Global Analysis and Discrete Mathematics
Volume 7, Issue 2, pp. 179–192
ISSN: 2476-5341 *Research Article*

A Solution for Sparse PDE Constrained Optimization by Partition of Unity and RBFs

Majid Darehmiraki · Arezou Rezazadeh

Received: 30 January 2023 / Accepted: 16 May 2023

Abstract In this paper, we propose a radial basis function partition of unity (RBF-PU) method to solve sparce optimal control problem governed by the elliptic equation. The objective function, in addition to the usual quadratic expressions, also includes an L_1-norm of the control function to compute its spatio sparsity. Meshless methods based on RBF approximation are widely used for solving PDE problems but PDE-constrained optimization problems have been barely solved by RBF methods. RBF methods have the benefits of being versatile in terms of geometry, simple to use in higher dimensions, and also having the ability to give spectral convergence. In spite of these advantages, when globally RBF collocation methods are used, the interpolation matrix becomes dens and computational costs grow with increasing size of data set. Thus, for overcome on these problemes RBF-PU method will be proposed. RBF-PU methods reduce the computational effort. The aim of this paper is to solve the first-order optimality conditions related to original problem.

Keywords Sparse · Optimal control · Radial basis functions · Partition of unity

Mathematics Subject Classification (2010) 49J20

1 Introduction

In areas like robotics, sports movement patterns, the control of chemical reactions, and power plants, the optimal control of ordinary differential equations

M. Darehmiraki (Corresponding Author)
Department of Mathematics, Behbahan Khatam Alanbia University of Technology, Khouzestan, Iran.
E-mail: darehmiraki@bkatu.ac.ir

A. Rezazadeh
Department of mathematics, University of Qom, Qom, Iran.
E-mail: rezazadeh_arezo@yahoo.com

is crucial. Partial differential equations must be used to describe the processes to be optimized in many situations where regular differential equations are no longer sufficient to characterize them. A crucial role is played by the optimal control problem (OCP), which is governed by partial differential equations, in many branches of science and engineering. Partial differential equations can be used to simulate a variety of physical phenomena, such as heat conduction, diffusion, electromagnetic waves, fluid flows, and freezing processes [8–10, 26].

Due to the widespread use of optimization problems with PDE constraints in various sciences and industries such as oil and gas industry, they are interesting for many researchers in various fields and many efforts have been made to provide efficient and useful solutions for them. When in the objective function of this category of optimization problems there is an expression with L_1 norm, they transform to sparse optimization problems. They were first examined, as far as we know, by Stedler in 2007 (Newton-typed algorithms were proposed for them), and then various methods have been proposed to solve them, some of which are mentioned below. Porcelli et al. [25] used a general semismooth Newton's algorithm and a preconditioner for solving them. Langer et al. [18] applied unstructured space-time finite element method for optimal sparse control of semilinear parabolic equations. The authors of [33] solve dual of main problem instead of itself using a majorized accelerated block coordinate descent. Recently, the author of [29] have provided an an adaptive finite element method for the sparse optimal control of fractional diffusion. Pearson et al. proposed an efficient method based on the interior method [24]. For an optimal control problem that includes a nondifferentiable cost functional, the Poisson problem as the state equation, and control constraints, the authors of [1] presented and examined trustworthy and effective a posteriori error estimators. While three different approaches are used to approximate the control variable: piecewise constant discretization, piecewise linear discretization, and the so-called variational discretization approach. The unstructured space-time finite element method was used by Langer et al. [18] for the best sparse management of semilinear parabolic equations. Instead of solving the dual of the primary problem themselves, the authors of [33] use a majorized accelerated block coordinate descent. A recent adaptive finite element approach for the sparse optimal control of fractional diffusion has been presented by the author of [29]. Based on the interior method, Pearson et al. suggested an effective method; [24].

In this paper we will consider the distributed convection-diffusion control problem as follows:

$$\min_{y,w} \quad \frac{1}{2}\|y-\widehat{y}\|^2_{L_2} + \frac{\beta}{2}\|w\|^2_{L_2} + \gamma\|w\|_{L_1} \tag{1}$$

$$Ay = w + g \quad in\, \Omega, \tag{2}$$

$$y = 0 \quad on\, \partial\Omega, \tag{3}$$

$$w \in W_{ad} = \{h(x) | a \leq w \leq b,\, a.e.\, on\, \Omega\} \tag{4}$$

where A denotes the Laplacian operator, $g \in L_2(\Omega)$, the domain $\Omega \subset \mathbb{R}^d, d = 2$ or 3. y denotes the state variable and $\widehat{y}$ is desired state, w denotes the control variable, and $\beta > 0$ is a regularization parameter.
Optimality conditions for this problem is as follows (for more details see [34]):

$$Ay - w - g = 0,\ A^*p + y - \hat{y} = 0,\ -p + \beta w + \mu = 0, \tag{5}$$

$$\begin{aligned} &w - max(0, w + c(\mu - \gamma)) - min(0, w + c(\mu + \gamma)) + \\ &max(0, (w - b) + c(\mu - \gamma)) + min(0, (w - a) + c(\mu + \gamma)) = 0 \end{aligned} \tag{6}$$

where $c > 0$ and A^*, p and μ are duals A, y and w respectively. Stadler in [34] was able to rewrite the above conditions by performing some calculations and introducing an operator:

$$\begin{aligned} &w - \beta^{-1} max(0, \Upsilon w - \gamma) - \beta^{-1} min(0, \Upsilon w + \gamma) + \\ &\beta^{-1} max(0, \Upsilon w - \gamma - \beta b) - \beta^{-1} min(0, \Upsilon w + \gamma - \beta a) = 0 \end{aligned} \tag{7}$$

where $\Upsilon w = Sw + h$, $S = -A^{-*}A^{-1}$ and $h = -A^{-*}(A^{-1}g - \hat{y})$.

The following two methods are available for solving PDE-constrained optimization problems: The first option is to discretize first, then optimize, which entails creating a discrete cost functional and then deriving discrete optimality requirements from it. In the continuous situation, one can also derive optimality conditions, which they can then discretize. The optimize-then-discretize strategy is what it is called.
These two approaches overlap and result in the same discrete equation for many PDE situations, especially those that are self-adjoint. There will generally be a difference between the optimize-then-discretize and discretize-then-optimize techniques because the convection diffusion equation is not self-adjoint.

In [5,15], we have investigated the streamline upwind Petrov-Galerkin (SUPG) stabilized finite element approach for the convection-diffusion control problem. The adjoint-consistent Local Projection Stabilization method is used by Pearson and colleagues in [23] and also is discussed in [2–4]. In this study, the RBF-PU approach is the main topic.

Radial Basis Function (RBF) methods are basic tools for interpolating scattered data spicially in higher dimensional domains. This method was first introduced by Kansa [17] for solving partial differential equations (PDEs) in 1990. In [27], the authors applied RBF method to find the optimal control of a parabolic distributed parameter system with a quadratic cost function. In [22] Pearson used RBF collocation methods to the problem of Poisson control.

Basic tools for interpolating scattered data, particularly in higher dimensional domains, include Radial Basis Function (RBF) algorithms. In 1990, Kansa [17] invented this technique for solving partial differential equations (PDEs). To identify the best control for a parabolic distributed parameter system with a quadratic cost function, the authors of [27] used the RBF approach. Pearson applied RBF collocation techniques to the Poisson control problem in [22].

Radial basis functions have gained a lot of popularity in the last twenty years as a method for solving partial differential equations. Despite the general simplicity of RBF approaches in tackling many issues, global methods, regrettably, have a disadvantage in that, as the size of the problem increases, the computational cost of solving dense linear systems increases. Good attempts have been made to localize RBF collocation algorithms in order to address these drawbacks. One of the useful techniques in this regard is the partition unity method, which has been used in a number of articles. Since around 1960, the Partition of Unity (PU) approach has been utilized for interpolation [32]; more recently, the PU method has been integrated with RBFs [36, 30]. Through the combination of the Partition of Unity method and rational Radial Basis Function (RBF) interpolants, Marchi et al. suggested a localized approach [11]. The authors of [6] used an adaptive refinement approach to address Poisson problems using a collocation scheme based on the radial basis function partition of unity (RBF-PU). For the initial-boundary value problem, Garmanjani et al. [13] used the RBF partition of unity approach (RBF-PUM) based on a finite difference (FD) scheme. To price American and European options under the Lévy model, Fereshtian et al. [12] developed the RBF-PU approach for spatial discretization of partial integro-differential equations. Using the RBF division of unity local approach, the ellipitic interface difficulties were resolved in [14]. A RBF partition of unity method employing a direct discretization approach for PDEs was just recently proposed by Mirzaee [21].

RBF-PUM is a local mesh-free approach that divides the original domain into numerous overlapping subdomains or patches that cover it. In the RBF-PU technique, each overlapping patch receives a local RBF-approximation that is then combined with compactly supported PU weight functions to generate the global approximation. A comparison between global RBF and RBF-PUM has been made for the option pricing issues in [31]. The RBF-PUM has been shown to be the most effective solution for these problems.

This study develops and applies the RBF-PU approach to the sparse PDE constrained optimization problem. The remainder of this paper is structured as follows: We provide a brief summary of RBF approaches in Section 2. Deal with the RBF-PU method's statement next, which is that it can solve constrained sparse PDE optimization problems. Finally, to demonstrate the precision and effectiveness of the suggested strategy, some numerical results are presented in Section 4.

2 Radial basis function (RBF) method

In this section, we introduced RBF methods for interpolating of scattered data.

Definition 1 A function $\Phi : \mathbb{R}^s \to \mathbb{R}$, s is the dimension of the interpolation space, is called radial basis function provided there exists a univariate function $\varphi : [0, \infty) \to \mathbb{R}$ such that

$$\Phi(x) = \varphi(r), \qquad where \qquad r = \|x\|,$$

Table 1 Som well-known RBFs

Name of functions	Definition
cubic	r^3
Quintic	r^5
Gaussian(GA)	$exp(-\varepsilon^2 r^2)$
Inverse quadrics (IQ)	$1/(r^2+\varepsilon^2)$
Wendland functions,where ρ is a polynomial	$(1-r)^m \rho(r)$
Multiquadratic (MQ)	$\sqrt{1+(cr)^2}$
Shifted logarithm	$\log(r^2+c^2)$
Thin plate Splines(TPS)	$(-1)^{k+1} r^{2k} \log(r)$

and $\|.\|$ is the Euclidean norm on $\mathbb{R}^s$.
Some well-known radial basis functions are listed in Tabel 1.

Assume that $x_1, \cdots, x_N$ are a given collection of dispersed nodes in $\Omega \subseteq \mathbb{R}^S$. The RBF approximation for $U(x)$ is denoted by $U^N(x)$ and has the following form: In Table 2 ϵ, c, k and m are shape parameter, constant coefficient, the number of control points and degree of polynomial repectively.

$$U^N(x) = \sum_{j=1}^{N} \lambda_j \varphi(\|x - x_j\|) = \Phi^T(x)\lambda, \qquad x \in \Omega \tag{8}$$

where $\{\lambda_j\}_{j=1}^N$ are the unknown coefficient to be determined, $\|.\|$ is Eucludian norm and $\phi(\|x - x_j\|)$ can be any radial basis function, and

$$\varphi(x) = [\varphi(\|x - x_1\|), \varphi(\|x - x_2\|), \cdots, \varphi(\|x - x_N\|)]^T.$$

By applying the interpolation criteria, the cofficient $\{\lambda_j\}_{j=1}^N$ is discovered.

$$U^N(x_j) = U(x_j), \qquad j = 1, \cdots, N. \tag{9}$$

and we obtain a linear system as follow:

$$A\lambda = \mathbf{U} \tag{10}$$

where

$$\lambda = \begin{bmatrix}\lambda_1 \ \lambda_2 \cdots \ \lambda_N\end{bmatrix}^T, U = \begin{bmatrix}U(x_1) \ U(x_2) \cdots \ U(x_N)\end{bmatrix}^T \mathit{and} \tag{11}$$

$$A = \begin{bmatrix} \varphi(\|x_1 - x_1\|) & \varphi(\|x_1 - x_2\|) & \cdots & \varphi(\|x_1 - x_N\|) \\ \varphi(\|x_2 - x_1\|) & \varphi(\|x_2 - x_2\|) & \cdots & \varphi(\|x_2 - x_N\|) \\ \vdots & \vdots & & \vdots \\ \varphi(\|x_N - x_1\|) & \varphi(\|x_N - x_2\|) & \cdots & \varphi(\|x_N - x_N\|) \end{bmatrix} \tag{12}$$

from (8) and (10) we can write

$$U^N(x) = \phi^T(x) A^{-1} U = D(x) U \tag{13}$$

where $D(x) = \phi^T(x) A^{-1}$.

3 RBF partition of unity method

Let $\{\Omega_i\}_{i=1}^M$ be an open covering of an open set Ω i.e., $\Omega \subseteq \cup_{i=1}^M \Omega_i$. The RBF partition of unity method is a local method based on subdividing the domain Ω on M subdomains $\Omega_1, \ldots, \Omega_M$ called patches.
Now, we define a partition of unity $\{\omega_i\}_{i=1}^M$ subordinated to the covering $\{\Omega_i\}_{i=1}^M$ such that

$$\sum_{i=1}^{M} \omega_i(x) = 1, \qquad x \in \Omega, \tag{14}$$

where the weight function $\omega_i : \Omega_i \to \mathbb{R}$ is compactly supported, nonnegative and continuous with $supp(\omega_i \subseteq \Omega_i)$.
We create a local RBF interpolant of the type $\gamma_u^i : \Omega_i \to \mathbb{R}$ for each subdomain.

$$\gamma_u^i = \sum_{j=1}^{N_i} \lambda_j^i \phi(\|x - x_j^i\|), \tag{15}$$

In here, N_i is the number of collocation points in Ω_i. So, over the entire domain Ω, the global RBF-PUM interpolant is defined as

$$\gamma_u(x) = \sum_{i=1}^{M} \omega_i(x)\gamma_u^i = \sum_{i=1}^{M} \omega_i(x) \sum_{j=1}^{N_i} \lambda_j^i \phi(\|x - x_j^i\|), \qquad x \in \Omega. \tag{16}$$

The partition of unity functions ω_i can be constructed using shepard method [32] given by

$$\omega_i(x) = \frac{\phi_i(x)}{\sum_{k=1}^M \phi_k(x)}, \qquad i = 1, \cdots, M, \tag{17}$$

where $\phi_i(x)$ is compactly supported function with support on Ω_i. We have used the following compactly supported the wendland's C^2 function [35]:

$$\phi(r) = \begin{cases} (1-r)^4(4r+1), & 0 \le r \le 1 \\ 0, & r > 1. \end{cases} \tag{18}$$

The elements of the open cover of will be chosen as circular patches. Thus, the Wendland functions will be scaled to get

$$\phi_i(x) = \phi(\frac{\|x - c_i\|}{r_i}), \qquad i = 1, \cdots, M, \tag{19}$$

where r_i and c_i are, respectively, the centers and the radial of patches $\Omega_i, i = 1, \cdots, M$.
The global interpolant is approximated using the following formulat:

$$y(x) = \sum_{i=1}^{M} \omega_i(x)\gamma_y^i, \tag{20}$$

$$\gamma_y^i = A_i \bar{\lambda}^i \longrightarrow \bar{\lambda}^i = A_i^{-1}\gamma_y^i, \tag{21}$$

γ_y^i is a local interpolant on Ω_i that defined above. By enforcing the interpolation condition, for global interpolant we have the following form [31]

$$\bar{y}(x) = \sum_{i=1}^{M} R_i W_i A_i \bar{\lambda}^i, \tag{22}$$

where W_i is a diagonal matrix with element $\omega_i(x_j)$ on it and A_i is the local RBF matrix.
The approximation of the first and second dervative can be obtained respectively:

$$\frac{\partial \bar{y}}{\partial x} = \sum_{i=1}^{M} R_i[(W_i)_x A_i + W_i(A_i)_x]\bar{\lambda}^i \tag{23}$$

$$= \sum_{i=1}^{M} R_i[(W_i)_x A_i + W_i(A_i)_x]A_i^{-1}\gamma_y^i \tag{24}$$

$$= \sum_{i=1}^{M} R_i(D_i)_x \gamma_y^i \tag{25}$$

we construct approximation for second derivative as follow:

$$\frac{\partial^2 \bar{y}}{\partial x^2} = \sum_{i=1}^{M} R_i[(W_i)_{xx} A_i + 2(W_i)_x(A_i)_x + W_i(A_i)_{xx}]\bar{\lambda}^i \tag{26}$$

$$= \sum_{i=1}^{M} R_i[(W_i)_{xx} A_i + 2(W_i)_x(A_i)_x + W_i(A_i)_{xx}]A_i^{-1}\gamma_y^i \tag{27}$$

$$= \sum_{i=1}^{M} R_i(D_i)_{xx} \gamma_y^i \tag{28}$$

Now, in order to solve the Problem (1) by the partition unity of RBF method, we refer to its optimality conditions, which are summarized in Equation (7). According to the Equation (7), it is sufficient to discrete only function w and its derivatives using Formulas (22), (23) and (26), and then place them in the Equation (7). In this case we reach a nonlinear equation that can be easily solved using Newton's method.

4 Numerical Example

In this section, two examples are provided to clarify and exemplify our theoretical findings as well as the capabilities of the suggested methodology. We take into account the domain $\Omega = [0,1] \times [0,1]$ for the examples that were provided, and we also give their precise state and exact adjoint. Assume that $A = -\Delta$ as well. We select $\nu = 10^{-8}$ and $\gamma = 10^{-6}$ for each of the two examples.

Example 1 The data for this example are as follows: $a = -30$, $b = 30$, $\hat{y} = sin(2\pi x)sin(2\pi y)exp(2x)/6$, $g = 0$ and $\beta = 10^{-5}$.
Figure 1 contains numerical solutions of state, control, adjoint and μ functions with $\gamma = 10^{-4}$ in Example 1. Control functions with various γ values are compared in Figure 2.

Example 2 In this example, the same data as in Example 1 without the control function boundaries ($a = -\infty$, $b = \infty$) are considered.
Plots of numerical solution of state, control, adjoint and μ functions with $\gamma = 10^{-4}$ are depicted in Figure 3 for the case where control is not bounded. Finally, Figure 4 shows plot of all the functions in the control mode without bound and $\gamma = 0$. Table 2 shows $\|y - \hat{y}\|_2$ due to various values of N.

Table 2 Compare $\|y - \hat{y}\|_2$ for different values of N in Example 1.

N	11	21	31
$\|y - \hat{y}\|_2$	0.0380	0.0148	0.0133

5 Conclusion

RBFs are a very powerful tool for solving differential equations numerically, and when they are combined with the unit differentiation method, their approximation power is increased. For linear elliptic optimal sparse control problems, we have taken into consideration a partition of unity approach in conjunction with an RBF method in our study. To the best of our knowledge, this is the first time that radial basis functions are used to solve a sparse optimal control problem. The objective functional includes the conventional L_2-regularization term in addition to the well-known L_1 -norm of the control. Our numerical studies have shown that the suggested technique can capture spatio-temporal sparsity. Future research will conduct a thorough convergence and error analysis of our partition unity of RBF approach for such optimal sparse control problems.

References

1. A. Allendes, F. Fuica, E. Otarola, Adaptive finite element methods for sparse PDE-constrained optimization. IMA Journal of Numerical Analysis, 40, 2106–2142 (2020).
2. R. Becker, M. Braack, A finite element pressure gradient stabilization for the Stokes equations based on local projections. Calcolo, 38, 173–199 (2001).
3. R. Becker, R., B. Vexler, Optimal control of the convection-diffusion equation using stabilized finite element methods. Numerische Mathematik, 106, 349–367 (2007).
4. M. Braack, E. Burman, Local projection stabilization for the Oseen problem and its interpretation as a variational multiscale method. SIAM Journal on Numerical Analysis, 43, 2544–2566 (2006).

5. S. S. Collis, M. Heinkenschloss, Analysis of the streamline upwind/Petrov Galerkin method applied to the solution of optimal control problems. CAAM TR02-01, 108, (2002).
6. R. Cavoretto, A. De Rossi, Error indicators and refinement strategies for solving Poisson problems through a RBF partition of unity collocation scheme. Applied Mathematics and Computation, 369, 124824 (2020).
7. M. A. Darani, The RBF partition of unity method for solving the Klein-Gordon equation. Engineering with Computers, 1–13 (2020).
8. M. Darehmiraki, M. H. Farahi, S. Effati, A novel method to solve a class of distributed optimal control problems using Bezier curves. Journal of Computational and Nonlinear Dynamics, 11, (2016).
9. M. Darehmiraki, M. H. Farahi, S. Effati, Solution for fractional distributed optimal control problem by hybrid meshless method. Journal of Vibration and Control, 24, 2149–2164 (2018).
10. M. Darehmiraki, An efficient solution for stochastic fractional partial differential equations with additive noise by a meshless method. International Journal of Applied and Computational Mathematics, 4, 1–13 (2018).
11. S. De Marchi, A. Martínez, E. Perracchione, Fast and stable rational RBF-based partition of unity interpolation. Journal of Computational and Applied Mathematics, 349, 331–343 (2019).
12. A. Fereshtian, R. Mollapourasl, F. Avram, RBF approximation by partition of unity for valuation of options under exponential Lévy processes. Journal of computational science, 32, 44–55 (2019).
13. G. Garmanjani, R. Cavoretto, M. Esmaeilbeigi, A RBF partition of unity collocation method based on finite difference for initial–boundary value problems. Computers & Mathematics with Applications, 75, 4066–4090 (2018).
14. F. Gholampour, E. Hesameddini, A. Taleei, A stable RBF partition of unity local method for elliptic interface problems in two dimensions. Engineering Analysis with Boundary Elements, 123, 220–232 (2021).
15. M. Heinkenschloss, D. Leykekhman, Local error estimates for SUPG solutions of advection-dominated elliptic linear-quadratic optimal control problems. SIAM Journal on Numerical Analysis, 47, 4607–4638 (2010).
16. M. Hinze, N. Yan, Z. Zhou, Variational discretization for optimal control governed by convection dominated diffusion equations. Journal of Computational Mathematics, 237–253 (2009).
17. E. J. Kansa, Multiquadrics-A scattered data approximation scheme with applications to computational fluid-dynamics-I surface approximations and partial derivative estimates. Computers & Mathematics with applications, 19, 127–145 (1990).
18. U. Langer, O. Steinbach, F. Tröltzsch, H. Yang, Unstructured space-time finite element methods for optimal control of parabolic equations. arXiv preprint arXiv:2004.02014 (2020).
19. D. Leykekhman, M. Heinkenschloss, Local error analysis of discontinuous Galerkin methods for advection-dominated elliptic linear-quadratic optimal control problems. SIAM Journal on Numerical Analysis, 50, 2012–2038 (2012).
20. A. Martínez, C. Rodríguez, M. E. Vázquez-Méndez, Theoretical and numerical analysis of an optimal control problem related to wastewater treatment. SIAM Journal on Control and Optimization, 38, 1534–1553 (2000).
21. D. Mirzaei, The direct radial basis function partition of unity (D-RBF-PU) method for solving PDEs. SIAM Journal on Scientific Computing, 43, A54–A83 (2021).
22. J. W. Pearson, A radial basis function method for solving PDE-constrained optimization problems. Numerical Algorithms, 64, 481–506 (2013).
23. J. W. Pearson, A. J. Wathen, Fast iterative solvers for convection-diffusion control problems, (2011).
24. J. W. Pearson, M. Porcelli, M. Stoll, Interior-point methods and preconditioning for PDE-constrained optimization problems involving sparsity terms. Numerical Linear Algebra with Applications, 27, e2276 (2020).
25. M. Porcelli, V. Simoncini, M. Stoll, Preconditioning PDE-constrained optimization with L1-sparsity and control constraints. Computers & Mathematics with Applications, 74, 1059–1075 (2017).

26. A. Rezazadeh, M. Mahmoudi, M. Darehmiraki, Space–time spectral collocation method for one-dimensional PDE constrained optimisation. International Journal of Control, 93, 1231–1241 (2020).
27. J. A. Rad, S. Kazem, K. Parand, Optimal control of a parabolic distributed parameter system via radial basis functions. Communications in Nonlinear Science and Numerical Simulation, 19, 2559–2567 (2014).
28. T. Rees, Preconditioning iterative methods for PDE constrained optimization (Doctoral dissertation, University of Oxford) (2010).
29. E. Otárola, An adaptive finite element method for the sparse optimal control of fractional diffusion. Numerical Methods for Partial Differential Equations, 36, 302–328 (2020).
30. A. Safdari-Vaighani, A. Heryudono, E. Larsson, A radial basis function partition of unity collocation method for convection–diffusion equations arising in financial applications. Journal of Scientific Computing, 64, 341–367 (2015).
31. V. Shcherbakov, E. Larsson, Radial basis function partition of unity methods for pricing vanilla basket options. Computers & Mathematics with Applications, 71, 185–200 (2016).
32. D. Shepard, A two-dimensional interpolation function for irregularly-spaced data. In Proceedings of the 1968 23rd ACM national conference, 517–524.
33. X. Song, D. Sun, K. C. Toh, Mesh Independence of a Majorized ABCD Method for Sparse PDE-constrained Optimization Problems. arXiv preprint arXiv:2001.02118 (2020).
34. G. Stadler, Elliptic optimal control problems with L 1-control cost and applications for the placement of control devices. Computational Optimization and Applications, 44, 159 (2009).
35. H. Wendland, Sobolev-type error estimates for interpolation by radial basis functions. Surface fitting and multiresolution methods, 337–344 (1997).
36. H. Wendland, Fast evaluation of radial basis functions: Methods based on partition of unity. In Approximation theory X: wavelets, splines, and applications, (2002).
37. N. Yan, Z. Zhou, A Priori and A Posteriori Error Estimates of Streamline Diffusion Finite Element Method for Optimal Control Problem Governed by Convection Dominated Diffusion Equation. Numerical Mathematics: Theory, Methods & Applications, 1, (2008).
38. Z. Zhou, N. Yan, The local discontinuous Galerkin method for optimal control problem governed by convection diffusion equations. International Journal of Numerical Analysis & Modeling, 7, (2010).
39. J. Zhu, Q. Zeng, A mathematical formulation for optimal control of air pollution. Science in China Series D: Earth Sciences, 46, 994–1002 (2003).

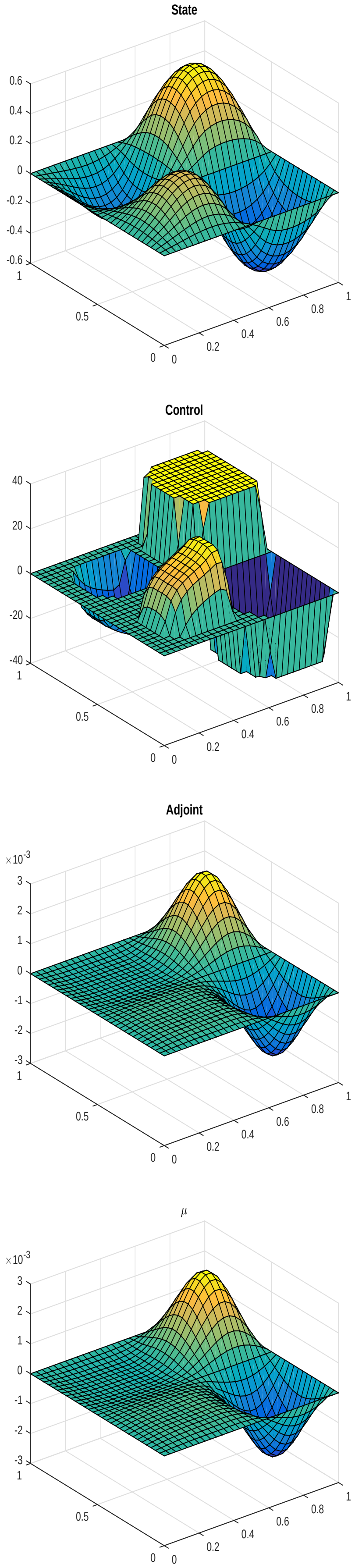

Fig. 1 Optimal control w, corresponding multiplier μ, Optimal state y and corresponding adjoint p in Example 1 with $\gamma = 10^{-4}$.

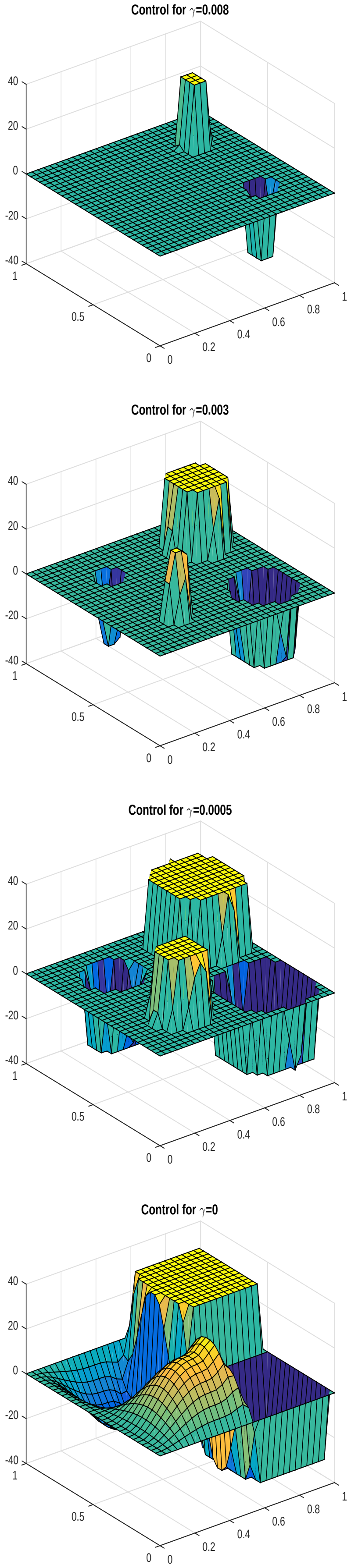

Fig. 2 Comparisons among optimal control w for $\gamma = 0.008$s, $\gamma = 0.003$s, $\gamma = 0.0005$, $\gamma = 0$ in Example 1.

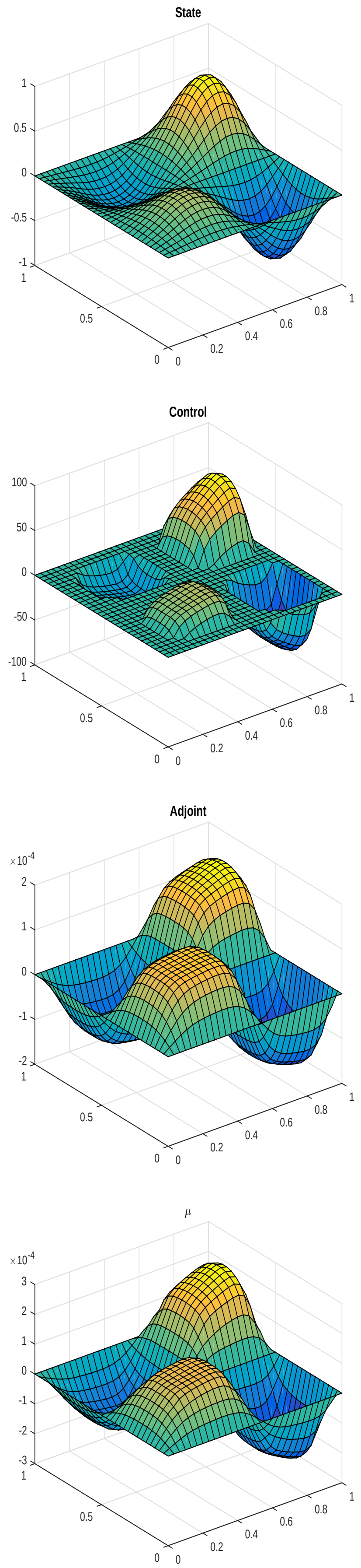

Fig. 3 Optimal control w, corresponding multiplier μ, Optimal state y and corresponding adjoint p in Example 2 with $\gamma = 10^{-4}$.

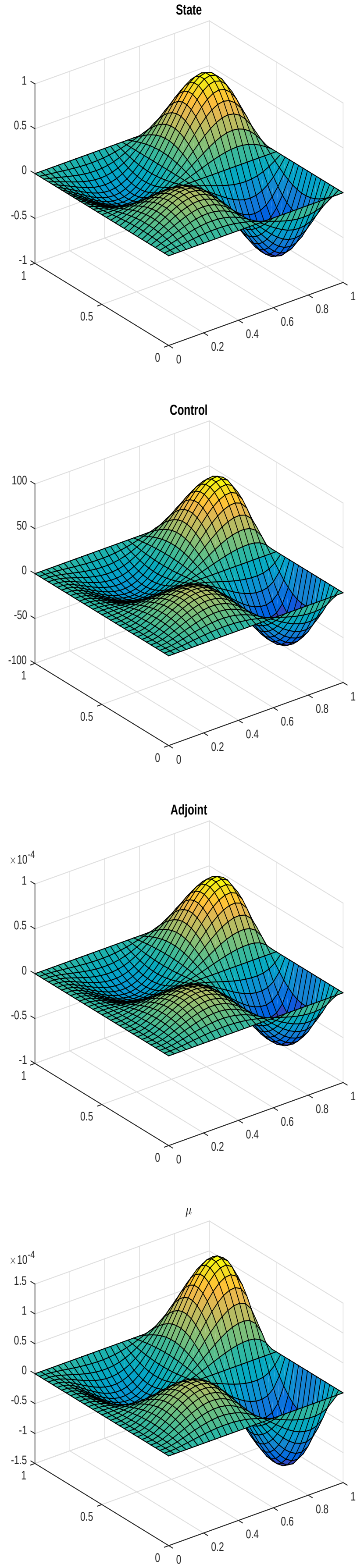

Fig. 4 Optimal control w, corresponding multiplier μ, Optimal state y and corresponding adjoint p in Example 2 with $\gamma = 0$.