_{k-1} - \xi|}. \tag{56}$$

We are going to use the following test functions:

$$\begin{cases} f_1(x) = t\log(1 + x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin(\pi x),\ \xi = 0,\ x_0 = 0.6, \\ f_2(x) = e^{x^3-x} - \cos(x^2-1) + x^3 + 1,\ \xi = -1,\ x_0 = -1.5, \\ f_3(x) = \log(1 + x^2) + e^{-3x+x^2}\sin(x),\ \xi = 0,\ x_0 = 0.4. \end{cases} \tag{57}$$

5 Summary

In this paper, we contributed further to the development of the theory of iteration processes and proposed an optimal three-point family ((7) and (24)) of iterative root-finding algorithms. We can conclude that the numerical experience confirms the theoretical study performed in Sections 2 and 3. It allows us to obtain the convergence order where the approximation to the solution of the nonlinear problem by using the most efficient iterative method. The family three-step iterative process does not implement the first derivative in (29), (35), (37), (39) and (46). We observe from the numerical results of Table 1-3 that the computational order of convergence COC agrees with theoretical

Table 1 Comparison improvment of convergence order the with-memory schemes

With memory methods	Optimal order	p	EI	Percentage increase	With memory methods	Optimal order	p	EI	Percentage increase
CLTAMM[7]	4.00	7.00	1.91	%75	CLTAMM[7]	4.00	7.00	1.91	%75
CCJTM[9]	8.00	10.00	1.78	%25	CJM[10]	8.00	10.00	1.78	%259
JM[16]	8.00	14.00	1.93	%75	JM[16]	4.00	7.00	1.91	%75
TM[44]	8.00	12.00	1.86	%50	LSGAM[23]	4.00	7.53	1.96	%88
LSSSAKM[25]	8.00	14.00	1.93	%75	NM[26]	8.00	10.82	1.81	%35
AZJNM[1]	8.00	14.00	1.93	%75	PNPDM[27]	16.00	21.69	1.85	%36
PIDM[29]	4.00	4.45	1.65	%11	SGGM[30]	8.00	12.00	1.86	%50
SSSM[34]	8.00	12.00	1.86	%50	SLTKM[36]	8.00	12.00	1.86	%50
TKM[43]	16.00	28.00	1.95	%75	WZM[47]	4.00	4.72	1.69	%20
WDZM[48]	8.00	12.00	1.86	%50	TM12 (24)	8.00	12.00	1.86	%50
TM14 (27)	8.00	14.00	1.93	%75	TM15 (29)	8.00	15.00	1.97	%88
TM15.5 (35)	8.00	15.52	1.99	%94	TM16 (37)	8.00	16	2	%100

results.

Therefore, by using a suitable approximation of the β_k parameter, a 50% improvement, and approximating the two parameters β_k and α_k by Newton's method, convergence order improves 75%. Using the three self-accelerators, β_k, α_k and γ_k, we showed a theoretical and practical improvement in the convergence of up to 87.5%. Also, by using the four accelerator parameters β_k, α_k, γ_k and λ_k, we have achieved a 100% convergence order improvement without any additional performance evaluation in each iteration. And the proposed methods with memory that possess the highest computational efficiency are obtained theoretically and practically in iterative prosses for solving nonlinear equations.

The efficiency index for the with memory methods (29), (35), (37), (39) and (46) are ($12^{\frac{1}{4}} = 1.86$), ($14^{\frac{1}{4}} = 1.93$), ($15^{\frac{1}{4}} = 1.97$), ($15.52^{\frac{1}{4}} = 1.99$) and ($16^{\frac{1}{4}} = 2.00$), respectively, which are better than those without and with memory methods [38]. Table 1 shows that the proposed method (46) has the highest convergence rate improvement (100%).

Further researches must be done to develop the proposed methods for system of nonlinear equations. These could be done in the next studies.

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Table 2 Comparison of various iterative schemes.

Methods	$\lvert x_1-\xi\rvert$	$\lvert x_2-\xi\rvert$	$\lvert x_3-\xi\rvert$	COC
SLTKM[36], $\gamma=0.1, f_1$	0.18(−1)	0.40(−19)	0.23(−233)	12.00
SLTKM[36], $\gamma=0.1, f_3$	0.43(−5)	0.39(−66)	0.61(−799)	12.00
SLTKM[36], $\gamma=0.1, f_5$	0.47(−4)	0.90(−48)	0.17(−572)	12.00
TM16 (37), f_1, G_{16}	0.16(−1)	0.22(−29)	0.16(−475)	16.00
TM16 (37), f_2, G_{16}	0.61(−4)	0.55(−69)	0.87(−1114)	16.00
TM16 (37), f_3, G_{16}	0.24(−2)	0.47(−39)	0.13(−623)	16.00
TM16 (37), f_1, G_{21}	0.13(−1)	0.37(−31)	0.11(−504)	16.00
TM16 (37), f_2, G_{21}	0.68(−4)	0.24(−68)	0.11(−1103)	16.00
TM16 (37), f_3, G_{21}	0.23(−2)	0.39(−39)	0.58(−625)	16.00
TM8 (3), f_1, G_1	0.96(−3)	0.14(−21)	0.24(−172)	8.00
TM8 (3), f_1, G_5	0.16(−2)	0.14(−20)	0.40(−165)	8.00
TM8 (3), f_1, G_{10}	0.15(−2)	0.11(−20)	0.10(−165)	8.00
TM8 (3), f_1, G_8	0.13(−2)	0.72(−21)	0.59(−167)	8.00
TM8 (3), f_1, G_7	0.12(−2)	0.53(−21)	0.60(−168)	8.00
TM8 (3), f_1, G_{11}	0.50(−3)	0.20(−23)	0.13(−186)	8.00
TM8 (3), f_1, G_{22}	0.12(−2)	0.41(−21)	0.80(−169)	8.00
TM8 (3), f_1, G_{12}	0.14(−2)	0.40(−21)	0.20(−169)	8.00
TM8 (3), f_3, G_8	0.71(−3)	0.34(−24)	0.94(−195)	8.00
TM8 (3), f_3, G_1	0.69(−3)	0.12(−24)	0.10(−198)	8.00
TM8 (3), f_3, G_5	0.52(−3)	0.34(−25)	0.13(−202)	8.00
TM8 (3), f_3, G_{12}	0.70(−3)	0.39(−24)	0.35(−194)	8.00
TM8 (3), f_3, G_7	0.72(−3)	0.35(−24)	0.11(−194)	8.00
TM8 (3), f_3, G_3	0.68(−3)	0.11(−24)	0.46(−199)	8.00
TM8 (3), f_3, G_{16}	0.70(−3)	0.32(−24)	0.58(−195)	8.00
TM8 (3), f_3, G_9	0.62(−3)	0.14(−24)	0.11(−197)	8.00
TM8 (3), f_3, G_2	0.67(−3)	0.99(−25)	0.20(−199)	8.00
TM8 (3), f_3, G_{17}	0.67(−3)	0.64(−25)	0.42(−201)	8.00
TM8 (3), f_3, G_{19}	0.89(−4)	0.29(−31)	0.33(−251)	8.00
TM15.5 (35), f_1, G_1	0.13(−1)	0.91(−32)	0.20(−495)	15.52
TM15.5 (35), f_2, G_1	0.30(−6)	0.75(−106)	0.13(−1648)	15.52
TM15.5 (35), f_3, G_1	0.16(−4)	0.16(−68)	0.51(−1069)	15.52
TM15.5 (35), f_1, G_8	0.16(−1)	0.14(−29)	0.85(−461)	15.52
TM15.5 (35), f_2, G_8	0.35(−6)	0.74(−105)	0.37(−1633)	15.52
TM15.5 (35), f_1, G_5	0.17(−1)	0.66(−29)	0.14(−450)	15.52
TM15.5 (35), f_2, G_5	0.39(−6)	0.52(−104)	0.52(−1620)	15.52
TM15.5 (35), f_3, G_5	0.16(−4)	0.11(−68)	0.28(−1071)	15.52
TM12 (24), f_1, G_1	0.96(−3)	0.23(−32)	0.71(−389)	12.00
TM12 (24), f_2, G_1	0.26(−1)	0.51(−25)	0.13(−298)	12.00
TM12 (24), f_3, G_1	0.66(−2)	0.17(−25)	0.20(−309)	12.00
TM12 (24), f_1, G_7	0.12(−2)	0.11(−31)	0.94(−381)	12.00
TM12 (24), f_2, G_7	0.33(−2)	0.35(−32)	0.37(−384)	12.00
TM12 (24), f_3, G_7	0.24(−2)	0.57(−31)	0.36(−375)	12.00
TM14 (27), f_3, G_1	0.11(0)	0.37(−19)	0.18(−272)	14.00
TM14 (27), f_3, G_1	0.34(−1)	0.13(−18)	0.80(−262)	14.00
TM15 (29), f_3, G_{17}	0.25(−2)	0.88(−35)	0.12(−521)	15.00
TM12 (24), f_1, G_8	0.12(−2)	0.12(−31)	0.18(−380)	12.00
TM12 (24), f_2, G_8	0.41(−2)	0.23(−31)	0.49(−374)	12.00
TM12 (24), f_3, G_8	0.20(−2)	0.62(−32)	0.10(−386)	12.00
TM15 (29), f_3, G_1	0.67(0)	0.26(0)	0.21(−9)	15.00
TM15 (29), f_1, G_{21}	0.28(−2)	0.14(−38)	0.54(−586)	15.00
TM15 (29), f_2, G_{21}	0.24(1)	0.37(−1)	0.18(−15)	15.00
TM16 (37), f_1, G_1	0.13(−1)	0.91(−32)	0.21(−514)	16.00
TM16 (37), f_2, G_1	0.35(−4)	0.18(−66)	0.65(−1064)	16.00

Table 3 Comparison of various iterative schemes.

Methods	$\lvert x_1-\xi\rvert$	$\lvert x_2-\xi\rvert$	$\lvert x_3-\xi\rvert$	COC
TM16 (37), f_1, G_5	0.17(−1)	0.66(−29)	0.96(−468)	16.00
TM16 (37), f_2, G_5	0.57(−4)	0.22(−69)	0.41(−1120)	16.00
TM16 (37), f_3, G_5	0.25(−2)	0.37(−39)	0.22(−625)	16.00
TM16 (37), f_1, G_8	0.16(−1)	0.14(−29)	0.15(−478)	16.00
TM16 (37), f_2, G_8	0.62(−4)	0.69(−69)	0.35(−1112)	16.00
TM16 (37), f_3, G_8	0.24(−2)	0.47(−39)	0.12(−623)	16.00
TM16 (37), f_1, G_7	0.16(−1)	0.16(−29)	0.11(−477)	16.00
TM16 (37), f_1, G_6	0.11(−1)	0.25(−30)	0.19(−493)	16.00
TM16 (37), f_2, G_6	0.74(−4)	0.67(−68)	0.15(−1096)	16.00
TM16 (37), f_3, G_6	0.21(−2)	0.23(−39)	0.17(−628)	16.00
TM16 (37), f_1, G_{18}	0.13(−1)	0.10(−30)	0.63(−498)	16.00
TM16 (37), f_2, G_{18}	0.70(−4)	0.30(−68)	0.42(−1102)	16.00
TM16 (37), f_3, G_{18}	0.23(−2)	0.35(−39)	0.13(−625)	16.00
ZLHM [50], f_1, $\gamma_0 = 0.1$	0.35(−1)	0.46(−12)	0.62(−99)	8.00
ZLHM [50], f_2, $\gamma_0 = 1$	0.23(−1)	0.51(−11)	0.23(−88)	8.00
ZLHM [50], f_3, $\gamma_0 = 1$	0.19(−3)	0.64(−27)	0.11(−214)	8.00
TPM [40], f_1, $a = b = 0$	0.16(−1)	0.32(−13)	0.81(−107)	8.00
TPM [40], f_2, $a = b = 0$	0.23(−5)	0.12(−44)	0.10(−358)	8.00
TPM [40], f_3, $a = b = 0$	0.91(−4)	0.48(−29)	0.28(−231)	8.00
CLM[12], Method 1, f_1	0.56(−2)	0.58(−17)	0.71(−37)	8.00
CLM[12], Method 1, f_2	0.13(−4)	0.26(−39)	0.43(−317)	8.00
CLM[12], Method 1, f_3	0.16(−4)	0.14(−35)	0.38(−284)	8.00
BWRM[4], $\beta = 0, \gamma = -2, \lambda = -2.5, f_1$	0.40(−2)	0.24(−18)	0.38(−148)	8.00
BWRM[4], $\beta = 0, \gamma = -2, \lambda = -2.5, f_3$	0.17(−4)	0.37(−39)	0.20(−316)	8.00
BWRM[4], $\beta = 0, \gamma = -2, \lambda = -2.5, f_3$	0.71(−6)	0.52(−46)	0.47(−367)	8.00
LSSSM[21], Method 1, f_1	0.42(−2)	0.78(−18)	0.10(−143)	8.00
LSSSM[21], Method 1, f_2	0.13(−4)	0.73(−39)	0.72(−313)	8.00
LSSSM[21], Method 1, f_3	0.74(−6)	0.94(−46)	0.67(−365)	8.00
KTM[19], f_1, $\gamma = 0.1$	0.23(−1)	0.33(−13)	0.14(−107)	8.00
KTM[19], f_2, $\gamma = 1$	0.38(−2)	0.30(−16)	0.36(−129)	8.00
KTM[19], f_3, $\gamma = 1$	0.35(−5)	0.21(−41)	0.42(−331)	8.00
SSSLM[33],Method 6, f_2	0.19(−10)	0.23(−170)	0.74(−2729)	16.00
SSSLM[33],Method 6, f_3	0.29(−5)	0.36(−79)	0.11(−1261)	16.00
TM16 (37), f_1, G_9	0.14(−1)	0.90(−29)	0.14(−465)	16.00
TM16 (37), f_2, G_9	0.57(−4)	0.19(−69)	0.39(−1121)	16.00
TM16 (37), f_3, G_9	0.25(−2)	0.41(−39)	0.13(−624)	16.00
TM16 (37), f_1, G_{22}	0.16(−1)	0.29(−29)	0.14(−473)	16.00
TM16 (37), f_2, G_{22}	0.62(−4)	0.60(−69)	0.40(−1113)	16.00
TM16 (37), f_3, G_{22}	0.24(−2)	0.46(−39)	0.10(−623)	16.00
TM16 (37), f_2, G_7	0.62(−4)	0.67(−69)	0.24(−1112)	16.00
TM16 (37), f_3, G_7	0.24(−2)	0.47(−39)	0.12(−323)	16.00
TM16 (37), f_1, G_3	0.14(−1)	0.28(−33)	0.21(−538)	16.00
TM16 (37), f_2, G_3	0.68(−4)	0.21(−68)	0.22(−1104)	16.00
TM16 (37), f_3, G_3	0.23(−2)	0.37(−39)	0.28(−625)	16.00
TM16 (37), f_1, G_{11}	0.34(−2)	0.89(−34)	0.96(−541)	16.00
TM16 (37), f_2, G_{11}	0.82(−4)	0.26(−67)	0.21(−1087)	16.00
TM16 (37), f_3, G_{11}	0.21(−2)	0.16(−39)	0.34(−631)	16.00
TM16 (37), f_2, G_1	0.68(−4)	0.22(−68)	0.48(−1104)	16.00
TM16 (37), f_3, G_1	0.23(−2)	0.38(−39)	0.41(−625)	16.00

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