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A Study on the Existence and Uniqueness of Uncertain Fractional Differential Equations with Monotone Condition

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Abstract The aim of this paper is to investigate the existence and uniqueness of solutions to uncertain fractional differential equations proposed by Canonical Liu's process. To this end, we provide and prove a a novel existence and uniqueness theorem for uncertain fractional differential equations under Local Lipschitz and monotone conditions is provided and proved. This result helps us to consider and analyze solutions to a wide range of nonlinear uncertain fractional differential equations driven by Canonical's process to be considered and analyzed.

Keywords Uncertainty theory · Fractional derivative · Uncertain fractional differential equation · Existence · Uniqueness

Mathematics Subject Classification (2010) 65L03 · 60J74 · 60J76

1 Introduction

A large number of physical processes such as real-life phenomena appear to display fractional order demeanor that may vary with space or time. The fractional calculus has authorized the operations of differentiation and integration to at all fractional order. The order may take on at all real or imaginary

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value. Also, multitude systems modeled with the support of fractional calculus demonstrate fractional dynamical conduct such as viscoelastic systems [21], colored noise [17], boundary layer effects in ducts and electromagnetic waves [16]. In recent decades, theory fractional differential equations (FDEs) has enticed many researchers such as [8] and [25] which, applied FDEs for model acoustics, thermal systems, signal processing, system identification, robotics and control, etc.

A FDEs is a differential equation including fractional derivatives. The results of various studies have clearly declared that the fractional derivatives seem to arise universally and generally from main mathematical reasons. There are different kinds of fractional derivatives like the Riemann-Liouville type and Caputo type. To some sources about FDEs are presented in [8] and [9].

With this in mind, we tried to investigate the theory of uncertain fractional differential equations.

As we already know, most of the phenomena and events in the real world like changes in economic and political systems, collapse of governments, conflicts between tribes, wars, terrorist attacks occur unexpectedly. Thus, it is not possible to anticipate or estimate, the price of stocks, valuable papers, monetary units and precious metals accurately. Therefore, focusing on the price of stocks seems to be the only way of finding out how this factor can affect the growth or drop in the value of companies. Investigations on the effects of the factors along with uncertainty theory can help better understanding and more exact modeling of these phenomena. The uncertainty theory was first introduced by Liu who then presented the concept of uncertainty measure which is a powerful tool for dealing with uncertain phenomena facilitates measuring of uncertain events that are based on normality, monotonicity, self-duality and maximality axioms.

Then the concept of uncertain process was proposed by Liu, [11] introducing a particular uncertain process with stationary and independent increment named canonical Liu's process which is exactly like a stochastic process described by Brownian motion.

Since then some articles have been published on the canonical Liu's process and its applications in other sciences such as economics and optimal control [25]. Then Liu was inspired by stochastic notions and ito process to introduce uncertain differential equations [12] driven by canonical Liu's process for better understanding of the uncertain phenomena.

Recently at [26], the concept of uncertain fractional differential equations (UFDEs) was introduced based on the uncertainty theory. The Riemann-Liouville type of uncertain fractional differential equation

$$D^p X(t) = f(t, X(t)) + g(t, X(t)) \frac{dC(t)}{dt}$$

and the Caputo type of uncertain fractional differential equation

$${}^c D^p X(t) = f(t, X(t)) + g(t, X(t)) \frac{dC(t)}{dt}$$

were dealt with where C_t is the canonical Liu process.

Regarding the importance of the existence and uniqueness of a solution to UFDEs driven by canonical Liu's process, Liu investigated the existence and uniqueness of solutions to the uncertain differential equations by employing Lipschitz and Linear growth conditions [6], and stability analysis of uncertain differential equations was given by Yao et al.[21]. Many researchers such as Chen and Liu have managed to find analytic solutions for some special types of uncertain differential equations [1]. Analytical solutions were presented only for some special uncertain fractional differential equations [26] in order to understand what kinds of uncertain fractional differential equations have solutions in [8]. The main goal of this paper is providing some weaker conditions to study the existence an uniqueness of solution to the uncertain fractional differential equations. In this regard, we tried to prove a novel existence and uniqueness theorem under the Local Lipschitz and monotone conditions.

In this paper, some concepts and results in uncertainty theory are first reviewed. Then, the fractional derivatives and uncertain fractional differential equations are also taken into consideration. Finally, an existence and uniqueness theorem under Local Lipschitz and monotone conditions are proved.

2 Preliminaries

The emphasis in this section is mainly on introducing some concepts such as uncertainty measure,uncertainty space, uncertain variables, independence of uncertain variables, expected value, variance, uncertain process, and canonical Liu's process.

Suppose that Θ is a non-empty set and $\mathcal{P}$ is the power set of Θ. Each element of κ in $\mathcal{P}$ is called an event. For the purpose of presenting a basic definition of uncertainty, it is needed to consider a number $\mathbf{M}\{\kappa\}$ to each event κ. In order to make sure the number $\mathbf{M}\{\kappa\}$ has certain mathematical features that is intuitively expect these four axioms are accepted [8]:

1. Axiom (Normality) $\mathbf{M}\{\Theta\} = 1$.
2. Axiom (Monotonicity) $\mathbf{M}\{\kappa\} \leq \mathbf{M}\{\beta\}$ whenever $\kappa \subset \beta$.
3. Axiom (Self-Duality) $\mathbf{M}\{\kappa\} + \mathbf{M}\{\kappa^c\} = 1$ for any event κ.
4. Axiom (Maximality) $\mathbf{M}\{\mathbf{U}_i\kappa_i\} = \sup_i \mathbf{M}\{\kappa_i\}$ for any events $\{\kappa_i\}$ with $\sup_i \mathbf{M}\{\kappa_i\} < 0.5$.

Definition 1 [18]. The set function $\mathbf{M}$ is called a uncertainty measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms.
A family $\mathcal{P}$ with these four properties is called a σ-algebra. The pair $(\Theta, \mathcal{P})$ is called a measurable space, and the elements of $\mathcal{P}$ is afterwards called $\mathcal{P}$-measurable sets instead of events.

Definition 2 [18]. The triple $(\Theta, \mathcal{P}, \mathbf{M})$ is a uncertainty space if Θ be a nonempty set, $\mathcal{P}$ the power set of Θ, and $\mathbf{M}$ a uncertainty measure.
Let $(\Theta, \mathcal{P}, \mathbf{M})$ be a uncertainty space. A filtration is a family $\{\mathcal{P}_t\}_{t\geq 0}$ of increasing sub-σ-algebras of $\mathcal{P}$ (i.e. $\mathcal{P}_t \subset \mathcal{P}_s \subset \mathcal{P}$ for all $0 \leq t < s < \infty$). The

filtration is said to be right continuous if $\mathcal{P}_t = \bigcap_{s>t} \mathcal{P}_s$ for all $t \leq 0$. When the uncertainty space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and $\mathcal{P}_0$ contains all $\mathbf{M}$-null sets.
We also define $\mathcal{P}_\infty = \sigma(U_{t\geq 0}\mathcal{P}_t)$ (i.e. σ-algebra generated by $U_{t\geq 0}\mathcal{P}_t$.)
$\mathcal{P}$-measurable uncertain variable are denoted by $\mathbf{L}^p(\Theta, \mathbf{R^d})$ that will be defined later.
A process is called $\mathcal{P}$-adapted, if for all $t \in [0, t]$ the uncertain variable $x(t)$ is $\mathcal{P}$-measurable.

Definition 3 [18]. An uncertain variable is defined as a (measurable) function $\xi : (\Theta, \mathcal{P}, \mathbf{M}) \longrightarrow \mathbf{R}$.

Definition 4 [18]. If we suppose that ς is an uncertain variable. Then the expected value of ς is as follows:

$$\mathbf{E}[\varsigma] = \int_0^{+\infty} \mathbf{M}\{\varsigma \geq s\}\mathrm{ds} - \int_{-\infty}^0 \mathbf{M}\{\varsigma \leq s\}\mathrm{ds}$$

these two integrals are finite.

Definition 5 [10]. The uncertainty distribution $\eta(x)$ of an uncertain variable ς is defined by

$$\eta(w) = \max\{1, 2\mathbf{M}(\varsigma = w)\},\ w \in \mathbf{R}.$$

Definition 6 [10]. An uncertainty distribution $\eta(w)$ is regular on condition that it is a continuous and strictly increasing function with respect to w at which $0 < \eta(w) < 1$, and

$$\lim_{w\to-\infty} \eta(w) = 0, \qquad \lim_{w\to+\infty} \eta(w) = 1.$$

In addition, the inverse function $\eta^{-1}(\alpha)$ can be called the inverse uncertainty distribution of ς.

Definition 7 [9]. Considering $\mathbf{T}$ be an index set and $(\Theta, \mathcal{P}, \mathbf{M})$ be an uncertainty space. An uncertain process can be described as a function from $\mathbf{T} \times (\Theta, \mathcal{P}, \mathbf{M})$ to the set of real numbers.
An uncertain process is basically a sequence of uncertain variables indexed by time or space.

Definition 8 [10]. An uncertain process $\mathbf{C}_t$ is a canonical Liu process if the following are met

1. $\mathbf{C}_0 = 0$,
2. $\mathbf{C}_t$ has stationary and independent increments,
3. every increment $\mathbf{C}_{t+s} - \mathbf{C}_s$ is a normally distributed uncertain variable with expected value **et** and variance $\sigma^2 t^2$ whose membership function is

$$\eta(w) = 2\Big(1 + \exp(\frac{\pi|w - et|}{\sqrt{6}\sigma t})\Big)^{-1},\ \ -\infty < w < +\infty.$$

Based on canonical Liu process, uncertain integral is defined as an uncertain counterpart of Ito integral as follows.

provided that the limit exists almost surely and is an uncertain variable. Let us define a sequence of uncertainty stopping times.

Definition 9 [9]. An uncertain variable $\tau : \Theta \to [0,\infty]$ (it may take the value ∞) is called a $\{\mathcal{P}_t\}$-stopping time (or simply, stopping time) if $\{\theta : \tau(\theta) \leq t\} \in \mathcal{P}_t$ for any $t \geq 0$

$$\begin{cases} \tau_h = \inf\{t \geq 0 : |w(t)| \geq k\}, \\ \sigma_1 = \inf\{t \geq 0 : x(w(t)) \geq 2\varepsilon\}, \\ \sigma_{2i} = \inf\{t \geq \sigma_{2i-1} : x(w(t)) \leq \varepsilon\} \quad i = 1,2,\ldots, \\ \sigma_{2i+1} = \inf\{t \geq \sigma_{2i} : x(w(t)) \geq 2\varepsilon\} \quad i = 1,2,\ldots, \end{cases}$$

where throughout this paper we set $\inf \phi = \infty$.

Definition 10 [9]. If $W = \{W_t\}_{t\geq 0}$ is a measurable process and τ is a stopping time, then $\{W_{\tau\wedge t}\}_{t\geq 0}$ is called a stopped process of W.

There are some useful inequalities for uncertain variables such as Hölder inequality and Chebyshev inequality. In the sequence, we introduce generalized inequalities for uncertain variables.

Theorem 1 *(Hölders Inequality) [4]. Let* $\mathbf{n}$ *and* $\mathbf{m}$ *be two positive real numbers with* $\frac{1}{\mathbf{n}} + \frac{1}{\mathbf{m}} = 1$, ξ *and* η *be independent uncertain variables with*

$$\mathbf{E}[|\varsigma|^{\mathbf{n}}] \leq +\infty \quad \text{and} \quad \mathbf{E}[|\rho|^{\mathbf{m}}] \leq +\infty.$$

We have

$$\mathbf{E}[|\varsigma\rho|] \leq \sqrt[\mathbf{n}]{\mathbf{E}[|\varsigma|^{\mathbf{n}}]}\ \sqrt[\mathbf{m}]{\mathbf{E}[|\rho|^{\mathbf{m}}]}.$$

Theorem 2 *(Chebychev's Inequality). Let* $\varsigma : \theta \to \mathbf{R}^k$ *be an uncertain variable such that* $\mathbf{E}[|\varsigma|^{\mathbf{n}}] \leq +\infty$ *for some* $\mathbf{n}, 0 \leq \mathbf{n} \leq \infty$.
Then Chebychev's inequality:

$$\mathbf{M}[|\varsigma| \geq \lambda] \leq \frac{1}{\lambda^{\mathbf{n}}}\mathbf{E}[|\varsigma|^{\mathbf{n}}] \textit{for all} \lambda \geq 0.$$

Proof Put $\mathbf{A} = \{x \mid |\varsigma(x)| \geq \lambda\}$. Then

$$\int_\theta |\varsigma(x)|^{\mathbf{p}} \mathrm{d}\mathbf{M}_x \geq \int_A |\varsigma(x)|^{\mathbf{n}} \mathrm{d}\mathbf{M}_x \geq \lambda^n \mathbf{M}_A.$$

Before, ending this section it is essential to introduce some symbols that are used in next sections.

Notation 1: $\mathbf{L}^{\mathbf{n}}(\theta, \mathbf{R}^{\mathbf{d}})$ the family of $\mathbf{R}^{\mathbf{d}}$-valued uncertain variables ς with $\mathbf{E}|\xi|^{\mathbf{p}} < \infty$.
Notation 2: $\ell^{\mathbf{p}}([a,b], \mathbf{R}^{\mathbf{d}})$ the family of $\mathbf{R}^{\mathbf{d}}$-valued $\mathcal{P}_t$-adapted processes $\{h(t)\}_{a\leq t\leq b}$ such that $\int_a^b |h(t)|^{\mathbf{n}} \mathrm{d}t < \infty$ almost surely.
Notation 3: $M^{\mathbf{n}}([a,b], \mathbf{R}^{\mathbf{d}})$ the family of processes $\{h(t)\}_{a\leq t\leq b}$ in $\ell^{\mathbf{n}}([a,b], \mathbf{R}^{\mathbf{d}})$ such that $\int_a^b |h(t)|^{\mathbf{n}} dt < \infty$.
Notation 4: $\ell^{\mathbf{n}}(\mathbf{R}_+, \mathbf{R}^{\mathbf{d}})$ the family of processes $\{h(t)\}_{t>0}$ such that for every $T > 0$, $\{h(t)\}_{a\leq t\leq T} \in \ell^{\mathbf{n}}([0,T], \mathbf{R}^{\mathbf{d}})$.

3 Uncertain fractional differential equations

In this section, we give some basic definitions, notations and lemmas which will be used throughout the paper, in order to establish our main results.

Let us introduce tow common notation for the fractional-order differential operator: the Riemann-Liouville and the Caputo-type. For more details see [1, 26].

Definition 11 [10]. The Riemann-Liouville fractional derivative of f is defined as

$$ {}^{R}D^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}f(s)ds \tag{1}$$

where $\Gamma(.)$ stands for the gamma function $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$, $\alpha\in(0,1]$, and $t>0$.

Definition 12 [9]. The Caputo-type derivative of order α for a function f can be written as

$$ {}^{C}D^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f'(s)ds \tag{2}$$

where $f'(s)$ is the first-order derivative of $f(s)$.

Remark 1 The relationship between the Riemann-Liouville derivative and the Caputo-type derivative can be written as

$$ {}^{R}D^{\alpha}f(t)={}^{C}D^{\alpha}f(t)+\frac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0). \tag{3}$$

In this study, we consider the Caputo-type fractional derivative of order α and the initial value problem of uncertain fractional-order differential equation is given as following:

$$\begin{cases} {}^{C}D^{\alpha}w(t)=f(w(t),t)+g(w(t),t)\dot{C}(t) \\ w(t)=w_t, \end{cases} \tag{4}$$

where the functions $f(w(t),t)$ and $g(w(t),t):[0,T]\times R\to R$. The term $\dot{C}(t)=\frac{dC}{dt}$ describes a state dependent random noise, $C(t)$ is a canonical Liu process defined on a given filtered uncertainty space $(\Theta,\mathcal{P},\{\mathcal{P}_t\}_{t\geq 0},\mathbf{M})$ with a normal filtration $\{\mathcal{P}_t\}_{t\geq 0}$, which is an increasing and continuous family of σ-algebras of $\mathcal{P}$, contains all of Θ-null sets, and $C(t)$ is $\mathcal{P}$-measurable for each $t>0$. Here, let us recall the definitions of fractional calculus, the fractional integral operator of order α is given as following

$$ I^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds, \quad t>0 \tag{5}$$

Applying the integral operator (5) to the both sides of initial value problem (4) we can obtain the Volterra integral equation:

$$ w(t)=w_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(w(s),s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(w(s),s)dC(s), \tag{6}$$

in which $\alpha\in(0,1)$ and $t>0$.

Now, the required tools to prove our proposed theorem for existence and uniqueness, such as Lemma's and inequalities our mentioned.

Lemma 1 *(Burkholder -Davis-Gundy inequality for* uncertain process*):*
Let $g \in \ell^p(\mathbf{R}_+, \mathbf{R}^{d\times m})$*. Define for* $t > 0$*,*

$$w(t) = \int_0^t q(s)d\mathbf{C}_s \quad and \quad \kappa(t) = \int_0^t |g(s)|^2 ds.$$

Then,

$$\mathbf{E}|\kappa(t)| \leq \mathbf{E}(\sup_{0\leq s\leq t} |w(s)|^2) \leq 4\mathbf{E}|\kappa(t)|. \tag{7}$$

Proof Without loss of generality, if $w(t)$ and $\kappa(t)$ are bounded. For each integer $n > 1$, we define the stopping time as

$$\tau_n = \inf\{t \geq 0 : |w(t)| \vee \kappa(t) \geq n\}.$$

If (7) can be state by the stopped processes $w(t \wedge \tau_n)$ and $\kappa(t \wedge \tau_n)$, then the general case follows by letting $n \to \infty$. Moreover, for simplicity we set

$$w^*(t) = \sup_{0\leq s\leq t} |w(s)|.$$

Consider, the following inequality

$$E|w(t)|^2 \leq E\int_0^t |g(s)|^2 ds = E(\kappa(t)), \tag{8}$$

then by the use of Doob martingale inequality[3],

$$E|w^*(t)| \leq 4E|w(t)|^2,$$

by substituting this into (8) yields,

$$E|w^*(t)| \leq 4E(\kappa(t)),$$

which is the right-hand-side inequality of (7). In order to demonstrate the left-hand-side one,

$$y(t) = \int_0^t q(s)ds.$$

Then,

$$\mathbf{E}|y(t)|^2 = \mathbf{E}\int_0^t |q(s)|^2 ds = \mathbf{E}\int_0^t d\kappa(s) = \mathbf{E}|\kappa(t)|. \tag{9}$$

On the other hand, the integration by parts formula yields,

$$\begin{aligned} w(t) &= \int_0^t dw(s) + \int_0^t w(s)ds \\ &= y(t) + \int_0^t w(s)ds. \end{aligned}$$

Therefore,

$$|y(t)| \leq |w(t)| + \int_0^t |w(s)|ds \leq 2w^*(t).$$

Here, by substituting this into (9) one sees that,

$$\mathbf{E}|\kappa(t)| \leq 4\mathbf{E}[|w^*(t)|^2].$$

This implies,

$$\frac{1}{4}\mathbf{E}|\kappa(t)| \leq \mathbf{E}[|w^*(t)|^2].$$

Finally, the proof is complete.

In the theory of ordinary differential equations, stochastic differential equations, and fuzzy differential equations the integral inequalities of Gronwall type have been used in a wide scope. In order to prove the results on stability, existence, and uniqueness.

Lemma 2 *(Gronwall's inequality for* canonical Liu process*): Let* $\mathrm{T} > 0$*,* $c \leq 0.$*, and* $\vartheta(.)$ *be an uncertainty measurable bounded nonnegative function on* $[0, \mathrm{T}]$*, and* $\beta(.)$ *be a nonnegative integrable function on* $[0, \mathrm{T}]$*. If*

$$\vartheta(t) \leq c + \int_0^t \beta(s)\vartheta(s)ds \quad \textit{for all} \quad 0 \leq t \leq T, \tag{10}$$

then

$$\vartheta(t) \leq c\exp(\int_0^t \beta(s)ds) \quad \textit{for all} \quad 0 \leq t \leq T. \tag{11}$$

Proof First, we may assume that $c > 0$. Set

$$e(t) = c + \int_0^t \beta(s)\vartheta(s)ds \quad \text{for} \quad 0 \leq t \leq T.$$

Then $\vartheta(t) \leq e(t)$. Moreover, by the chain rule of classical calculus, we have

$$\log(e(t)) = \log(c) + \int_0^t \frac{\beta(s)\vartheta(s)}{e(s)}ds \leq \log(c) + \int_0^t \beta(s)ds.$$

This implies

$$e(t) \leq c\exp(\int_0^t \beta(s)ds) \quad \text{for} \quad 0 \leq t \leq T.$$

According to $\vartheta(t) \leq e(t)$, the proof is complete.

Lemma 3 *If the function* $L(u, t)$ *is locally integrable in* t *for each fixed* $u \in [0, \infty]$ *and is continuous non-decreasing in* u *for each fixed* $t \in [0, T]$*, for all* $\lambda > 0$*,* $u_0 \geq 0$*, then the integral equation*

$$u(t) = u_0 + \lambda \int_0^t L(u(s), s)ds, \tag{12}$$

has a global solution on $[0, T]$*.*

Lemma 4 *The function $K(u,t)$ is locally integrable in t for each fixed $u \in [0,\infty]$ and is continuous non-decreasing in u for each fixed $t \in [0,T]$, for $K(0,t)=0$ and $\gamma > 0$, if a non-negative continuous function $z(t)$ satisfies*

$$\begin{aligned} &z(t) \le \gamma \int_0^t K(z(s),s)ds, \quad t \in R, \\ &z(0) = 0, \end{aligned} \tag{13}$$

then $z(t)] = 0$ for all $t \in [0,T]$.

Throughout this paper, we consider the uncertain fractional differential equations

$$\begin{cases} {}^C D^\alpha w(t) = f(w(t),t)dt + g(w(t),t)\frac{d\mathbf{C}_t}{dt}, \\ w(0) = w_0, \end{cases} \tag{14}$$

where $\mathbf{C}_t$ is a canonical Liu process and p, q are some given functions. $w(t)$ is the solution to the (14) which is an uncertain process in the sense of Liu. In order to consider the existence and uniqueness of the solution of equation (14), we attempt to use the following approximate technique, known as Picard s iteration. The sequence of stochastic process $\{w_n\}_{n\ge 0}$ is constructed as follows:

$$\begin{cases} w_{n+1}(t) = w_0 + \frac{1}{\Gamma(\alpha)}\int_0^t p(w_n(s),s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t q(w_n(s),s)dC(s), \ n \ge 1, \\ w(0) = w_0, \end{cases} \tag{15}$$

in which

$$\begin{cases} p(w_n(t),t) = (t-s)^{\alpha-1} f(w_n(t),t), \\ \\ q(w_n(t),t) = (t-s)^{\alpha-1} g(w_n(s),s). \end{cases} \tag{16}$$

As this point the following conditions, lemmas, and remarks for the proof of the uniqueness and existence should be elaborated on.

(I) Local Lipschitz condition: For each integer $n \ge 1$, there exists a positive constant number $\mathbf{L}_n$ such that

$$|p(w(t),t) - p(y(t),t)|^2 \vee |q(w(t),t) - q(y(t),t)|^2 \le \mathbf{L}_n |w(t) - y(t)|^2,$$

for those $w(t), y(t) \in \mathbf{R}^n$ with $|w(t)| \vee |y(t)| \le n$.

(II) Linear growth condition: There exists a positive number $\mathbf{L}$ such that

$$|p(w(t),t)|^2 \vee |q(w(t),t)|^2 \le \mathbf{L}(1 + |w(t)|^2),$$

(III) Monotone condition: there exists a positive constant $\mathbf{K}$ such that

$$w(t)^T f(w(t),t) + \frac{1}{2}|q(w(t),t)| \le \mathbf{K}(1 + |w(t)|^2)$$

for all $w(t) \in \mathbf{R}^n$.

The following Remark proves the exact solution to equation (14) under the monotone condition (**III**).

Remark 2 Assume the monotone condition (**III**), there exists a positive constant **G** such that the solution of (7) satisfies

$$\mathbf{E}(\sup_{t_0\le t\le T}|w(t)|^2)\vee\mathbf{E}(\sup_{t_0\le t\le T}|y(t)|^2)\le\mathbf{G}, \tag{17}$$

where $\mathbf{G}=\mathbf{G}(T,K,w_0)$ is a constant independent of h, ($h=\frac{1}{m}$ be a given step size with integer $m\ge 1$ and $T=Nh$).

Proof By uncertain'formula and condition (**III**), we can derive that for $t\in[t_0,T]$

$$|w(t)|^2=|w_0|^2+\int_{t_0}^{t}[2(w^Tp(w(s),s)) \\ +\frac{1}{2}|q(w(s),s)|^2]ds+\int_{t_0}^{t}2(w^Tq(w(s),s))d\mathbf{C}_s,$$

we have

$$1+|w(t)|^2\le 1+|w_0|^2+2k\int_{t_0}^{t}[1+|w(s)|^2]ds+2\int_{t_0}^{t}2(w^Tq(w(s),s))d\mathbf{C}_s,$$

we know

$$\mathbf{E}\sup(1+|w(t)|^2)\le(1+|w_0|^2)+\mathbf{E}\sup\int_{t_0}^{t}2k[1+|w(s)|^2]ds \\ +\mathbf{E}\sup|\int_{t_0}^{t}2(w^Tq(w(s),s))d\mathbf{C}_s|. \tag{18}$$

By the Lemma(3) (Burk holder -Davis-Gundy inequality for *canonical Liu process*) to show that

$$\begin{aligned}\mathbf{E}\sup|\int_{t_0}^{t}2(w(s)^Tq(w(s),s))d\mathbf{C}_s| &\le 2\mathbf{E}(\int_{t_0}^{t}4(|w(s)|^2|q(w(s),s)|^2)ds)^{\frac{1}{2}}\\ &\le 2\mathbf{E}(\sup(1+|w(t)|^2)\int_{t_0}^{t}4|q(w(s),s)|^2ds)^{\frac{1}{2}}\\ &\le 3\mathbf{E}(\sup(1+|w(t)|^2)\int_{t_0}^{t}4k(1+|w(t)|^2)ds)^{\frac{1}{2}}\\ &\le 0.5\,\mathbf{E}\sup(1+|w(t)|^2)\\ &\quad+\frac{9}{2}\mathbf{E}\int_{t_0}^{t}4k(1+|w(t)|^2)ds.\end{aligned} \tag{19}$$

Substituting (19) into (18) and using the Hölder inequality

$$\mathbf{E}\sup(1+|w(t)|^2)\le\left[2(1+|w(0)|^2)+40k\left(\mathbf{E}\int_{t_0}^{t}(1+|w(t)|^2)ds\right)\right],$$

so we obtain

$$\mathbf{E}\sup(1+|w(t)|^2) \leq \left[2(1+|w(0)|^2) + 80k\left(\int_{t_0}^{t}\mathbf{E}\sup(1+|w(t)|^2)ds\right)\right].$$

By the Gronwall inequality, we must get

$$\mathbf{E}\sup(|w(t)|) \leq \mathbf{G}$$

where $\mathbf{G} = (T, k, w_0)$ is a constant independent of h. Similarly, we can show that $\mathbf{E}\sup(|y(t)|) \leq \mathbf{G}$.

It is known that some functions, such as $\sin^2 w$ and $-|w|^2 w$, do not satisfy in Lipschitz and Linear growth conditions. Therefore, we prove the following theorem under weaker conditions that ensure the existence and uniqueness of the solution to equation (14).

Lemma 5 *The sequence of uncertain process* $\{w_n\}_{n\geq 0}$ *is a Cauchy sequence.*

Proof Using the same argument in Remark (2), we can obtain

$$\| w_m(t) - w_n(t) \|^2 \leq \mathbf{G}_1 \int_0^t k\left(\| w_m(t) - w_n(t) \|^2\right) ds, \tag{20}$$

in which $\mathbf{G}_1 = \frac{4T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}$. Let $\beta_n = \sup_{m\geq n}(\| w_m(t) - w_n(t) \|^2)$, we imply that

$$\beta_n \leq \mathbf{G}_1 \int_0^t k(\beta_{n-1}(t), s)ds. \tag{21}$$

It is obvious that the function $\beta_n(t), n \geq$ is well defined and bounded by Remark (2) and also monotone non-decreasing. So there exist a monotone non-decreasing function $\beta(t)$ such that $\lim_{n\to 1}\beta_n(t) = \beta(t)$. Using the Lebesgue convergence theorem and taking $n \to +\infty$ in above inequality, we get

$$\beta(t) \leq \gamma \int_0^t k(\beta(t), s). \tag{22}$$

It means that $\beta(t) = 0$ follows from Lemma (3), for all $t \in [0, T]$. However, we can see that $0 \leq\| w_m(t) - w_n(t) \|^2 \leq \beta_n(T)$ and $\beta_n(T) \to \beta(T) = 0$ when $n \to +\infty$. So $\{w_n\}_{n\geq 0}$ is a Cauchy sequence.

Theorem 3 *(Existence and Uniqueness of solution) Under the conditions Local Lipschitz condition,Monotone condition,(12) and (13), there exists a unique solution of equation (14).*

Proof Existence: If we denote $w(t)$ by the limit of the sequence $\{y_n\}_{n\geq 0}$, repeating the proof of Lemma 3.5, then we use the right side of Picard s iteration (15) converge to

$$w_0 + \frac{1}{\Gamma(\alpha)}\int_0^t p(w(s), s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t q(w(s), s)dC(s). \tag{23}$$

Now, we show that (24)satisfied equation (14). Note that

$$\mathbf{E}\Big|\int_{t_0}^{t}[p(w_{n+1}(s),s)]ds-\int_{t_0}^{t}[p(w_n(s),s)]ds\Big|^2 + \mathbf{E}|\int_{t_0}^{t}[q(w_{n+1}(s),s)]d\mathbf{C}s-\int_{t_0}^{t}[q(w_n(s),s)]d\mathbf{C}_s|^2$$
$$\leq \aleph(T-t_0+1)\int_{t_0}^{t}\mathbf{E}|(w_{n+1}(s),s)-(w_n(s),s)|^2ds\to 0 \quad \text{as} \quad n\to\infty.$$

Hence we can let $n\to\infty$ in (12) to obtain that

$$w(t)=w_0+\frac{1}{\Gamma(\alpha)}\int_0^t p(w(s),s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t q(w(s),s)dC(s). \tag{24}$$

as desired. The existence has been proved.
Uniqueness: Suppose $w(t)$ and $z(t)$ are two solutions of equation (12), using the same argument as in Lemma (5), we have

$$\| w(t)-z(t)\|^2\leq \mathbf{G}\int_0^t\aleph(\| w(s)-z(s)\|^2,s)ds. \tag{25}$$

Using the Lemma (3) again, we can obtain $\| w(t)-z(t)\|^2=0$ for all $t\in[0,T]$, which implies that $w(t)=z(t)$. The proof is completed.

4 Conclusion

The existence and uniqueness theorem is one of the most useful and basic theorems in the theory of uncertain fractional differential equations. However, there are few people who have considered weaker conditions. In the present paper, we have aimed to prove a novel existence and uniqueness theorem under the Local Lipschitz and monotone conditions.

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