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On Invariant Graph Of Γ-Near-Ring

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Abstract Let U be an invariant subset of finite Γ-near-ring M. There are many papers that consider the graph respect to the near-ring and the interplay between algebraic structures and graphs are studied. Indeed, it is worthwhile to relate algebraic properties of near-ring to the combinatorics properties of assigned graphs. In this paper the graph with respect to an invariant subset U of Γ-near-ring M, denoted by $\Gamma_U^\alpha(M)$ is introduced and the basic properties of it is investigated. Also the relation between the commutativity of M and properties of this graph is presented.

Keywords Invariant subset · Near-ring · Commutativity

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1 Introduction

Near-rings are generalized rings. In fact an algebraic structure $(M, +, .)$ (mostly abbreviated by M) is called a near-ring if $(M, +)$ is a (not necessarily abelian) group, $(M, .)$ is a semigroup and $(a + b).c = a.c + b.c$ for all $a,\ b,\ c \in M$. A standard reference of near-ring is Pilz [14].
A Γ-near-ring M is a triple $(M, +, \Gamma)$ where

- (i) $(M, +)$ is not a necessarily abelian group,
- (ii) Γ is a non-empty set of binary operations of M such that for each $\alpha \in \Gamma$, $(M, +, \alpha)$ is near-ring,
- (iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x,\ y,\ z \in M$ and $\alpha, \beta \in \Gamma$.

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A subset U of a Γ-near-ring M is said to be left (resp. right) invariant if $x\alpha a \in U$ (resp. $a\alpha x \in U$), for all $a \in U,\ \alpha \in \Gamma$ and $x \in M$. If U is both left and right invariant, we say that U is invariant.

Example 1 Let $(\mathbb{Z}_6,\ +_6,\ \Gamma)$ be a right Γ-near-ring when $\Gamma = \{\alpha,\ \beta\}$, where α and β defined by $\alpha = \cdot_6$ and $a\beta b = a$ for all $a,\ b \in \mathbb{Z}_6$. If $U \subset \mathbb{Z}_6$ such that $U = \{0,\ 2,\ 4\}$, then U is a right invariant subset of $\mathbb{Z}_6$.

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the set of vertices of G and $E(G)$ the set of edge of G. For graph theoretical concepts we refer to Bondy and Murty [6], and Godsil and Royle [7].

The concept of associating graphs to commutative rings, one of the most interesting concepts of algebraic structures in graph theory, was first introduced by Beck [5]. There are many papers about this subject, and you can see ([1, 3, 4, 10, 12, 13]).

Subsequently, Alan Cannon et al. [2] defined and studied the zero divisor graph corresponding to a near-ring. Recently Stayanarayana et al. [11] associated a graph to an ideal I of a near-ring M, denoted by $G_I(M)$. In this paper we define a graph respect to invariant finite subset of Γ-near-ring and investigate the properties of it.

2 Preliminaries

In this section, some definitions with more example were reviewed, that introduce in previous section, and some properties which are used in this work. The Γ-near-ring is defined in previous section.

Example 2 Let $M = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},\ \ a, b \in \mathbb{Z} \right\}$ and $\Gamma = \{\alpha, \beta\}$ defined by

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \alpha \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ac & bd \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \beta \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+b & a+b \\ 0 & 0 \end{pmatrix},$$

for all $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in M$. Then $(M, +, \Gamma)$ is a Γ-near-ring.

Definition 1 A Γ-near-ring M is called a prime if $x\Gamma M\Gamma y = (0)$ implies $x = 0$ or $y = 0$, where $x,\ y \in M$.

Example 3 A Γ-near-ring $(\mathbb{Z}_2, +_2, \Gamma)$ with $\Gamma = \{\alpha,\ \beta\}$ where $\alpha = +_2$ and $a\beta b = a$ for all $a,\ b \in \mathbb{Z}_2$ is a prime Γ-near-ring.

Definition 2 Let M be a Γ-near-ring, I define the product

$$[x, y]_\alpha = x\alpha y - y\alpha x,$$

which is called the commutator.

Definition 3 Let M be a Γ-near-ring, an additive endomorphism $D : M \to M$ is called a derivation of M if satisfying the product rule

$$D(x\alpha y) = D(x)\alpha y + x\alpha D(y),$$

for all $x, y \in M$, $\alpha \in \Gamma$, and is called a reverse derivation if

$$D(x\alpha y) = D(y)\alpha x + y\alpha D(x),$$

for all $x, y \in M$, $\alpha \in \Gamma$.

Example 4 Let M be a Γ-near-ring, as in Example 2, if we define $D : M \to M$ by $D\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then D is a Γ-derivation of M.

Lemma 1 *[9] Let M be a prime Γ-near-ring, $U(\neq \{0\})$ be an invariant subset of M, and D be a non-zero reverse derivation of M. If D is commutating on U, then M is commutative.*

Lemma 2 *[9] Let M be a prime Γ-near-ring, $U(\neq \{0\})$ be an invariant subset of M, and D be a non-zero reverse derivation of M. If $[D(v), D(u)]_\alpha = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$, then M is commutative.*

Lemma 3 *[9] Let M be a prime Γ-near-ring, $U(\neq \{0\})$ be an invariant subset of M, and D be a non-zero right reverse derivation of M. If*

$$[D(v), D(u)]_\alpha = [u, v]_\alpha,$$

for all $u, v \in U$ and $\alpha \in \Gamma$, then M is commutative.

Now, some definitions of graph theory were recalled that we need in the next section. Let G be a graph with vertex set $V(G)$. An edge between two vertices $x, y \in V(G)$ is denoted by xy. Recall that G is connected if there is a path between any two distinct vertices of G. For two vertices x and y of G, the distance $d(x, y)$ is the length of a shortest path from x to y. The diameter of G is $diam(G) = max\{d(x, y);\ x, y \in V(G)\}$ and the girth of G is the length of a smallest cycle of G and it is denoted by $gr(G)$. If $S \subset V(G)$ is any subset, I denote $G - S$ the graph whose vertex set is $V(G) - S$ and whose edge set is $E(G - S) = \{xy \mid \{x, y\} \cap S \neq \emptyset\}$. A vertex cut of G is a subset $S \subset V(G)$ such that $G - S$ is disconnected. If $T \subset E(G)$ is any subset, I denote by $G - T$, the graph whose vertex set is $V(G)$ and edge set is $E(G) - T$. An edge cut of G is a subset $T \subset E(G)$ such that the graph $G - T$ is disconnected. The (vertex) connectivity of G is defined by

$$k(G) = \min\{n \geq 0;\ \text{there exist a vertex cut } S \subset V(G) \text{ such that } |S| = n\}.$$

Similarly, the edge connectivity of G is defined by

$$\lambda(G) = \min\{n \geq 0;\ \text{there exists an edge cut } T \subset E(G) \text{ such that } |T| = n\},$$

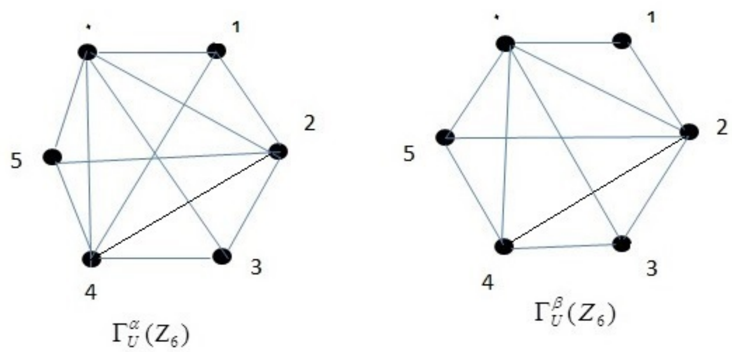
if G has a finite edge cut, and $\lambda(G) = \infty$ otherwise.

Let $\delta(G) = \min\{deg(v);\ v \in V(G)\}$. The following well-known result may be found in any standard textbook on graph theory, see for example Harary [8], $k(G) \leq \lambda(G) \leq \delta(G)$.
The chromatic number $\chi(G)$ of G is the minimum number of colors which can be assigned to the vertices of G in such a way that every pair of distinct adjacent vertices have different colors. The clique number $w(G)$ is the order of the maximum possible complete subgraph of G.

3 Basic properties of invariant graph of *Γ*-near-ring

Let $(M, +, \Gamma)$ be a Γ-near-ring M and U is an invariant subset of M. For each $\alpha \in \Gamma$, $(M, +, \alpha)$ is a near-ring. The researcher consider the invariant graph of a Γ-near-ring with vertices $V(\Gamma_U^\alpha(M))$ equals the element of M and for $x \in M$ and $a \in U$, $xa \in E(\Gamma_U^\alpha(M))$ if and only if $x\alpha a \in U$.

Example 5 Let $(\mathbb{Z}_6, +_6, \Gamma)$ be a right Γ-near-ring in Example 1. The graphs respect to $\alpha = \cdot_6$ and β are given below.



Remark 1 $\Gamma_U^\alpha(M)$ is a connected graph without self loops and multiple edges.

Proposition 1 *The maximum distance between any two vertices of $\Gamma_U^\alpha(M)$ is at most 2. That is $diam(\Gamma_U^\alpha(M)) \leq 2$ and $gr(\Gamma_U^\alpha(M)) = 3$.*

Proof Let $x, y \in M$ U, then there is $z \in U$ such that xz and yz are adjacent in $\Gamma_U^\alpha(M)$, hence $diam(\Gamma_U^\alpha(M)) \leq 2$. Now if x and y are two elements of U, then x, y and $z \in M$ U given a triangle in $\Gamma_U^\alpha(M)$, which implies that $gr((\Gamma_U^\alpha(M)) = 3$.

Proposition 2 *Let U_1 and U_2 be invariant subsets of M such that $U_1 \subset U_2$. Then*

$$\Gamma_{\{0\}}^\alpha(M) \subset \Gamma_{U_1}^\alpha(M) \subset \Gamma_{U_2}^\alpha(M) \subset \Gamma_M^\alpha(M),$$

where the notation $\Gamma^{\alpha}_{U_1}(M) \subset \Gamma^{\alpha}_{U_2}(M)$, I mean a graph $\Gamma^{\alpha}_{U_1}(M)$ is a subgraph of $\Gamma^{\alpha}_{U_2}(M)$.

Proposition 3 *Let $|M| = m$ and U be an invariant subset of M and $\alpha \in \Gamma$. Then*

$$1 \leq k(\Gamma^{\alpha}_{U}(M)) \leq \lambda(\Gamma^{\alpha}_{U}(M)) \leq \delta(\Gamma^{\alpha}_{U}(M)) \leq m - 1.$$

Proof For any graph G, it is well known that $k(G) \leq \lambda(G) \leq \delta(G)$. As $\Gamma^{\alpha}_{U}(M)$ is a connected graph, the minimum of vertices whose removal results is a disconnected or trivial graph is 1, hence $1 \leq k(\Gamma^{\alpha}_{U}(M))$. As $|M| = m$, $\delta(\Gamma^{\alpha}_{U}(M)) \leq deg(0) \leq m - 1$.

There is a connection between commutativity of invariant subset U and commutativity of M in Γ-near-ring respect of reverse derivation such D on it. So I associate weight $[x, a]_{\alpha}$ to an edge of graph $\Gamma^{\alpha}_{U}(M)$. Now consider the inductive weighted subgraphs on U for every $\alpha \in \Gamma$. If D is a reverse derivation on M, then I can consider the weighted graph with vertices set $D(x)$, for all $x \in M$. I investigate the weighted inductive subgraphs with weight $[D(u), D(v)]_{\alpha}$ on $D(u), D(v)$ for all $u, v \in U$ and $\alpha \in \Gamma$.

Theorem 1 *Let M be a prime Γ-near-ring, U be an invariant subset of M and D be a reverse derivation on M. If the weight of edges of induced graph on U and $D(U)$ are equal for every $\alpha \in \Gamma$. Then M is commutative.*

Proof It follows from Lemma 3.

Theorem 2 *Let the assumptions in the previous theorem be hold. If the weight of edges of induced graph on $D(U)$ are equal to zero for every $\alpha \in \Gamma$ ($[D(u), D(v)]_{\alpha} = 0$). Then M is commutative.*

Proof It follows from Lemma 2.

Theorem 3 *Let M be a Γ-near-ring, U be an invariant subset of M. Then $\chi(\Gamma^{\alpha}_{U}(M)) = w(\Gamma^{\alpha}_{U}(M))$, for every $\alpha \in \Gamma$.*

Proof For any graph G, $w(G) \leq \chi(G)$, so it is enough to prove that I can color the vertices of $\Gamma^{\alpha}_{U}(M)$ with $w(\Gamma^{\alpha}_{U}(M))$ color. The vertices of complete subgraph of $\Gamma^{\alpha}_{U}(M)$ can be color by $w(\Gamma^{\alpha}_{U}(M))$ colors. Since $U \neq M$, there is one vertex x such that it does not adjacent to every vertex of complete subgraph, and it can color with the vertex that is not adjacent to it. If there is another vertex y, such that it does not belong to the vertices of complete subgraph and x and y are adjacent. If x and y are not adjacent to the same vertex such a, then the clique number will be $w(\Gamma^{\alpha}_{U}(M)) + 1$, with complete subgraph induced by x, y and the other vertices except a. Hence the vertices of $\Gamma^{\alpha}_{U}(M)$ can be color with $w(\Gamma^{\alpha}_{U}(M))$ colors.

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