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Isoclinism of n-Hom-Lie Algebras

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Abstract In the present article, we define the concept of isoclinism in the context of n-Hom-Lie algebras and investigate some of its properties. Also, we introduce factor sets on n-Hom-Lie algebras. By restricting these structures to semisimple linear operators of these structures, it is shown that the equivalency between isoclinism and isomorphism of two finite-dimensional n-Hom-Lie algebras just depends on whether one of the operators of them is onto.

Keywords Isoclinism · n-Hom-Lie algebra · Factor set · Semisimple linear operator

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1 Introduction

In 2002, Hartwig, Larsson, and Silvestrov introduced the notion of Hom-Lie algebras [14], and outlined some of their fundamental properties which are studied in mathematical physics, for generalizing the Yang-Barter equation and braid group representations [24]. A Hom-Lie algebra is an F-vector space equipped with a bilinear skew-symmetric bracket that satisfies the Jacobi identity twisted by a linear operator φ. When φ is the identity map, the definition of Hom-Lie algebras coincides with Lie algebras. The construction of Hom-Lie

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algebras as a generalization of Lie algebras, leads to the category of Hom-Lie algebras which is denoted by HomLie [5]. In fact, Hom-Lie algebras can be considered as the objects of the category HomLie, and its morphisms are Lie algebra homomorphisms $f : (V, \varphi) \longrightarrow (W, \psi)$ such that $f \circ \varphi = \psi \circ f$. Hom-Lie algebras are studied in various areas related to Lie algebras such as semisimple Lie algebras, (co)homology theory, representation theory, universal central extension, non-abelian tensor product, and simple Lie algebra, respectively in [16, 27, 1, 9, 25, 7, 8, 18].

Philip Hall, in 1940, introduced group isoclinism [13], and Kay Moneyhun extended this notion to Lie algebras in 1994 and defined factor sets for Lie algebras. As a result, it was shown that for a given finite dimension, isomorphism and isoclinism are equivalent [16].

The concept of n-Lie algebras was defined by Filippov in 1987. Also, he proved all n-Lie algebras of dimension $n+1$ over an algebraically closed field were classified [12]. Eshrati and Moghaddam presented similar results of isoclinism in n-Lie algebras. Utilizing the notion of isoclinism, they proved that isomorphism and isoclinism are identical on n-Lie algebras of the same finite dimension [11]. The notion of an n-Hom-Lie algebra which is a generalization of an n-Lie algebra was introduced in 2011, [2]. Then several aspects of algebraic structures about n-Hom-Lie algebras, for example, the cohomologies, central extensions and deformations were studied.

The first purpose of this paper, is to provide a definition of isoclinism for an n-Hom-Lie algebra. In order to investigate its properties, we concentrate on such an n-Hom-Lie lgebra whose linear map is semisimple linear operator. Then by defining Hom-stem n-Hom-Lie algebras, we prove that the Hom-centers of two isoclinic Hom-stem n-Hom-Lie algebras are isomorphic. Finally, we introduce the notion of factor sets on n-Hom-Lie algebras. As a conclusive result, we show that the equivalency between isoclinism and isomorphism of two finite-dimensional n-Hom-Lie algebras with semisimple linear operator, depend on that only one of the operators of them be onto.

2 Preliminaries

Throughout this paper, we fix F as a ground field and all the vector spaces are considered over F and linear operators are F-linear operators.

Definition 1 A Lie algebra $(V, [-,-])$ with a linear operator $\varphi : V \longrightarrow V$ is called *Hom-Lie algebra* provided

(i) $[x, y] = -[y, x], \quad (skew - symmetry)$

(ii) $[\varphi(x), [y, z]] + [\varphi(y), [z, x]] + [\varphi(z), [x, y]] = 0, \quad (Hom - Jacobi\ identity)$

for all $x, y, z \in V$.
In this paper, we assume that φ preserves the bracket i.e. $\varphi([x, y]) = [\varphi(x), \varphi(y)]$, for all $x, y \in V$ and it is called *Lie algebra endomorphism.*

In the case of $\varphi = id_V$, Hom-Lie algebras are exactly Lie algebras. A vector space V endowed with a linear operator $\varphi : V \longrightarrow V$ is called *Hom-vector space*. A Hom-vector space (V, φ) with the trivial bracket and any linear operator $\varphi : V \longrightarrow V$ constructs a Hom-Lie algebra which is called *abelian* Hom-Lie algebra.

Example 1 For any Lie algebra V and Lie algebra endomorphism

$$\varphi : V \longrightarrow V,$$

we have Hom-Lie algebra (V, φ), if we define the bracket by

$$[x, y]_\varphi := [\varphi(x), \varphi(y)],$$

for all x, y in V.

Definition 2 A *Hom-Lie subalgebra* of (V, φ) is a vector subspace W of V, which is closed under bracket and φ, i.e. $[w, w'], \varphi(w) \in W$, for all $w, w' \in W$. A Hom-Lie subalgebra $(W, \varphi_|)$ which $\varphi_|$ is restriction of φ to W, is said to be an *ideal* if $[w, v] \in W$, for all $w \in W, v \in V$. A Hom-Lie algebra (V, φ) is said to be *regular* if φ is bijective. Recall that the center of a Lie algebra, $Z(V)$, is defined as $Z(V) = \{x \in V |\ [x, v] = 0, \quad \forall v \in V\}$.

As a generalization, the set $Z_\varphi(V) = \{x \in V |\ [\varphi^k(x), v] = 0, \quad \forall v \in V, k \geq 0\}$, where $\varphi^0 = id_V$ and φ^k, $k \geq 1$ is the k times composition of φ with itself, is the largest central ideal of (V, φ) which is called the *Hom-center* of (V, φ). Let (V, φ) and (W, ψ) be two Hom-Lie algebras. A linear map $f : V \longrightarrow W$ is a *Hom-Lie algebra morphism*, if $f([v_1, v_2]) = [f(v_1), f(v_2)]$, for all $v_1, v_2 \in V$ and $f \circ \varphi = \psi \circ f$. This property may be more palatable by asserting that the following diagram is commutative.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ V & \xrightarrow{f} & W \end{array}$$

In 1987, Filippov introduced the notion of n-Lie algebras which we recall that in the following definition [12].

Definition 3 Let $n \in \mathbb{N}$ and $n \geq 2$. An *n-Lie algebra* is a pair $(V, [-, \cdots, -])$ where V is a vector space and

$$\begin{aligned} [-, \ldots, -] : V \oplus \cdots \oplus V &\longrightarrow V, \\ (v_1, \ldots, v_n) &\longmapsto [v_1, \ldots, v_n], \end{aligned}$$

is a skew-symmetric n-linear map, called n-Lie bracket, that satisfies the following generalized Jacobi identity

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$

Clearly, such an algebra becomes an ordinary Lie algebra, when $n = 2$.

A generalization of Hom-Lie algebras is the following notion which is defined by H. Ataguema, A. Makhlouf, and S. Silvestrov in [3].

Definition 4 An *n-Hom-Lie algebra* is a triple $(V, [-, \cdots, -], \varphi)$ in which $(V, [-, \cdots, -])$ is an n-Lie algebra and $\varphi : V \to V$ is a linear operator such that for all $v_1, \dots, v_n, w_2, \dots, w_n \in V$ the following identity holds

$$[[v_1, \dots, v_n], \varphi(w_2), \dots, \varphi(w_n)] = \sum_{i=1}^{n} [\varphi(v_1), \dots, [v_i, w_2, \dots, w_n], \varphi(v_{i+1}), \dots, \varphi(v_n)].$$

It is denoted by (V, φ) briefly.
A subspace W of n-Hom-Lie algebra (V, φ), which is closed under the n-Lie bracket and φ is called *n-Hom-Lie subalgebra*. An n-Hom-Lie subalgebra W of V is called an *n-Hom-Lie ideal*, provided

$$[W, \underbrace{V, \dots, V}_{(n-1)-times}] \subseteq W.$$

The n-Hom-Lie ideal generated by $\langle [v_1, \dots, v_n] \mid v_i \in V \rangle$ is the *derived* ideal and denoted by V^2. The *Hom-center* of n-Hom-Lie algebra (V, φ) is defined as

$$Z_\varphi(V) = \{x \in V : [\varphi^k(x), v_1, \dots, v_{n-1}] = 0 \;\; \forall\, v_i \in V, 1 \le i \le n-1, k \ge 0\},$$

which is an ideal.

Definition 5 Let (V, φ) and (W, ψ) be two n-Hom-Lie algebras. A linear map $f : V \longrightarrow W$ is an *n-Hom-Lie algebra morphism*, if

$$f([v_1, \dots, v_n]) = [f(v_1), \dots, f(v_n)],$$

for all $v_1, \dots, v_n \in V$ and $f \circ \varphi = \psi \circ f$.

In our investigation the following definition is fundamental.

Definition 6 Let (V, φ) and (W, ψ) be two n-Hom-Lie algebras and

$$\alpha : V/Z_\varphi(V) \longrightarrow W/Z_\psi(W),$$

and $\beta : V^2 \longrightarrow W^2$ be two Hom-Lie algebra morphisms such that the following diagram commutes

$$\begin{array}{ccc} V/Z_\varphi(V) \oplus \cdots \oplus V/Z_\varphi(V) & \xrightarrow{\rho} & V^2 \\ {\scriptstyle \alpha^n}\downarrow & & \downarrow{\scriptstyle \beta} \\ W/\psi(W) \oplus \cdots \oplus W/\psi(W) & \xrightarrow{\sigma} & W^2 \end{array}$$

in which ρ and σ are defined by $\rho(\overline{v}_1, \dots, \overline{v}_n) = [v_1, \dots, v_n]$, for all

$$\overline{v}_i = v_i + Z_\varphi(V) \in V/Z_\varphi(V),$$

and $\sigma(\widetilde{w}_1, \ldots, \widetilde{w}_n) = [w_1, \ldots, w_n]$, for all

$$\widetilde{w}_i = w_i + Z_\psi(W) \in W/Z_\psi(W), \quad 1 \leqslant i \leqslant n.$$

In other words, $\beta([v_1, \ldots, v_n]) = [w_1, \ldots, w_n]$, whenever $w_i \in \alpha(v_i + Z_\varphi(V))$ for $i = 1, \ldots, n$. Then the pair (α, β) is called *homoclinism* and if they are both isomorphism, then (α, β) is *isoclinism* and we write $V \sim W$.

The proof of the following lemmas are straightforward, so we refer the reader to [4] for obtaining more information.

Lemma 1 *If (V, φ) is an n-Hom-Lie algebra and (W, ψ) is an abelian n-Hom-Lie algebra, then $V \sim V \oplus W$.*

Lemma 2 *Let N be an ideal of n-Hom-Lie algebra (V, φ). Then we have*

(i) $N \cap V^2 = 0$ implies $V \sim V/N$.
(ii) if (V, φ) is of finite dimension and $V \sim V/N$, then $N \cap V^2 = 0$.

Lemma 3 *If (α, β) is the isoclinism pair between two n-Hom-Lie algebras (V, φ) and (W, ψ), then*

(i) $\alpha(a + Z_\varphi(V)) = \beta(a) + Z_\psi(W)$.

(ii) $\beta([a, v_2, \ldots, v_n]) = [\beta(a), w_1, \ldots, w_n]$, for all $a \in V^2$, $v_i \in V$, and

$$w_i \in \alpha(v_i + Z_\varphi(V)), \quad 2 \leq i \leq n.$$

In 1994, Moneyhun defined the notion of stem Lie algebra, [16]. Now, we define *Hom-stem* n-Hom-Lie algebras which some results in the next section are given based on this concept. An n-Hom-Lie algebra $(V, [-, \cdots, -], \varphi)$ is called *Hom-stem* if $Z_\varphi(V) \subseteq V^2$.
The existance of a Hom-stem n-Hom-Lie algebra in each isoclinism family of n-Hom-Lie algebras, is stated in the following lemma which can be proved easily.

Lemma 4 *Let $\mathcal{V}$ be an isoclinism family of n-Hom-Lie algebras. Then*
(i) $\mathcal{V}$ contains a Hom-stem n-Hom-Lie algebra.
(ii) any finite-dimensional n-Hom-Lie algebra (V, φ) in $\mathcal{V}$ is Hom-stem if and only if (V, φ) has a minimal dimension in $\mathcal{V}$.

The following proposition shows that the Hom-centers of two isoclinic Hom-stem n-Hom-Lie algebras are isomorphic.

Proposition 1 *If (V, φ) and (W, ψ) are two isoclinic Hom-stem n-Hom-Lie algebras, then $Z_\varphi(V) \cong Z_\psi(W)$.*

Proof Let (α, β) be an isoclinism pair between (V, φ) and (W, ψ). Let $v \in Z_\varphi(V)$ be arbitrary. Since $Z_\varphi(V) \subseteq V^2$, by using Lemma 3 (i), we have

$$\alpha(v + Z_\varphi(V)) = \beta(v) + Z_\psi(W),$$

which implies $\beta(v) \in Z_\psi(W)$ and thus $\beta(Z_\varphi(V)) \subseteq Z_\psi(W)$.
On the other hand, for $z \in Z_\psi(W)$, since β is onto, there exists $x \in V^2$ such that $\beta(x) = z$. By Lemma 3 (i), we can write

$$\alpha(x + Z_\varphi(V)) = \beta(x) + Z_\psi(W) = z + Z_\psi(W) = 0.$$

Now, since α is an isomorphism we conclude $x \in Z_\varphi(V)$ and so

$$z = \beta(x) \in \beta(Z_\varphi(V)), \quad \text{or} \quad Z_\psi(W) \subseteq \beta(Z_\varphi(V)).$$

Hence $Z_\psi(W) = \beta(Z_\varphi(V))$ and consequently $Z_\varphi(V) \cong Z_\psi(W)$.

3 Factor sets in n-Hom-Lie algebras

In studying n-Hom-Lie algebras, the concept of factor sets is a basic tool. In 1994, the factor sets in Lie algebras are defined by Moneyhun, [16]. In this section, we introduce them for n-Hom-Lie algebras and investigate some of their properties.

Definition 7 Let (V, φ) be a finite-dimensional n-Hom-Lie algebra. The n-linear map

$$r : \frac{V}{Z_\varphi(V)} \oplus \cdots \oplus \frac{V}{Z_\varphi(V)} \longrightarrow Z_\varphi(V)$$

is said to be a *factor set* when

(i) $[\overline{v}_1, \ldots, \overline{v}_i, \ldots, \overline{v}_j, \ldots, \overline{v}_n] = 0$, for all $\overline{v}_k = v_k + Z_\varphi(V) \in V/Z_\varphi(V)$ with $\overline{v}_i = \overline{v}_j$,

(ii) $r([\overline{v}_1, \ldots, \overline{v}_n], \widetilde{\varphi}\,(\overline{w}_2), \ldots, \widetilde{\varphi}\,(\overline{w}_n)) = \sum\limits_{i=1}^{n} r(\widetilde{\varphi}\,(\overline{v}_1), \ldots, [\overline{v}_i, \overline{w}_2, \ldots, \overline{w}_n], \ldots, \widetilde{\varphi}$ $(\overline{v}_n))$,

for all $\overline{v}_i, \overline{w}_j \in V/Z_\varphi(V), 1 \leq i \leq n$, $2 \leq j \leq n$, where

$$\widetilde{\varphi}\colon V/Z_\varphi(V) \longrightarrow V/Z_\varphi(V),$$

defined by $\widetilde{\varphi}\,(\overline{v}) := \varphi(v) + Z_\varphi(V), \quad \forall\, \overline{v} \in V/Z_\varphi(V)$. The factor set r is said to be *multiplicative* if

$$r(\widetilde{\varphi}\,(\overline{v}_1), \ldots, \widetilde{\varphi}\,(\overline{v}_n)) = \varphi r(\overline{v}_1, \ldots, \overline{v}_n), \quad \forall\, \overline{v}_i \in V/Z_\varphi(V), \quad (1 \leq i \leq n).$$

Lemma 5 *Let (V, φ) be an n-Hom-Lie algebra and r be a factor set on (V, φ). Define*

$$R = (Z_\varphi(V), \tfrac{V}{Z_\varphi(V)}, r) = \Big\{(a, \overline{v}) : a \in Z_\varphi(V), \overline{v} \in \tfrac{V}{Z_\varphi(V)}\Big\}.$$

Then

(i) (R, ψ) *is an* n*-Hom-Lie algebra with an* n*-linear map defined by*

$$[(a_1, \overline{v}_1), \ldots, (a_n, \overline{v}_n)] := (r(\overline{v}_1, \ldots, \overline{v}_n), [\overline{v}_1, \ldots, \overline{v}_n]), \quad (1)$$

for all $(a_1, \overline{v}_1), \ldots, (a_n, \overline{v}_n) \in R$ *and the linear operator* $\psi : R \longrightarrow R$ *is given by*

$$\psi((a, \overline{v})) := (\varphi(a), \widetilde{\varphi}\,(\overline{v})), \quad \forall (a, \overline{v}) \in R. \quad (2)$$

(ii) $Z_R := \{(a, 0) \in R : a \in Z_\varphi(V)\} \cong Z_\varphi(V)$.

Proof (i) We need to check only the properties of being n-Hom-Lie algebra. Clearly, the first identity holds. To check the Hom-Jacobi identity, we have

$$\Big[[(a_1, \overline{v}_1), \ldots, (a_n, \overline{v}_n)], \psi(b_2, \overline{w}_2), \ldots, \psi(b_n, \overline{w}_n)\Big] \overset{(1),(2)}{=}$$

$$\Big[(r(\overline{v}_1, \ldots, \overline{v}_n), [\overline{v}_1, \ldots, \overline{v}_n]), (\varphi(b_2), \widetilde{\varphi}\,(\overline{w}_2)), \ldots, (\varphi(b_n), \widetilde{\varphi}\,(\overline{w}_n))\Big] \overset{(1)}{=}$$

$$\Big(r([\overline{v}_1, \ldots, \overline{v}_n], \widetilde{\varphi}\,(\overline{w}_2), \ldots, \widetilde{\varphi}\,(\overline{w}_n)), [[\overline{v}_1, \ldots, \overline{v}_n], \widetilde{\varphi}\,(\overline{w}_2), \ldots, \widetilde{\varphi}\,(\overline{w}_n)]\Big) =$$

$$\Big(\sum_{i=1}^{n} r(\widetilde{\varphi}\,(\overline{v}_1), \ldots, \widetilde{\varphi}\,(\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \ldots, \overline{w}_n], \widetilde{\varphi}\,(\overline{v}_{i+1}), \ldots, \widetilde{\varphi}\,(\overline{v}_n)),$$

$$\sum_{i=1}^{n} [\widetilde{\varphi}\,(\overline{v}_1), \ldots, \widetilde{\varphi}\,(\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \ldots, \overline{w}_n], \widetilde{\varphi}\,(\overline{v}_{i+1}), \ldots, \widetilde{\varphi}\,(\overline{v}_n)]\Big) =$$

$$\sum_{i=1}^{n} \Big(r(\widetilde{\varphi}\,(\overline{v}_1), \ldots, \widetilde{\varphi}\,(\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \ldots, \overline{w}_n], \widetilde{\varphi}\,(\overline{v}_{i+1}), \ldots, \widetilde{\varphi}\,(\overline{v}_n)),$$

$$[\widetilde{\varphi}\,(\overline{v}_1), \ldots, \widetilde{\varphi}\,(\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \ldots, \overline{w}_n], \widetilde{\varphi}\,(\overline{v}_{i+1}), \ldots, \widetilde{\varphi}\,(\overline{v}_n)]\Big) \overset{(1)}{=}$$

$$\sum_{i=1}^{n} \Big[(\varphi(a_1), \widetilde{\varphi}\,(\overline{v}_1)), \ldots, (\varphi(a_{i-1}), \widetilde{\varphi}\,(\overline{v}_{i-1})), [(a_i, \overline{v}_i), (b_2, \overline{w}_2), \ldots, (b_n, \overline{w}_n)],$$

$$\ldots, (\varphi(a_{i+1}), \widetilde{\varphi}\,(\overline{v}_{i+1})), \ldots, (\varphi(a_n), \widetilde{\varphi}\,(\overline{v}_n))\Big] \overset{(2)}{=}$$

$$\sum_{i=1}^{n} \Big[\psi(a_1, \overline{v}_1), \ldots, \psi(a_{i-1}, \overline{v}_{i-1}), [(a_i, \overline{v}_i), (b_2, \overline{w}_2), \ldots, (b_n, \overline{w}_n)],$$

$$\ldots, \psi(a_{i+1}, \overline{v}_{i+1}), \ldots, \psi(a_n, \overline{v}_n)\Big],$$

for all $(a_i, \overline{v}_i), (b_i, \overline{w}_i) \in R$. Thus (R, ψ) is an n-Hom-Lie algebra.
The proof of (ii) is obvious.

Definition 8 A linear operator $\varphi : V \longrightarrow V$ on a vector space V is *semisimple* if every φ-invarient subspace has a complementary φ-invarient subspace.

From now, we suppose that each n-Hom-Lie algebra is equipped with a semisimple linear operator. In special case there exists a complement ideal for the ideal $Z_\varphi(V)$.
The following lemma proves the existence of the factor set for a given n-Hom-Lie algebra and gives the connection between them.

Lemma 6 *For an n-Hom-Lie algebra (V, φ), there exists a factor set r such that*

$$V \cong (Z_\varphi(V), \frac{V}{Z_\varphi(V)}, r).$$

Proof Let K be a complement of $Z_\varphi(V)$ in V, i.e. $V = K \oplus Z_\varphi(V)$. Now, we define the map $\theta : V/Z_\varphi(V) \longrightarrow V$ such that

$$\theta(\overline{v}) = \theta(v + Z_\varphi(V)) = \theta(k + a + Z_\varphi(V)) = k,$$

when $v \in V, a \in Z_\varphi(V), k \in K$. Clearly, $\overline{\theta(\overline{v})} = \overline{v}$ and so

$$[\theta(\overline{v}_1), \dots, \theta(\overline{v}_n)] - \theta[\overline{v}_1, \dots, \overline{v}_n] \in Z_\varphi(V), \tag{1}$$

for all $\overline{v}_1, \dots, \overline{v}_n \in V/Z_\varphi(V)$. Now, define

$$r : \frac{V}{Z_\varphi(V)} \oplus \cdots \oplus \frac{V}{Z_\varphi(V)} \longrightarrow Z_\varphi(V)$$

given by

$$r(\overline{v}_1, \dots, \overline{v}_n) := [\theta(\overline{v}_1), \dots, \theta(\overline{v}_n)] - \theta[\overline{v}_1, \dots, \overline{v}_n].$$

First, we have $\theta \widetilde{\varphi} = \varphi\theta$, because

$$\theta \widetilde{\varphi} (\overline{v}) = \theta \widetilde{\varphi} (k+a+Z_\varphi(V)) = \theta(\varphi(k)+\varphi(a)+Z_\varphi(V)) = \theta(\varphi(k)+Z_\varphi(V)) = \varphi(k),$$

and

$$\varphi\theta(\overline{v}) = \varphi(\theta(k + a + Z_\varphi(V))) = \varphi(k),$$

for all $\overline{v} = \overline{k + a} \in V/Z_\varphi(V)$, where $k \in K, a \in Z_\varphi(V)$.

To show that r is a factor set, we only need to check the second condition in definition 7. Suppose that $\overline{v}_i, \overline{w}_j \in V/Z_\varphi(V), 1 \leq i \leq n$ and $2 \leq j \leq n$. By (1), the elemnet $z \in Z_\varphi(V)$ exists such that

$$\theta[\overline{v}_1, \dots, \overline{v}_n] = [\theta(\overline{v}_1), \dots, \theta(\overline{v}_n)] + z,$$

and one can write

$$r([\overline{v}_1, \dots, \overline{v}_n], \widetilde{\varphi} (\overline{w}_2), \dots, \widetilde{\varphi} (\overline{w}_n)) =$$

$$[\theta[\overline{v}_1, \dots, \overline{v}_n], \theta(\widetilde{\varphi} (\overline{w}_2)), \dots, \theta(\widetilde{\varphi} (\overline{w}_n))] - \theta[[\overline{v}_1, \dots, \overline{v}_n], \widetilde{\varphi} (\overline{w}_2), \dots, \widetilde{\varphi} (\overline{w}_n)] =$$

$$[[\theta(\overline{v}_1), \dots, \theta(\overline{v}_n)] + z, \varphi\theta(\overline{w}_2), \dots, \varphi\theta(\overline{w}_n)] - \theta[[\overline{v}_1, \dots, \overline{v}_n], \widetilde{\varphi} (\overline{w}_2), \dots, \widetilde{\varphi} (\overline{w}_n)]$$

$$= \sum_{i=1}^{n} [\varphi\theta(\overline{v}_1), \dots, \varphi\theta(\overline{v}_{i-1}), [\theta(\overline{v}_i), \theta(\overline{w}_2), \dots, \theta(\overline{w}_n)], \varphi\theta(\overline{v}_{i+1}), \dots, \varphi\theta(\overline{v}_n)]$$

$$- \theta(\sum_{i=1}^{n} [\widetilde{\varphi} (\overline{v}_1), \dots, \widetilde{\varphi} (\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \dots, \overline{w}_n], \widetilde{\varphi} (\overline{v}_{i+1}), \dots, \widetilde{\varphi} (\overline{v}_n)]) =$$

$$\sum_{i=1}^{n} \Big([\theta \widetilde{\varphi} (\overline{v}_1), \dots, \theta \widetilde{\varphi} (\overline{v}_{i-1}), \theta([\overline{v}_i, \overline{w}_2, \dots, \overline{w}_n]), \theta \widetilde{\varphi} (\overline{v}_{i+1}), \dots, \theta \widetilde{\varphi} (\overline{v}_n)]$$

$$- \theta([\widetilde{\varphi} (\overline{v}_1), \dots, \widetilde{\varphi} (\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \dots, \overline{w}_n], \widetilde{\varphi} (\overline{v}_{i+1}), \dots, \widetilde{\varphi} (\overline{v}_n)]) \Big) =$$

$$\sum_{i=1}^{n} r(\widetilde{\varphi} (\overline{v}_1), \dots, \widetilde{\varphi} (\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \dots, \overline{w}_n], \widetilde{\varphi} (\overline{v}_{i+1}), \dots, \widetilde{\varphi} (\overline{v}_n)).$$

Now, we define $T : (Z_\varphi(V), V/Z_\varphi(V), r) \longrightarrow V$ such that $T(a, \overline{v}) = a + \theta(\overline{v})$, for all $a \in Z_\varphi(V), \overline{v} = v + Z_\varphi(V) \in V/Z_\varphi(V), k \in K$. T is well-defined and it is injective, because if $T(a_1, \overline{v}_1) = T(a_2, \overline{v}_2)$, in which $\theta(v_1) = k_1$ and $\theta(v_2) = k_2$, then $a_1 + k_1 = a_2 + k_2$ and $a_1 - a_2 = k_2 - k_1 \in Z_\varphi(V) \cap K = 0$ implies $(a_1, \overline{v}_1) = (a_2, \overline{v}_2)$. Also, T is n-Hom-Lie algebra morphism as

$$\begin{aligned} T[(a_1, \overline{v}_1), \ldots, (a_n, \overline{v}_n)] &= T(r(\overline{v}_1, \ldots, \overline{v}_n), [\overline{v}_1, \ldots, \overline{v}_n]) \\ &= r(\overline{v}_1, \ldots, \overline{v}_n) + \theta([\overline{v}_1, \ldots, \overline{v}_n]) \\ &= [\theta(\overline{v}_1), \ldots, \theta(\overline{v}_1)] \\ &= [a_1 + \theta(\overline{v}_1), \ldots, a_n + \theta(\overline{v}_n)] \\ &= [T(a_1, \overline{v}_1), \ldots, T(a_n, \overline{v}_n)], \end{aligned}$$

for $(a_i, \overline{v}_i) \in R, (1 \le i \le n)$. Also, the following diagram commutes

$$\begin{array}{ccc} Z_\varphi(V) \oplus V/Z_\varphi(V) & \xrightarrow{T} & V \\ \psi \big\downarrow & & \big\downarrow \varphi \\ Z_\varphi(V) \oplus V/Z_\varphi(V) & \xrightarrow{T} & V \end{array}$$

since

$$\begin{aligned} \varphi T(a, \overline{v}) &= \varphi(a + \theta(\overline{v})) = \varphi(a + k), \\ T\psi(a, \overline{v}) &= T(\varphi(a), \widetilde{\varphi}\,(\overline{v})) = \varphi(a) + \theta(\varphi(v) + Z_\varphi(V)) \\ &= \varphi(a) + \varphi(k) = \varphi(a + k), \end{aligned}$$

for all $a \in Z_\varphi(V), \overline{v} = v + Z_\varphi(V) \in V/Z_\varphi(V), k \in K$.

The next lemma gives the connection between two isoclinic Hom-stem n-Hom-Lie algebras.

Lemma 7 *Let (V, φ_1) be a Hom-stem n-Hom-Lie algebra in an isoclinism family of n-Hom-Lie algebras $\mathfrak{C}$. Then for any Hom-stem n-Hom-Lie algebra (W, φ_2) of $\mathfrak{C}$, there exists a factor set r over (V, φ_1) such that*

$$W \cong (Z_{\varphi_1}(V), V/Z_{\varphi_1}(V), r).$$

Proof Let (α, β) be an isoclinism pair of n-Hom-Lie algebras (V, φ) and (W, ψ). Proposition 1 states $\beta(Z_{\varphi_1}(V)) = Z_{\varphi_2}(W)$. By Lemma 6, there exists a factor set s such that $W \cong (Z_{\varphi_2}(W), W/Z_{\varphi_2}(W), s)$. Now, we define the following factor set

$$\begin{aligned} r: \ V/Z_{\varphi_1}(V) \oplus \cdots \oplus V/Z_{\varphi_1}(V) &\longrightarrow Z_{\varphi_1}(V) \\ (\overline{v}_1, \ldots, \overline{v}_n) &\longmapsto \beta^{-1}(s(\alpha(\overline{v}_1), \ldots, \alpha(\overline{v}_n))), \end{aligned}$$

for all $\overline{v}_i \in V/Z_{\varphi_1}(V), 1 \leq i \leq n$.
Now, by noting that $\alpha \widetilde{\varphi}_1 = \widetilde{\varphi}_2 \alpha$, we show that r is a factor set by the following way

$$
\begin{aligned}
& r\Big([\overline{v}_1, \ldots, \overline{v}_n], \widetilde{\varphi}_1 (\overline{w}_2), \ldots, \widetilde{\varphi}_1 (\overline{w}_n)\Big) \\
& = \beta^{-1}\Big(s\big(\alpha[\overline{v}_1, \ldots, \overline{v}_n], \alpha(\widetilde{\varphi}_1 (\overline{w}_2)), \ldots, \alpha(\widetilde{\varphi}_1 (\overline{w}_n))\big)\Big) \\
& = \beta^{-1}\Big(s([\alpha(\overline{v}_1), \ldots, \alpha(\overline{v}_n)], \widetilde{\varphi}_2 \alpha(\overline{w}_2), \ldots, \widetilde{\varphi}_2 \alpha(\overline{w}_n))\Big) \\
& = \beta^{-1}\Big(\sum_{i=1}^{n} s(\widetilde{\varphi}_2 \alpha(\overline{v}_1), \ldots, \widetilde{\varphi}_2 \alpha(\overline{v}_{i-1}), [\alpha(\overline{v}_i), \alpha(\overline{w}_2), \ldots, \alpha(\overline{w}_n)], \\
& \qquad \widetilde{\varphi}_2 \alpha(\overline{v}_{i+1}), \ldots, \widetilde{\varphi}_2 \alpha(\overline{v}_n))\Big) \\
& = \sum_{i=1}^{n} \beta^{-1} s(\alpha \widetilde{\varphi}_1 (\overline{v}_1), \ldots, \alpha \widetilde{\varphi}_1 (\overline{v}_{i-1}), \alpha[\overline{v}_i, \overline{w}_2, \ldots, \overline{w}_n], \alpha \widetilde{\varphi}_1 (\overline{v}_{i+1}), \ldots, \alpha \widetilde{\varphi}_1 (\overline{v_n})) \\
& = \sum_{i=1}^{n} r(\widetilde{\varphi}_1 (\overline{v}_1), \ldots, \widetilde{\varphi}_1 (\overline{v}_{i-1}), [\overline{v}_i, \overline{w}_2, \ldots, \overline{w}_n], \widetilde{\varphi}_1 (\overline{v}_{i+1}), \ldots, \widetilde{\varphi}_1 (\overline{v}_n)),
\end{aligned}
$$

for all $\overline{v}_i, \overline{w}_j \in V/Z_{\varphi}(V), 1 \leq i \leq n$ and $2 \leq j \leq n$. Put

$$R = (Z_{\varphi_1}(V), V/Z_{\varphi_1}(V), r),$$

and

$$S = (Z_{\varphi_2}(W), W/Z_{\varphi_2}(W), s).$$

By Lemma 5, (R, ψ_1) and (S, ψ_2) are n-Hom-Lie algebras.
We define $\eta : R \longrightarrow S$ given by $\eta(a, \overline{v}) = (\beta(a), \alpha(\overline{v}))$. Clearly, η is a well-defined bijection and also,

$$
\begin{aligned}
\eta[(a_1, \overline{v}_1), \ldots, (a_n, \overline{v}_n)] &= \eta(r(\overline{v}_1, \ldots, \overline{v}_n), [\overline{v}_1, \ldots, \overline{v}_n]) \\
&= \Big(\beta(r(\overline{v}_1, \ldots, \overline{v}_n)), \alpha([\overline{v}_1, \ldots, \overline{v}_n])\Big) \\
&= \Big(s(\alpha(\overline{v}_1), \ldots, \alpha(\overline{v}_n)), [\alpha(\overline{v}_1), \ldots, \alpha(\overline{v}_n)]\Big) \\
&= [(\beta(a_1), \alpha(\overline{v}_1)), \ldots, (\beta(a_n), \alpha(\overline{v}_n))] \\
&= [\eta(a_1, \overline{v}_1), \ldots, \eta(a_n, \overline{v}_n)],
\end{aligned}
$$

for all $a \in Z_{\varphi}(V), \overline{v} \in V/Z_{\varphi}(V)$. Also, the following diagram is commutative

$$
\begin{array}{ccc}
R & \xrightarrow{\ \eta\ } & S \\
{\scriptstyle \psi_1}\big\downarrow & & \big\downarrow{\scriptstyle \psi_2} \\
R & \xrightarrow{\ \eta\ } & S
\end{array}
$$

because $\beta\varphi_1 = \varphi_2\beta$ and $\alpha \widetilde{\varphi}_1 = \widetilde{\varphi}_2 \alpha$ implies

$$\begin{aligned}
&\eta\psi_1(a,\overline{v}) = \eta(\varphi_1(a), \\
&\widetilde{\varphi}_1 (\overline{v})) = (\beta\varphi_1(a), \alpha \widetilde{\varphi}_1 (\overline{v})), \\
&\psi_2\eta(a,\overline{v}) = \psi_2(\beta(a), \alpha(\overline{v})) = (\varphi_2\beta(a), \widetilde{\varphi}_2\, \alpha(\overline{v})),
\end{aligned}$$

for all $a \in Z_\varphi(V), \overline{v} \in V/Z_\varphi(V)$. So η is our desired isomorphism and $R \cong S$.

Lemma 8 *Let (V,φ) be an n-Hom-Lie algebra, r and s be two multiplicative factor sets over (V,φ). Assume that*

$$R = (Z_\varphi(V), \frac{V}{Z_\varphi(V)}, r), \quad Z_R = \{(a,0) \in R : a \in Z_\varphi(V)\}$$

and

$$S = (Z_\varphi(V), \frac{V}{Z_\varphi(V)}, s), \quad Z_S = \{(a,0) \in S : a \in Z_\varphi(V)\}.$$

If η is an isomorphism from R to S satisfying $\eta(Z_R) = Z_S$, then the restrictions of η on $V/Z_\varphi(V)$ and $Z_\varphi(V)$ define the automorphisms $\mu \in Aut(V/Z_\varphi(V))$ and $\nu \in Aut(Z_\varphi(V))$, respectively.

Proof By Lemma 5, (R,ψ) and (S,ψ) are n-Hom-Lie algebras, so we have n-Hom-Lie algebras R/Z_R and S/Z_S and since η is isomorphism and $\eta(Z_R) = Z_S$, thus η induces $\overline{\eta} : (R/Z_R, \psi_1) \longrightarrow (S/Z_S, \psi_2)$ by $(a,\overline{v}) + Z_R \longmapsto \eta(a,\overline{v}) + Z_S$ is an isomorphism in which $\psi_1 : R/Z_R \longrightarrow R/Z_R$ and $\psi_2 : S/Z_S \longrightarrow S/Z_S$ are linear maps defined by $\psi_1((a,\overline{v}) + Z_R) = \psi(a,\overline{v}) + Z_R$ and $\psi_2((a,\overline{v}) + Z_S) = \psi(a,\overline{v}) + Z_S$, for $(a,\overline{v}) \in R/Z_R$, respectively. Consider σ_1 and σ_2 as two projection maps in the following diagram given by $\sigma_1(\overline{v}) = (0,\overline{v}) + Z_R$ and $\sigma_2(\overline{v}) = (0,\overline{v}) + Z_S$, for $\overline{v} \in V/Z_\varphi(V)$. Now, we define μ such that the following diagram commutes.

$$\begin{array}{ccc}
V/Z_\varphi(V) & \xrightarrow{\ \mu\ } & V/Z_\varphi(V) \\
{\scriptstyle\sigma_1}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_2} \\
R/Z(R) & \xrightarrow{\ \bar{\eta}\ } & S/Z(S)
\end{array}$$

where $\eta(0,\overline{v}) + Z_S = (0, \mu(\overline{v})) + Z_S$, for all $\overline{v} \in V/Z_\varphi(V)$. We prove $\mu \widetilde{\varphi} = \widetilde{\varphi} \mu$; For each $v \in V$, $\overline{v} \in V/Z_\varphi(V)$, one can write

$$(0, \mu \widetilde{\varphi} (\overline{v})) + Z_S = \eta(0, \widetilde{\varphi} (\overline{v})) + Z_S = \eta\psi(0,\overline{v}) + Z_S.$$

On the other hand, if $\eta(0,\overline{v}) - (0,\mu(\overline{v})) = t$, for some $t \in Z_S$, then $\psi(t) \in Z_S$ and so

$$\begin{aligned}
(0, \widetilde{\varphi}\, \mu(\overline{v})) + Z_S = \psi(0,\mu(\overline{v})) + Z_S &= \psi(\eta(0,\overline{v}) + t) + Z_S \\
&= \psi\eta(0,\overline{v}) + \psi(v) + Z_S = \psi\eta(0,\overline{v}) + Z_S,
\end{aligned}$$

Since $\eta\psi = \psi\eta$, we have $(0, \mu \widetilde{\varphi} (\overline{v})) + Z_S = (0, \widetilde{\varphi} \mu(\overline{v})) + Z_S$. By the definition, $\sigma_2(\mu \widetilde{\varphi} (\overline{v})) = \sigma_2(\widetilde{\varphi} \mu(\overline{v}))$, and surjectivity of σ_2 implies $\mu \widetilde{\varphi} (\overline{v}) = \widetilde{\varphi} \mu(\overline{v})$. Also,

$$\begin{aligned}
(0, \mu([\overline{v}_1, \ldots, \overline{v}_n])) + Z_S &= \eta(0, [\overline{v}_1, \ldots, \overline{v}_n]) + Z_S \\
&= \eta([(0, \overline{v}_1), \ldots, (0, \overline{v}_n)]) + Z_S \\
&= [\eta(0, \overline{v}_1), \ldots, \eta(0, \overline{v}_n)] + Z_S \\
&= [\eta(0, \overline{v}_1) + Z_S, \ldots, \eta(0, \overline{v}_n) + Z_S] \\
&= [(0, \mu(\overline{v}_1)) + Z_S, \ldots, (0, \mu(\overline{v}_n)) + Z_S] \\
&= [(0, \mu(\overline{v}_1)), \ldots, (0, \mu(\overline{v}_n))] + Z_S \\
&= (0, [\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)]),
\end{aligned}$$

for $\overline{v}_i \in V/Z_\varphi(V), 1 \leq i \leq n$. Hence $\mu([\overline{v}_1, \ldots, \overline{v}_n]) = [\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)]$ and μ is an automorphism i.e. $\mu \in Aut(V/Z_\varphi(V))$. Now define ν such that the following diagram is commutative

$$\begin{array}{ccc}
Z_\varphi(V) & \xrightarrow{\ \nu\ } & Z_\varphi(V) \\
{\scriptstyle \bar{\sigma}_1}\downarrow & & \downarrow{\scriptstyle \bar{\sigma}_2} \\
Z_R & \xrightarrow{\ \tilde{\mu}\ } & Z_S
\end{array}$$

where $\overline{\sigma_1}$ and $\overline{\sigma_2}$ are projection maps and $\eta(a, 0) = (\nu(a), 0)$, for all $a \in Z_\varphi(V)$. Similarly, one can easily check that ν is automorphism.

Lemma 9 *Let (V, φ) be an n-Hom-Lie algebra and $(R, \psi), (S, \psi), Z_R$ and Z_S be as in Lemma 8.*

(i) Consider $\eta : R \longrightarrow S$ is a Hom-Lie algebra isomorphism such that $\eta(Z_R) = Z_S$. Let $\mu \in Aut(V/Z_\varphi(V))$ and $\nu \in Aut(Z_\varphi(V))$ be the automorphisms induced by η. Then there exists a linear map $\gamma : V/Z_\varphi(V) \longrightarrow Z_\varphi(V)$ such that

$$\nu(r(\overline{v}_1, \ldots, \overline{v}_n)) + \gamma[\overline{v}_1, \ldots, \overline{v}_n] = s(\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)).$$

(ii) If $\mu \in Aut(V/Z_\varphi(V))$ and $\nu \in Aut(Z_\varphi(V))$ and $\delta : V/Z_\varphi(V) \longrightarrow Z_\varphi(V)$ is a linear map such that

$$\nu(r(\overline{v}_1, \ldots, \overline{v}_n)) + \delta[\overline{v}_1, \ldots, \overline{v}_n] = s(\mu(\overline{v}_1), \ldots, \mu(\overline{v}_n)), \quad \delta \widetilde{\varphi} = \varphi\delta,$$

then there exists an isomorphism $\eta : R \longrightarrow S$ which is induced by μ and ν satsfying $\eta(Z_R) = Z_S$.

Proof (i) For all $a \in Z_\varphi(V)$ and $\overline{v} \in V/Z_\varphi(V)$ we have $\eta(a, 0) = (\nu(a), 0)$ and $\eta(0, \overline{v}) + Z_S = (0, \mu(\overline{v})) + Z_S$. Hence

$$\eta(0, \overline{v}) - (0, \mu(\overline{v})) \in Z_S \Rightarrow \eta(0, \overline{v}) - (0, \mu(\overline{v})) = (a_{\overline{v}}, 0),$$

for some $a_{\overline{v}} \in Z_\varphi(V)$. Now, define the map $\gamma : V/Z_\varphi(V) \longrightarrow Z_\varphi(V)$ such that $\gamma(\overline{v}) = a_{\overline{v}}$, for all $\overline{v} = v + Z_\varphi(V) \in V/Z_\varphi(V)$. It is a well-defined linear map and we have

$$\begin{aligned}\eta(a,\overline{v}) &= \eta(a,0) + \eta(0,\overline{v})\\ &= (\nu(a),0) + (0,\mu(\overline{v})) + (\gamma(\overline{v}),0)\\ &= (\nu(a)+\gamma(\overline{v}),\mu(\overline{v})).\end{aligned}$$

Hence,

$$\begin{aligned}\eta[(0,\overline{v}_1),\ldots,(0,\overline{v}_n)] &= [\eta(0,\overline{v}_1),\ldots,\eta(0,\overline{v}_n)]\\ &= [(\gamma(\overline{v}_1),\mu(\overline{v}_1)),\ldots,(\gamma(\overline{v}_n),\mu(\overline{v}_n))]\\ &= (s(\mu(\overline{v}_1),\ldots,\mu(\overline{v}_n)),[\mu(\overline{v}_1),\ldots,\mu(\overline{v}_n)]).\end{aligned}$$

On the other hand

$$\begin{aligned}\eta[(0,\overline{v}_1),\ldots,(0,\overline{v}_n)] &= \eta(r(\overline{v}_1,\ldots,\overline{v}_n),[\overline{v}_1,\ldots,\overline{v}_n])\\ &= \nu(r(\overline{v}_1,\ldots,\overline{v}_n)) + \gamma[\overline{v}_1,\ldots,\overline{v}_n],\end{aligned}$$

so

$$\nu(r(\overline{v}_1,\ldots,\overline{v}_n)) + \gamma[\overline{v}_1,\ldots,\overline{v}_n] = s(\mu(\overline{v}_1),\ldots,\mu(\overline{v}_n)).$$

(ii) We only check that the following diagram commutes, in which $\eta : R \longrightarrow S$ is defined by $\eta(a,\overline{v}) = (\nu(a)+\delta(\overline{v}),\mu(\overline{v}))$,

$$\begin{array}{ccc} R & \xrightarrow{\ \eta\ } & S\\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi}\\ R & \xrightarrow{\ \eta\ } & S\end{array}$$

$$\begin{aligned}\eta\psi(a,\overline{v}) &= \eta(\varphi(a),\widetilde{\varphi}(\overline{v})) = (\nu\varphi(a) + \delta\,\widetilde{\varphi}(\overline{v}),\mu\,\widetilde{\varphi}(\overline{v})),\\ \psi\eta(a,\overline{v}) &= \psi(\nu(a)+\gamma(\overline{v}),\mu(\overline{v})) = (\varphi\nu(a) + \varphi\delta(\overline{v}),\widetilde{\varphi}\,\mu(\overline{v})).\end{aligned}$$

Since μ and ν are isomorphisms such that $\delta\,\widetilde{\varphi} = \varphi\delta$, $\varphi\nu = \nu\varphi$ and $\mu\,\widetilde{\varphi} = \widetilde{\varphi}\,\mu$, one concludes $\eta\psi = \psi\eta$.

The following theorem plays a major role which leads us to deduce the main theorems of this section.

Theorem 1 *Let (V,φ_1) and (W,φ_2) be two finite-dimensional Hom-stem n-Hom-Lie algebras and φ_1 be onto. Then $V \sim W$ if and only if $V \cong W$.*

Proof Suppose that $V \sim W$. By Lemmas 6 and 7,

$$V \cong (Z_{\varphi_1}(V), V/Z_{\varphi_1}(V), r) = R,$$

and also

$$W \cong (Z_{\varphi_2}(W), W/Z_{\varphi_2}(W), s) = S.$$

Now, let (α,β) be isoclinism pair between the n-Hom-Lie algebras (R,ψ_1) and (S,ψ_2). Certainly $Z_R=Z(R)$ and $Z_S=Z(S)$. Let the map $\mu\in Aut(V/Z_{\varphi_1}(V))$ is defined by

$$\alpha((0,\overline{v})+Z_R)=(0,\mu(\overline{v}))+Z_S,$$

for all $\overline{v}\in V/Z_{\varphi_1}(V)$.
Also, $\nu\in Aut(Z_{\varphi_1}(V))$ is the map defined by $\beta(a,0)=(\nu(a),0)$, for all $a\in Z_{\varphi_1}(V)$. Let us consider the following commutative diagram

$$\begin{array}{ccccc} V/Z_{\varphi_1}(V)\times\cdots\times V/Z_{\varphi_1}(V) & \xrightarrow{\rho} & R/Z_R\times\cdots\times R/Z_R & \xrightarrow{\theta} & R^2 \\ \mu^n\big\downarrow & & \big\downarrow\alpha^n & & \big\downarrow\beta \\ V/Z_{\varphi_1}(V)\times\cdots\times V/Z_{\varphi_1}(V) & \xrightarrow{\sigma} & S/Z_S\times\cdots\times S/Z_S & \xrightarrow{\xi} & S^2 \end{array}$$

in which

$$\begin{aligned}
&\rho(\overline{v}_1,\dots,\overline{v}_n)=((0,\overline{v}_1)+Z_R,\dots,(0,\overline{v}_n)+Z_R)),\\
&\sigma(\overline{v}_1,\dots,\overline{v}_n)=((0,\overline{v}_1)+Z_S,\dots,(0,\overline{v}_n)+Z_S)),\\
&\xi((a_1,\overline{v}_1)+Z_S,\dots,(a_n,\overline{v}_n)+Z_S)=[(a_1,\overline{v}_1),\dots,(a_{n+1},\overline{v}_n)]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad=(s(\overline{v}_1,\dots,\overline{v}_n),[\overline{v}_1,\dots,\overline{v}_n]),\\
&\theta((a_1,\overline{v}_1)+Z_R,\dots,(a_n,\overline{v}_n)+Z_R)=[(a_1,\overline{v}_1),\dots,(a_n,\overline{v}_n)]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad=(r(\overline{v}_1,\dots,\overline{v}_n),[\overline{v}_1,\dots,\overline{v}_n]).
\end{aligned}$$

We have

$$\begin{aligned}
\beta\theta((0,\overline{v}_1)+Z_R,\dots,(0,\overline{v}_n)+Z_R))&=\beta(r(\overline{v}_1,\dots,\overline{v}_n),[\overline{v}_1,\dots,\overline{v}_n])\\
&=\beta[(0,\overline{v}_1),\dots,(0,\overline{v}_n)],
\end{aligned}$$

and further

$$\begin{aligned}
\xi\alpha^n((0,\overline{v}_1)+Z_R,\dots,(0,\overline{v}_n)+Z_R))&=\xi((0,\mu(\overline{v}_1))+Z_S,\dots,(0,\mu(\overline{v}_n))+Z_S))\\
&=[(0,\mu(\overline{v}_1)),\dots,(0,\mu(\overline{v}_n))]\\
&=(s(\mu(\overline{v}_1)),\dots,\mu(\overline{v}_n)),[\mu(\overline{v}_1)),\dots,\mu(\overline{v}_n]).
\end{aligned}$$

Hence we have $\beta[(0,\overline{v}_1),\dots,(0,\overline{v}_n)]=(s(\mu(\overline{v}_1)),\dots,\mu(\overline{v}_n)),[\mu(\overline{v}_1)),\dots,\mu(\overline{v}_n])$.
The map $\delta:(V/Z_{\varphi_1}(V))^2\longrightarrow Z_{\varphi_1}(V)$ such that

$$\beta(0,[\overline{v}_1,\dots,\overline{v}_n])=(\delta([\overline{v}_1,\dots,\overline{v}_n]),t),$$

where $t\in V/Z_{\varphi_1}(V)$ is considered. Thus we get

$$\nu(r(\overline{v}_1,\dots,\overline{v}_n))+\delta[\overline{v}_1,\dots,\overline{v}_n]=s(\mu(\overline{v}_1),\dots,\mu(\overline{v}_n)).$$

To apply Lemma 9, we may extend δ to $V/Z_{\varphi_1}(V)$ by assuming that it vanishes on the complement of $(V/Z_{\varphi_1}(V))^2$ in $V/Z_{\varphi_1}(V)$. Now, we need only to show $\delta\,\widetilde{\varphi_1}=\varphi_1\delta$. For all $\overline{v}_1,\dots,\overline{v}_n,t\in V/Z_{\varphi_1}(V)$

$$\delta(\widetilde{\varphi_1}\,[\overline{v}_1,\dots,\overline{v}_n],t)=\beta(0,\widetilde{\varphi_1}\,([\overline{v}_1,\dots,\overline{v}_n])=\beta\psi_1(0,[\overline{v}_1,\dots,\overline{v}_n]).$$

Further $\widetilde{\varphi_1}$ is onto, i.e. $\widetilde{\varphi_1}\ (t') = t$ for some $t' \in V/Z_{\varphi_1}(V)$. So

$$\begin{aligned}(\varphi_1\delta[\overline{v}_1,\ldots,\overline{v}_n],t) &= (\varphi_1\delta[\overline{v}_1,\ldots,\overline{v}_n],\widetilde{\varphi_1}\ (t'))\\ &= \psi_1(\delta[\overline{v}_1,\ldots,\overline{v}_n],t')\\ &= \psi_1\beta(0,[\overline{v}_1,\ldots,\overline{v}_n]).\end{aligned}$$

Hence $\varphi_1\delta(\overline{v}) = \delta\ \widetilde{\varphi_1}\ (\overline{v})$, for all $\overline{v} \in V^2$. Consider $V = V^2 \oplus U$ and define δ to be zero in U. Then $\varphi_1\delta(\overline{u}) = 0$, for all $u \in U$. Since φ_1 is semisimple, $\varphi_1(u) \in U$, thus

$$\delta\ \widetilde{\varphi_1}\ (\overline{u}) = \delta(\varphi_1(u) + Z_{\varphi_1}(V)) = 0.$$

Consequently, $\varphi_1\delta = \delta\ \widetilde{\varphi_1}$ and now we can use Lemma 9 to obtain the result.

Theorem 2 *Let $\mathfrak{C}$ be an isoclinism family of finite-dimensional regular n-Hom-Lie algebras. Then any $V \in \mathfrak{C}$ can be expressed as $V = T \oplus A$, where T is a Hom-stem n-Hom-Lie algebra and A is some finite-dimensional abelian n-Hom-Lie algebra.*

Theorem 3 *Let (V,φ_1) and (W,φ_2) be two n-Hom-Lie algebras with same dimension. Then $V \sim W$ if and only if $V \cong W$.*

The following example shows that the above theorem does not valid for two different dimension n-Hom-Lie algebras.

Example 2 Let (V,φ) be an $(n+1)$-dimensional n-Hom-Lie algebra over a field F defined by

$$[e_2,\ldots,e_{n+1}] = e_1, \quad [e_1,e_3,\ldots,e_{n+1}] = e_2,$$

where $\{e_1,\ldots,e_{n+1}\}$ is a basis for V and all other commutator relations are zero. The linear map φ is defined as follows

$$\varphi(e_1) = e_1, \quad \varphi(e_{2i}) = e_{2i+1}, \quad \varphi(e_{2i+1}) = e_{2i}, \quad 1 \le i \le n.$$

Then $V^2 = \langle e_1, e_2\rangle$ and $Z_\varphi(V) = 0$ and hence $V/Z_\varphi(V) \cong V$.
Now, let (W,ψ) be an $(n+2)$-dimensional n-Hom-Lie algebra with the basis $\{e_1,\ldots,e_{n+2}\}$ and the commutator relations are defined by

$$[e_2,\ldots,e_{n+1}] = e_1, \quad [e_1,e_3,\ldots,e_{n+1}] = e_2,$$

and all other commutator relations are zero. Also, the linear map is given by

$$\psi(e_1) = e_1, \quad \psi(e_{n+2}) = e_{n+2}, \quad \psi(e_{2i}) = e_{2i+1}, \quad \psi(e_{2i+1}) = e_{2i},$$

for $1 \le i \le n/2$. Then $W^2 = \langle e_1, e_2\rangle$ and $Z_\psi(W) = \langle e_{n+2}\rangle$ and so

$$W/Z_\psi(W) = \langle \overline{e}_1,\ldots,\overline{e}_{n+1}\rangle,$$

where $\overline{e}_i = e_i + Z_\psi(W)$. We conclude that $V^2 \cong W^2$ and $V/Z_\varphi(V) \cong W/Z_\psi(W)$ and hence $V \sim W$ while $dim(V) \neq dim(W)$.

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