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Numerical Integration of Symmetric Multivariate Function

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Abstract In this paper, we introduce a method for finding the integral of symmetric multivariate function. We compute the number of nodes which the method use them, and also by using the Gauss-Legendre integrating, we obtain the approximate value the generalized symmetric function. Theoretical consideration has been discussed and some examples were presented to show the ability of the method for approximate value of integral of the symmetric functions. In this numerical integration approach, for a symmetric function that has the same calculations at a number of different node points, only calculations are performed for a node and the result is multiplied by the number of repetitions of similar cases. In addition to modulating errors due to rounding and expanded error during calculations, much less memory is used than the numerical integration method. Also in this approach, the time of numerical integration is reduced and the numerical results confirm this.

Keywords Numerical integration · Symmetric function · Permutation

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1 Introduction

In this paper we consider the following integral

$$I = \int_{[a,b]^n} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n, \tag{1}$$

which $f : [a, b]^n \to \Re$ is a multivariate integrable function, and a and b are finite real number. It is difficult or impossible to find the value of integral (1) analytically, and so we have to approximate it. k-points quadrature formula for numerical integration are considered. Gaussian-Legendre quadrature is used to develop a family of approximation formulas for the accurate numerical computation of the many of integrals. Furthermore, symmetry properties related to zeros of Legendre polynomials and weights are available [4, 7]. In [11], Exact Gaussian quadrature methods have been applied for near-singular integrals in the boundary element method. In [2], estimates for the zeroes of the $n-$th Legendre polynomial are numerically studied, and alternative Gaussian-Legendre quadrature formula are considered.

There are many methods for finding the approximate value of (1), [10, 12]. Let the following formula is a k-points quadrature formula for computing the approximate of (1):

$$I \simeq \sum_{i_1=1}^{k} \sum_{i_2=1}^{k} \cdots \sum_{i_n=1}^{k} w_{i_1, i_2, \cdots, i_n} f(x_{i_1}, x_{i_2}, \dots, x_{i_n}) \tag{2}$$

that $w_{i_1, i_2, \cdots, i_n}$ denote for $w_{i_1} w_{i_2} \cdots w_{i_n}$ and in the case $n = 2$ we have

$$\int_{[a,b]^2} f(x, y) dx dy \simeq \sum_{i=1}^{k} \sum_{j=1}^{k} w_{i,j} f(x_i, x_j). \tag{3}$$

In the special case which f is symmetric, we can do this by fewer computations, because some points are iterative. Symmetric functions are important in several branches of mathematics, especially in approximation theory, probability theory, combinatorics, algebra, integral equations, and they have many applications in different areas [3, 9].

This article is organized in this form. In section 2, the number of nodes required to use the k-point Gaussian Legendre formula for multivariate symmetric functions is obtained, in order to avoid repeated calculations. Then the corresponding formula is rewritten with the aim of reducing the amount of calculations, errors and time. In section 3, numerical examples are solved using the proposed method and the results are compared with the Gaussian-Legendre method. Finally, in section 4, the results are stated.

2 Integral formula

Now, we define equivalence relation $\sim$: $[a, b]^n \to [a, b]^n$.

Definition 1 Two points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equivalent and denote by $(x_1, x_2, \ldots, x_n) \sim (y_1, y_2, \ldots, y_n)$ if and only if there is a permutation on $\{1, 2, \ldots, n\}$ such as σ such that

$$\forall i = 1, 2, \ldots, n : y_i = x_{\sigma(i)}.$$

We denote the equivalence class of $[a, b]^n$ respect to $\sim$ by $\frac{[a,b]^n}{\sim}$ and its element by $\overline{(x_1, x_2, \ldots, x_n)}$.

Definition 2 The multivariate function $f : [a, b]^n \to \Re$ is symmetric, if for each $(x_1, x_2, \ldots, x_n) \in [a, b]^n$ and each permutation $\sigma \in S_n$,

$$f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}),$$

which S_n is a symmetric group of order n [1,3,8].

Definition 3 Let f is a symmetric multivariate function. We denote number of nodes which the k-points quadrature formula (2) need them by $N_{n,k}$.

Lemma 1 *Let f is a symmetric two variables function. We have*

$$N_{2,k} = \binom{k+1}{2}$$

Proof Clearly.

Theorem 1 *For every k and $n > 1$ we have*

$$N_{n,k} = \sum_{j=1}^{k} N_{n-1,j} \tag{4}$$

Proof Using induction on n. By using $N_{1,k} = k$ and Lemma 1, the proposition is correct for $n = 2$. Let 4 hold for up to n. If the first component of $(i_1, i_2, \ldots, i_n)$ equal to (1) [6], then by using the hypothesis, the corresponding permutations is $N_{n,k}$. Also, if the first component of $(i_1, i_2, \ldots, i_n)$ equal to i, the corresponding permutations is $N_{n,k-i+1}$. So, the corresponding permutations for $n + 1$ is

$$N_{n,k} + N_{n,k-1} + \cdots + N_{n,1} = \sum_{j=1}^{k} N_{n,j}$$

therefore, statement (4) holds for $n + 1$.

Theorem 2 *For every k and $n > 1$ we have*

$$N_{n,k} = \binom{n+k-1}{n} \tag{5}$$

Table 1 Number of nodes required in n-point numerical integration formula, for symmetric multivariate functions

k	$N_{1,k}$	$N_{2,k}$	$N_{3,k}$	$N_{4,k}$	$N_{5,k}$	$N_{6,k}$	$N_{7,k}$
1	1	1	1	1	1	1	1
2	2	3	4	5	6	7	8
3	3	6	10	15	21	28	36
4	4	10	20	35	56	84	120
5	5	15	35	70	126	210	330
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Proof By using induction on n and using the following relation

$$\sum_{j=1}^{k} \binom{n+j-1}{n} = \binom{n+k}{n+1},$$

we can conclude (5).
By using Theorems 1 and 2, we can write the Table (1) for different n and k:

In Table 1, For $k = 5$ and $n = 5$, we have $N_{5,5} = 126$ points in symmetric form, while in the classic quadrature, we need to $5^5 = 3125$ points.

Definition 4 Let n_i is the number of iterative of x_i in $(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$, we define

$$\binom{n}{n_1, n_2, \ldots, n_n} = \frac{n!}{n_1! n_2! \cdots n_n!}$$

where $n_1 + n_2 + \cdots + n_n = n$ [13].

By using above definition we can write

$$\int_{C_n(a,b)} f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n$$

$$\simeq \sum_{i_1=1}^{k} \sum_{i_2=i_1+1}^{k} \cdots \sum_{i_n=i_{n-1}+1}^{k} \binom{n}{n_1, n_2, \ldots, n_n} w_{i_1,i_2,\cdots,i_n} f(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$$

Definition 5 A function f is called generalized symmetric if f is symmetric and for all x and $y \in [-1, 1]$:
i) $f(x, x) = 0$
ii) $f(x, -y) = f(x, y)$

Theorem 3 *The Gauss-Legendre quadrature for generalized symmetric function f of two variables and $k \geq 2$ are:*
i) If we use $2k$ points we have

$$\int_{-1}^{1} \int_{-1}^{1} f(x, y) dx dy \simeq 8 \sum_{i=1}^{k} \sum_{j=i+1}^{k} w_{i,j} f(x_i, x_j)$$

ii) If we use $2k+1$ points we have

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy \simeq 8\sum_{i=1}^{k}\sum_{j=i+1}^{k} w_{i,j} f(x_i,x_j) + 4\sum_{j=1}^{k} w_{0,j} f(0,x_j)$$

Proof for $n = 2k$ since that the Legendre polynomial $p_n(x)$ is even function, so if $p_n(x) = 0$ then $p_n(-x) = 0$ [14]. Zeros of $p_{2k}(x)$ on $[-1,1]$ and weights are listed

$$-x_k, -x_{k-1}, \ldots, -x_2, -x_1, x_1, x_2, \ldots, x_{k-1}, x_k$$

$$w_k, w_{k-1}, \ldots, w_2, w_1, w_1, w_2, \ldots, w_{k-1}, w_k$$

for generalized symmetric function f ,we have $f(x_i,x_i) = 0$ and

$$\begin{aligned} f(x_i,x_J) &= f(x_i,-x_J) = f(-x_i,x_J) = f(-x_i,-x_J) = f(x_J,x_i) \\ &= f(x_J,-x_i) = f(-x_J,x_i) = f(-x_J,-x_i). \end{aligned}$$

So, the Gauss-Legendre quadrature for generalized symmetric function f of two variables for $2k$ points is:

$$8\sum_{i=1}^{k}\sum_{j=i+1}^{k} w_{i,j} f(x_i,x_j).$$

Similarly to for $n = 2k+1$ zeros of $p_{2k+1}(x)$ and weights the following are listed

$$-x_k, -x_{k-1}, \ldots, -x_2, -x_1, 0, x_1, x_2, \ldots, x_{k-1}, x_k$$

$$w_k, w_{k-1}, \ldots, w_2, w_1, w_0, w_1, w_2, \ldots, w_{k-1}, w_k$$

but, $f(0,x_i) = f(0,-x_i) = f(x_i,0) = f(-x_i,0)$. So, the Gauss-Legendre quadrature for generalized symmetric function f of two variables for $2k+1$ points is:

$$8\sum_{i=1}^{k}\sum_{j=i+1}^{k} w_{i,j} f(x_i,x_j) + 4\sum_{j=1}^{k} w_{0,j} f(0,x_j).$$

3 Computation Results

In this section, the following two examples for numerical integration using the Definitions and Theorems of the previous section and comparing the amount of memory and time used, compared to the method of numerical integration of Gaussian-Legendre were examined.

Table 2 The approximate value obtained from the proposed method and Gaussian-Legendre for Example 1, with different n.

n	Gaussian-Legendre quadrature formula	propsed method
4	1.136017156	1.136017156
5	1.240434694	1.240434694
6	1.248384751	1.248384752
7	1.279186715	1.279186714
8	1.285482627	1.285482628
9	1.298529611	1.298529610
10	1.302519696	1.302519696
11	1.307494901	1.309216805
12	1.311799291	1.311799290
13	1.315679426	1.315679431
14	1.317423410	1.317423414
15	1.318986118	1.319867674
16	1.321093554	1.321093557
17	1.322730782	1.322730782
18	1.323622627	1.323622631
19	1.324772046	1.324772048
20	1.325439990	1.325439989

Table 3 The amount of memory used the proposed method compared to the Gaussian-Legendre method, for Example 1, with different n.

n	memory used with proposed method (KiB)	memory used with Gaussian-Legendre quadrature (KiB)
4	2.73	15.56
5	4.16	20.94
6	4.37	31.64
7	6.79	39.68
8	6.80	52.23
9	10.20	69.49
10	10.70	85.54
11	14.23	100.14
12	14.79	119.73
13	19.90	141.22
14	20.12	164.65
15	24.66	180.16
16	26.55	220.10
17	31.69	233.58
18	33.21	273.34
19	39.44	294.89
20	41.42	340.40

Example 1 We Consider

$$\int_{-1}^{1}\int_{-1}^{1}|x^2-y^2|dxdy.$$

The exact solution is $\frac{4}{3}$. Table 2, shows the obtained values by using Theorem 3, and n-points quadrature formula.

Table 4 The time used in the proposed method compared to the Gaussian-Legendre method, for Example 1, with different n.

n	real time for proposed method (ms)	real time for Gaussian-Legendre quadrature (ms)
4	2	0
5	0	0
6	0	1
7	1	0
8	0	2
9	0	1
10	0	1
11	0	2
12	5	2
13	0	2
14	1	2
15	1	2
16	0	3
17	1	3
18	0	5
19	0	4
20	2	4

In Table 3, for different values of n, the amount of memory required in the introduced method is much smaller than the Gaussian-Legendre method. Table 4, shows the amount of real time for the proposed method and the Gaussian-Legendre method.

Example 2 Consider the following integration problem

$$\int_{-1}^{1}\int_{-1}^{1}\frac{|cos(x)-cos(y)|}{(1+x^2)(1+y^2)}dxdy. \tag{6}$$

Using Maple software, the value of the above integral is 0.3471432304. In the Table 5, for different values of n and using Theorem 3, the approximate values of the integral is given.

In Table 6, the amount of memory required by the proposed method is much less compared to Gaussian-Legendre method. Table 7, shows the amount of real time for Example 2 with the proposed method and the Gaussian-Legendre method.

4 Conclusion

In this work, using the properties of symmetric functions and the Gaussian-Legendre quadrator, the multivariate numerical integration formula was simplified. In this quadrator, the computational volume is greatly reduced. The Gauss-Legendre quadrature for generalized symmetric function of two variables was obtained and with numerical examples, the efficiency of this rule was examined. Since, the integral of multivariate functions plays a critical

Table 5 The approximate value obtained from the proposed method and Gaussian-Legendre, for Example 2, with different n.

n	Gaussian-Legendre quadrature formula	propsed method
4	0.2719947831	0.2719947829
5	0.3262859197	0.3262859199
6	0.3197361154	0.3197361155
7	0.3324682055	0.3324682054
8	0.3324640169	0.3324640170
9	0.3373150893	0.3373150892
10	0.3378535141	0.3378535142
11	0.3400445547	0.3402280539
12	0.3407046302	0.3407046300
13	0.3420420997	0.3420420996
14	0.3424086464	0.3424086450
15	0.3431430365	0.3432347498
16	0.3435116943	0.3435116942
17	0.3440568769	0.3440568770
18	0.3442678775	0.3442678786
19	0.3446461533	0.3446461533
20	0.3448093436	0.3448093436

Table 6 The amount of memory used the proposed method compared to the Gaussian-Legendre method, for Example 2, with different n.

n	memory used with proposed method (KiB)	memory used with Gaussian-Legendre quadrature (KiB)
4	47.16	72.56
5	49.88	82.00
6	51.63	107.84
7	55.62	122.48
8	57.97	154.59
9	63.61	180.02
10	65.98	219.31
11	72.07	244.44
12	75.61	293.68
13	83.27	335.59
14	87.07	384.86
15	94.55	423.75
16	100.80	496.05
17	109.07	532.48
18	115.17	614.40
19	124.92	655.36
20	132.84	747.52

role in the cost function of the optimal control problem and in the integral equations, methods based on numerical integration methods are of particular importance because the values of some integrals are not accurately calculated. When the cost function in the optimal control or the kernel in the integral equations are symmetric, we can use the results of this research to design a numerical method with less and more efficient calculations to compute the integral and finally for the main problem. This issue needs further research.

Table 7 The time used in the proposed method compared to the Gaussian-Legendre method, for Example 2, with different n.

n	real time for proposed method (ms)	real time for Gaussian-Legendre quadrature (ms)
4	3	1
5	2	1
6	1	2
7	1	2
8	2	2
9	1	3
10	2	4
11	3	4
12	2	6
13	2	5
14	2	3
15	2	7
16	0	8
17	2	8
18	3	8
19	1	6
20	3	11

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