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On Clique Mantel's Theorem

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Abstract A complete subgraph of any simple graph G on k vertices is called a *k-clique* of G. In this paper, we first introduce the concept of the value of a k-clique ($k > 1$) as an extension of the idea of the degree of a given vertex. Then, we obtain the generalized version of handshaking lemma which we call it clique handshaking lemma. The well-known classical result of Mantel states that the maximum number of edges in the class of triangle-free graphs with n vertices is equal to $\frac{n^2}{4}$. Our main goal here is to find an extension of the above result for the class of $K_{\omega+1}$-free graphs, using the ideas of the value of cliques and the clique handshaking lemma.

Keywords Maximum independent set · value of a clique · handshaking lemma · double-counting

Mathematics Subject Classification (2010) 05C30 · 05C35

1 Introduction

Finding the *maximum* values of some key *invariants* in discrete structures with *forbidden* (finite) family of substructures is an interesting problem in the area of *extremal combinatorics* with potential applications in theoretical and applied computer science. One of the classical problems of these kind is the well-known *Mantel's* theorem [1] which answers the question about the maximum number of edges in any simple graph in which the family of forbidden subgraphs consists of only the triangle graph K_3. There are many interesting proofs of the well-known Mantel's theorem and one of those beautiful proofs is

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based on the idea of *maximum* independent set of vertices. Roughly speaking, the basic idea is to partition the vertex set of a given graph $G = (V, E)$ into two sets A and B. The first set A is an independent set of maximum cardinality (maximum number of vertices) and B is the rest of vertices. Now, considering the maximality of A, the *triangle-freeness* of G and the well-known *handshaking* lemma, one can obtain an upper bound for the number of edges based on the sum of degrees of vertices lie in B. Finally combining all previous findings with the well-known *arithmetic-geometric mean* inequality, we get the classical Mantel's theorem.

It seems that the next step is to consider the set A of vertices of maximum cardinality for which the graph induced on A is *triangle-free*. Then, using a similar argument, one can get a generalization of Mantel's result for the class of K_4-free graphs which we call it *edge Mantel's theorem*. Next, we generalize the concept of the degree of a vertex to a higher $k-$clique $(k > 1)$ by introducing the idea of the *value of a clique*. This simply means that a value of a clique can be defined as the number of common neighbors of it's vertices. In this direction, we also obtain a higher clique generalization of the handshaking lemma which we call it *clique handshaking lemma*. Finally, using the same machinery introduced for proving the classical Mantel's theorem, we obtain the so called *clique Mantel's theorem*.

2 Basic Definitions and Notations

Throughout this paper, we will assume that our graphs are finite, simple and undirected. For terminologies which are not defined here, one can refer to the book [5].

For a give graph $G = (V, E)$, the vertex set and the edge set will be denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, it's *open neighborhood* denoted by $N_G(v)$ is the set of vertices *adjacent* to v. A subgraph of G consisting of all those vertices that are pairwise adjacent is called a *complete* subgraph (clique) of G. A complete subgraph with k vertices will be called a *k-clique*. The set of all k-cliques in G is denoted by $\Delta_k(G)$. We will also denote the number of k-cliques of a graph G by $c_k(G)$. A complete subgraph on three vertices is called a *triangle*. A subset of vertices with no edges among them is called an *independent* set of G.

A generalization of the concept of the *degree* of a vertex can be extended to the *value* of an edge, as follows.

Definition 1 For a given graph $G = (V, E)$ and an edge $e = \{u, v\} \in E(G)$, the value of e denoted by $val_G(e)$ is defined as the number of common neighbors of two end vertices u and v of the edge e. More precisely, we have

$$val_G(e) = \Big| N_G(u) \cap N_G(v) \Big|.$$

Remark 1 It is interesting to note that a definition similar to the value of an edge has been given in the literature (see for instance [3]). Indeed, the *co-degree* of two vertices $u, v \in V(G)$, not necessarily adjacent, is defined as the number of their common neighborhoods.

Example 1 Let $G = (V, E)$ be a graph depicted in Fig. 1. Then, the values of the edges of G are, as follows

$$val_G(e_{12}) = val_G(e_{13}) = val_G(e_{23}) = 1, \quad val_G(e_{34}) = 0. \tag{1}$$

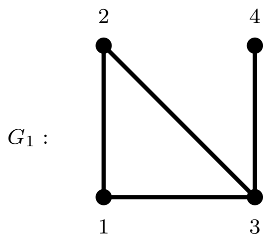

Fig. 1 The values of edges for the graph G_1

Next, we generalize the above idea for any k-clique $q_k \in \Delta_k(G)$ $(k > 1)$ of G.

Definition 2 (Value of a Clique) Let $G = (V, E)$ be a simple graph and q_k be a k-clique of G. Then, we define the *value* of the clique q_k denoted by $val_G(q_k)$, as follows

$$val_G(q_k) = \Big| \bigcap_{v \in V(q_k)} N_G(v) \Big|. \tag{2}$$

As an extension of the well-known *handshaking lemma*, we have the following key result.

Lemma 1 *[Clique Handskaing Lemma] For a simple graph $G = (V, E)$, we have*

$$\sum_{q_k \in \Delta_k(G)} val_G(q_k) = (k+1)c_{k+1}(G), \quad (k \geq 1). \tag{3}$$

One can give several proofs of the above lemma. Here, we present a proof which based on the idea of *double-counting* technique.

Proof Let $G = (V, E)$ be any simple graph. Define the set $I_k(G)$, as follows.

$$I_k(G) = \{(q_k, q_{k+1}) \in \Delta_k(G) \times \Delta_{k+1}(G) \mid q_k \textbf{ is a subgraph of } q_{k+1}\}. \tag{4}$$

The proof proceeds by counting the set $I_k(G)$ in two different ways.
Case I. We first fix the clique q_k. Then, it is clear that the number of those $(k+1)$-cliques containing q_k is exactly $val_G(q_k)$. Now, summing over all those k-cliques q_k will result in $\sum_{q_k \in \Delta_k(G)} val_G(q_k)$.
Case II. Next, we fix the $(k+1)$-clique q_{k+1}. Then, it is obvious that the number of such k-cliques which are the subgraph of q_{k+1} is equal to $k+1$. Thus, by summing over all $(k+1)$-cliques, we get $(k+1)c_{k+1}(G)$.
Finally, the proof is complete by the *double-counting* technique.

Remark 2 In is worthy to note that the above lemma is called the *transfer equations* by Knill in [6] which is even true for the generalized discrete structures like *simplicial complexes*. Indeed, the transfer equations are used to obtain a *graph-theoretical* version of the well-known *Gauss-Bonnet* formula in differential geometry.

We conclude this section by recalling the following well-known *arithmetic-geoemtric* mean ineqaulity.

Lemma 2 *For a sequence of* n *non-negative real numbers* $a_1, a_2, \ldots, a_n$, *we have*

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}, \tag{5}$$

and the equality holds iff $a_1 = a_2 = \cdots = a_n$.

3 Main Results

In this section, we will use the idea of the value of cliques to generalize the following clique-counting inequality due to Mantel [1].

Theorem 1 (Mantel's Theorem for Triangle-free Graphs) *For a given triangle-free graph* $G = (V, E)$ *with* n *vertices and* m *edges, we have*

$$m \leq \frac{n^2}{4}.$$

The motivation of this paper originates from the proof of the above classical result which is based on the idea of *maximality*. Thus, we also include the proof.

Proof Let $A \subseteq V(G)$ be an independent set of maximum cardinality (a maximum independent set). Next, we put $B = V(G) - A$. Since, G is triangle-free, the open neighborhood of any arbitrary vertex $v \in V(G)$ is an independent set. Hence, by the maximality of A, we immediately conclude that

$$\deg_G(v) = |N_G(v)| \leq |A|, \qquad \forall\ v \in V(G). \tag{6}$$

On the other hand, since A is an independent set of vertices, we obviously have

$$\sum_{v \in A} \deg_G(v) \leq m. \tag{7}$$

Considering the well-known handshaking lemma, we also get

$$\sum_{v \in A} \deg_G(v) + \sum_{v \in B} \deg_G(v) = 2m. \tag{8}$$

Form identities (7) and (8), we conclude that

$$m \leq \sum_{v \in B} \deg_G(v). \tag{9}$$

Thus, from relations (6), (9) and the *arithmetic-geometric mean* inequality (Lemma 2 for $n = 2$), we finally obtain

$$\begin{aligned} m &\leq \sum_{v \in B} \deg_G(v) \\ &\leq \sum_{v \in B} |A| \\ &= |A||B| \\ &\leq \Big(\frac{|A| + |B|}{2}\Big)^2 \\ &\leq \frac{n^2}{4}, \end{aligned}$$

as required.

Remark 3 It is important to note that from the arithmetic-geometric mean inequality in the above proof, it immediately follows that the *extremal graph* for Mantel's classical result is the balanced complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. But here, we are only interested in extremal bounds (inequalities) and not the extremal graphs themselves.

4 Edge Mantel's Theorem

The next result is a slight generalization of the Mantel's theorem and is based on the idea of the value of an edge and the generalized *handshaking* lemma (Lemma 1) in graphs.

Theorem 2 *[Edge Mantel's Theorem] For a given K_4 -free graph $G = (V, E)$ with n vertices and t triangles, we have*

$$t \leq \frac{n^3}{27}.$$

Proof Let G be a K_4-free graph. We also let A be a set of vertices of G with maximum cardinality in which $G[A]$ is a *triangle-free* graph.

Now, we note that for any K_4-free graph G, the open neighborhood of v, for each $v \in V(G)$, is a triangle-free graph. This immediately implies that

$$c_2(G[N_G(v)]) \leq \frac{|A|^2}{4}, \quad (\forall v \in V(G)). \tag{10}$$

On the other hand, we clearly have

$$\sum_{v\in A} c_2(G[N_G(v)]) \leq |T(G)| = t. \tag{11}$$

By using the *clique handsahking* lemma (Lemma 1 for $k = 3$)

$$\sum_{v\in V(G)} c_2(G[N_G(v)]) = \binom{2+1}{2} c_3(G) = 3|T(G)| = 3t. \tag{12}$$

From (11) and (12), we get

$$2t \leq \sum_{v\in B} c_2(G[N_G(v)]). \tag{13}$$

Thus, using arithmetic-geometric mean inequality for $n = 3$, we finally get

$$\begin{aligned} t &\leq \frac{1}{2}\sum_{v\in B} c_2(G[N_G(v)]) \\ &\leq \frac{1}{2}\sum_{v\in B} \frac{|A|^2}{4} \\ &\leq \frac{1}{8}|A|^2 \sum_{v\in B} 1 = \frac{1}{8}|A|^2|B|^1 \\ &\leq \frac{1}{8}\left(\frac{2|A|+|B|}{3}\right)^3 \\ &\leq \left(\frac{|A|+|B|}{3}\right)^3 \\ &\leq \left(\frac{n}{3}\right)^3. \end{aligned}$$

5 Clique Mantel's Theorem

Now, considering the idea of the value of a clique and the clique handshaking lemma and using similar arguments as above, we obtain the following generalization of Theorem 2.

Theorem 3 (Clique Mantel's Theorem) *Let G be a $K_{\omega+1}$-free graph with n vertices. Then, we have*

$$c_\omega(G) \leq \left(\frac{n}{\omega}\right)^\omega. \tag{14}$$

Proof We proceed by induction on the clique number $\omega = \omega(G) \geq 2$ of the graph G. The base case $\omega(G) = 2$ is ture (the classical Mantel's theorem). Let A be the set of vertices of G with the maximum cardinality in which $G[A]$ is a K_ω-free graph. We also put $B = V(G) - A$. Now, by induction hypothesis, we have

$$c_\omega(G[N_G(v)]) \le \left(\frac{|A|}{\omega-1}\right)^{\omega-1}. \tag{15}$$

On the other hand, by the maximality of A, we clearly get

$$\sum_{v\in A} c_\omega(G[N_G(v)]) \le |\Delta_\omega(G)| = c_\omega(G). \tag{16}$$

Now, considering the clique handshaking lemma for $k = \omega(G)$ and the inequality (16), we obtain

$$(\omega-1)c_\omega(G) \le \sum_{v\in B} c_\omega(G[N_G(v)]). \tag{17}$$

Thus, considering the inequality (15) and the well-known arithmetic-geometric mean inequality (Lemma 2), we finally get

$$\begin{aligned}
c_\omega(G) &\le \frac{1}{\omega-1}\sum_{v\in B} c_\omega(G[N_G(v)]) \\
&\le \frac{1}{\omega-1}\sum_{v\in B}\left(\frac{|A|}{\omega-1}\right)^{\omega-1} \\
&\le \frac{1}{(\omega-1)^\omega}|A|^{\omega-1}\sum_{v\in B} 1 = \frac{1}{(\omega-1)^\omega}|A|^{\omega-1}|B|^1 \\
&\le \frac{1}{(\omega-1)^\omega}\left(\frac{(\omega-1)|A|+|B|}{\omega}\right)^\omega \\
&\le \left(\frac{|A|+|B|}{\omega}\right)^\omega \\
&\le \left(\frac{n}{\omega}\right)^\omega.
\end{aligned}$$

6 Concluding Remarks and Future Works

In this paper, we obtain an upper bound for the number of k-clique in the class of $(k+1)$-cliques-free graphs; that is, the class of those graphs not containing any complete subgraph of on $(k+1)$ vertices. The basic ideas were maximality, clique handshaking identity and using the arithmetic-geometric mean inequality.

Our future project is to consider a more general class of graphs that we will call them $\mathcal{H}$-free graphs. We recall that the *increasing family* [4] of graphs $\mathcal{H}$ is the following

$$\mathcal{H} = \{H_1, H_2, \ldots, H_k, H_{k+1}, \ldots\},$$

in which $H_1 = K_1$ and each H_i is an *induced subgraph* of H_{i+1}, for all i. Our main *goal* is to find an upper bound similar to that of Theorem 3 for the maximum number of copies of H_k in the class of those graphs not containing any subgraph *isomorphic* to H_{k+1} (for any integer $k > 1$). To achieve this goal, we need two main steps. We have to first define a similar notion of the value of a clique for any graph H_k in $\mathcal{H}$. Then, we need to find an analogue of our key lemma; the clique handshaking lemma 1. We will call it $\mathcal{H}$-handshaking lemma. The following result, due to $Kelly$ [2], will play an essential role.

Proposition 1 *Let $G = (V, E)$ be an n-vertex graph with no* isolated *vertices. Then for any graph H on k vertices, we have*

$$(n-k)s(H,G) = \sum_{v \in V} s(H, G-v),$$

where $s(H, G)$ denotes the number of subgraphs of G isomorphic to H.

Note that in particular case where H_k is a k-clique, it is not hard to show that Proposition 1 is equivalent to our clique handshaking lemma.

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