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Infinitely Many Solutions for Discrete Fourth-Order Boundary Value Problem with Four Parameters

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Abstract The aim of this paper is to study the existence of infinitely many solutions for discrete fourth-order boundary value problem with four parameters involving oscillatory behaviors of nonlinearity at infinity. The approach is based on variational methods. In addition, one example is presented to illustrate the feasibility and effectiveness of the main results.

Keywords Discrete boundary value problem · Fourth order boundary value problem · Infinitely many solutions · Variational methods

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1 Introduction

Let $\mathbb{T}$ be a time scale, that is, a nonempty closed subset of $\mathbb{R}$. In particular, $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ are examples of time scales corresponding to differential and difference equations, respectively. Let $T > 0$ be fixed and suppose $0, T \in \mathbb{T}$. The aim of this paper is to investigate the existence of infinitely many solutions for the following second-order Sturm-Liouville type boundary value problem on time scales: Let $T > 2$ be a positive integer and $[2, T]_{\mathbb{Z}}$ be the discrete interval given by $\{2, 3, 4, \ldots, T\}$. In this paper, we will examine a discrete nonlinear fourth order boundary value problems (BVP) with four parameters with intention of proving the existence of three solutions. The problem to be studied can be viewed as a discrete version of the generalized beam equation.

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Consider the fourth BVP:

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f(k, u(k)), \quad k \in [2,T]_{\mathbb{Z}}, \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0, \end{cases} \tag{P_λ^f}$$

where Δ denotes the forward difference operator defined by

$$\Delta u(k) = u(k+1) - u(k), \qquad \Delta^{i+1}u(k) = \Delta(\Delta^i u(k)),$$

$\lambda \geq 0$, $f : [2,T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is a continuous functions, and α and β are real parameters and satisfy:

$$1 + (T-1)T\alpha_- + T(T-1)^3\beta_- > 0, \tag{1}$$

where

$$\alpha_- = \min\{\alpha, 0\},$$

and

$$\beta_- = \min\{\beta, 0\}.$$

Discrete boundary value problems have been intensively studied in the last decade. The modeling of certain nonlinear problems from biological neural networks, economics, optimal control and other areas of study have led to the rapid development of the theory of difference equations; see [3,5,16,18] for an overview on this subject.

Much interest has lately shown in fourth-order difference equations derived from various discrete elastic beam problems. Many researches have investigated into the existence and multiplicity of solutions for discrete fourth-order boundary value problems through classical methods, including the fixed point theory, the critical point theory, Krein–Rutman theorem, and the bifurcation theory. For instance you can see [1,6–13,15,17,19,22] and the references therein. For example, Graef et al. in [9] by applying some recent results from mixed monotone operator theory, investigated uniqueness and dependence of positive solutions on the parameters for the following nonlinear discrete fourth-order Lidstone BVP

$$\begin{cases} \Delta^4 u(t-2) + \beta\Delta^2(t-1) = \lambda\left[f(t,u(t),u(t)) + r(t,u(t))\right], \ t \in [a+1, b-1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \ u(b) = \Delta^2 u(b-1) = 0, \end{cases}$$

where

$$f : [a+1, b-1]_{\mathbb{Z}} \times [0,\infty) \times (0,\infty) \to [0,\infty),$$

and

$$r : [a+1, b-1]_{\mathbb{Z}} \times [0,\infty) \to [0,\infty),$$

are continuous functions. Ousbika and El Allali in [19] based on the critical point theory, discussed the existence of at least two solutions for the following discrete nonlinear fourth order boundary value problems with four parameters

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f(k,u(k)) + \mu g(k,u(k)), \ k \in [2,T]_{\mathbb{Z}}, \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0. \end{cases}$$

In [12] by employing variational methods, the existence of at least three classical solutions for the problem (P_λ^f) was studied, also in [13] by using variational methods, the existence of least one solution under an asymptotical behaviour of the potential of the nonlinear term at zero for the problem (P_λ^f) was discussed.

Graef et al. in [9] by applying some recent results from mixed monotone operator theory, obtained the existence, uniqueness and dependence of positive solutions on the parameters for the following nonlinear discrete fourth-order Lidstone BVP

$$\begin{cases} \Delta^4 u(t-2) - \beta\Delta^2 u(t-1) = \lambda[f(t,u(t),u(t)), r(t,u(t))], \ t \in [a+1,b-1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \ u(b) = \Delta^2 u(b-1) = 0, \end{cases}$$

where

$$f : [a+1,b-1]_{\mathbb{Z}} \times [0,\infty) \times (0,\infty) \to [0,\infty),$$

and

$$r : [a+1,b-1]_{\mathbb{Z}} \times [0,\infty) \to [0,\infty),$$

are continuous functions. Heidarkhani et al. in [11], by using a consequence of the local minimum theorem due Bonanno, studied the existence one solution and two solutions for the perturbed problem (P_λ^f). Unlike the mentioned works, we look for for the existence of the infinitely many solutions for the problem (P_λ^f) considering the oscillating behaviour condition of the nonlinear term at infinity. We are also able to achieve a sequence of pairwise distinct solutions which converges to zero for the problem by replacing the oscillating behaviour condition at infinity, by a similar one at zero.

Motivated by the above works, in the present paper, employing a smooth version of [2, Theorem 2.1], under an appropriate oscillating behaviour of the nonlinear term f, we determine the exact collections of the parameter λ in which the problem (P_λ^f), admits infinitely many solutions (Theorem 2). The applicability of our results is illustrated by an example.

The present paper is arranged as follows. In Section 2 we recall some basic definitions and preliminary results, while Section 3 is devoted to the existence of multiple solutions for the eigenvalue problem (P_λ^f).

2 Preliminaries

In this section, we formulate our main results on the existence infinitely many solutions for the problem (P_λ^f). Our main tool to ensure the results is a smooth version of Theorem 2.1 of [2] which is a more precise version of Ricceri's Variational Principle [20, Theorem 2.5] that we now recall here.

Theorem 1 *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower* *semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) - \Psi(u)}{r - \Phi(u)}$$

and

$$\theta := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\theta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\theta}[$, the following alternative holds: either*

(b_1) *I_λ possesses a global minimum,*
or

(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds:*

(c_1) *there is a global minimum of Φ which is a local minimum of I_λ,*

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ.*

We refer the interested reader to the paper [4, 14] in which Theorem 1 has been successfully employed to the existence of infinitely many solutions for boundary value problems.

This section is devoted to introduce some basic notations and results which will be used in the proofs of our main results. In this section, we will introduce several basic definitions, notations, lemmas, and propositions used all over this paper.

We define the real vector space E

$$E = \Big\{u : [0, T+2]_Z \to \mathbb{R},\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0\Big\},$$

which is a $(T-1)$-dimentional Hilbert space, see [21] with the inner product

$$(u, v) = \sum_{k=2}^{k=T} u(k)v(k).$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=2}^{k=T} |u(k)|^2\right)^{\frac{1}{2}}.$$

Lemma 1 *[19, Lemma 2.5] For any $u, v \in E$, we have*

$$\sum_{k=2}^{T} \Delta^4 u(k-2)v(k) = \sum_{k=2}^{T+1} \Delta^2 u(k-2)\Delta^2 v(k-2),$$

$$\sum_{k=2}^{T} \Delta u(k-1)\Delta v(k-1) = -\sum_{k=2}^{T} \Delta^2 u(k-1)v(k).$$

We consider the functional as follows:

$$\Phi(u) = \frac{1}{2}\left(\sum_{k=2}^{T+1} |\Delta^2 u(k-1)|^2 + \alpha \sum_{k=2}^{T} |\Delta u(k-1)|^2 + \beta \sum_{k=2}^{T} |u(k)|^2\right), \quad (2)$$

$$\Psi(u) = \sum_{k=1}^{T} F(k, u(k)), \quad (3)$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

for every $u \in E$.

Lemma 2 *[19, Lemma 2.6] For any $u \in E$, we have*

$$\Phi(u) \geq 0,$$

and

$$\Phi(u) \geq \frac{1}{2}\rho\|u\|^2,$$

where

$$\rho = \left(1 + (T-1)T\alpha_- + T(T-1)^3\beta_-\right)T^{-1}(T-1)^3.$$

Lemma 3 *[19, Lemma 2.7] If $u \in E$ is a critical point of the functional I then u is a solution of BVP (P_λ^f).*

Put

$$F(k,t) := \int_0^t f(k,\xi)d\xi \quad \text{for all } (k,t) \in [2,T]_{\mathbb{Z}} \times \mathbb{R}.$$

2.1 Main Result

In this section, we will state and prove our main results.

For convenience, put

$$A = \liminf_{\xi\to+\infty} \frac{\sum_{k=2}^{T} \max_{|t|\le\xi} F(k,t)}{\xi^2},$$

$$B = \frac{2}{2+\alpha+(T-1)\beta} \limsup_{\xi\to+\infty} \frac{\sum_{k=2}^{T} F(k,\xi)}{\xi^2},$$

$$\lambda_1 = \frac{1}{B}$$

and

$$\lambda_2 = \frac{\rho}{2A}.$$

Theorem 2 *Assume that*

(A_1) $F(k,t) \ge 0$ *for each* $(k,t) \in [2,T]_{\mathbb{Z}} \times [0,+\infty)$*;*

(A_2)

$$A < \frac{\rho}{2}B.$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$*, the problem* (P_λ^f) *has an unbounded sequence of solutions in* E*.*

Proof Our aim is to apply Theorem 1 to the problem (P_λ^f). Define the functionals Φ and Ψ as given in (2) and (3), respectively. Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 1. By lemma 2, we prove that Φ is coercive, sequentially weakly lower semicontinuous and is bounded on each bounded subset of E. On the other hand, Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in E^*$, given by

$$\Phi'(u)(v) = \sum_{k=2}^{T+1} \Delta^2 u(k-2)\Delta^2 v(k-2) + \alpha \sum_{k=2}^{T} \Delta u(k-1)\Delta^2 v(k-1) + \beta \sum_{k=2}^{T} u(k)v(k),$$

for every $u, v \in E$. It is well known that Ψ is a differentiable functional whose differential at the point $u \in E$ is

$$\Psi'(u)(v) = \sum_{k=2}^{T} f(k,u(k))v(k),$$

for any $v \in E$ as well as it is sequentially weakly upper semicontinuous. We show that Ψ' is compact. Suppose that $u_n \to u \in E$ then since f is continuous and from (3), we deduce that $\Psi'(u_n) \to \Psi(u_n)$, thus Ψ' is compact. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested in Theorem 1, are verified.

Let $\{\xi_n\}$ be a real sequence of positive numbers such that $\lim\limits_{n\to+\infty} \xi_n = +\infty$ and

$$A = \lim_{n\to+\infty} \frac{\sum\limits_{k=2}^{T} \max\limits_{|t|\le\xi_n} F(k,t)}{\xi_n^2}.$$

Put

$$r_n = \frac{\rho}{2}\xi_n^2.$$

If $u \in \Phi^{-1}(-\infty, r_n)$, then $\Phi(u) < r_n$. From the definition of r_n, it follows that

$$\Phi^{-1}(-\infty, r_n) = \{u \in E;\ \Phi(u) < r_n\} \subseteq \{u \in E;\ |u| \le \xi_n\}.$$

Hence, we have

$$\sup_{u\in\Phi^{-1}(-\infty,r_n)} \Psi(u) \le \sum_{k=2}^{T} \max_{|t|\le\xi_n} F(k,t).$$

Therefore, since $0 \in \Phi^{-1}(-\infty, r_n)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}\varphi(r_n) &\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_n)} \Psi(u)}{r_n} \\ &= \frac{\sum_{k=2}^{T} \max_{|t|\le\xi_n} F(k,t)}{r_n} \\ &= \frac{2}{\rho}\frac{\sum\limits_{k=2}^{T} \max\limits_{|t|\le\xi_n} F(k,t)}{\xi_n^2}\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, from assumption (A_2), one has

$$\theta \le \liminf_{n\to+\infty} \varphi(r_n) \le \frac{2}{\rho} A < +\infty.$$

Now, let $\{\eta_n\}$ be positive real sequences and for all $n \in \mathbb{N}$, and

$$\lim_{n\to+\infty} \eta_n = +\infty.$$

Define $w_n(k) = \eta_n$ for all $k \in [2, T]_{\mathbb{Z}}$. Clearly, $w_n \in E$, from (2), we have

$$\Phi(w_n) = \frac{1}{2}\left(2 + \alpha + (T-1)\beta\right)\eta_n^2. \tag{4}$$

On the other hand, bearing the assumption (A_1) in mind, from (3) one has

$$\Psi(w_n) = \sum_{k=2}^{T} F(k, \eta_n).$$

Then,

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) \le \frac{1}{2}\left(2+\alpha+(T-1)\beta\right)\eta_n^2 - \lambda\sum_{k=2}^{T} F(k,\eta_n).$$

Now, consider the following cases.
If $B < +\infty$, let $\epsilon \in]0, B - \frac{1}{\lambda}[$. There exists ν_ϵ such that

$$\sum_{k=2}^{T} F(k,\eta_n) > (B-\epsilon)\frac{1}{2}\left(2+\alpha+(T-1)\beta\right)\eta_n^2,$$

for all $n > \nu_\epsilon$, and so

$$\begin{aligned} I_\lambda(w_n) <& \frac{1}{2}\left(2+\alpha+(T-1)\beta\right)\eta_n^2 - \lambda\sum_{k=2}^{T} F(k,\eta_n) \\ =& \frac{1}{2}\left(2+\alpha+(T-1)\beta\right)\eta_n^2(1-\lambda(B-\epsilon)). \end{aligned}$$

Since $1-\lambda(B-\epsilon) < 0$, and taking into account (4) one has

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty.$$

If $B = +\infty$, fix $N > \frac{1}{\lambda}$. There exists ν_N such that

$$\sum_{k=2}^{T} F(k,\eta_n) > \frac{N}{2}\left(2+\alpha+(T-1)\beta\right)\eta_n^2,$$

for all $n > \nu_N$, and moreover,

$$I_\lambda(w_n) < \frac{1}{2}\left(2+\alpha+(T-1)\beta\right)\eta_n^2(1-\lambda N).$$

Since $1-\lambda N < 0$, and arguing as before, we have

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty.$$

Taking into account that $]\frac{1}{B}, \frac{\rho}{2A}[\subset]0, \frac{1}{\theta}[$ and that I_λ does not possess a global minimum, from part (b) of Theorem 1, there exists an unbounded sequence $\{u_n\}$ of critical points which are the solutions of (P_λ^f). So, our conclusion is achieved.

We present an example to illustrate Theorem 2.

Example 1 Let $T = 4$. We consider the following problem

$$\begin{cases} \Delta^4 u(k-2) - \Delta^2 u(k-1) + u(k) = \lambda f(u), \quad k \in [2,4]_{\mathbb{Z}}, \\ u(1) = \Delta u(0) = \Delta u(4) = \Delta^3 u(0) = \Delta^3 u(3) = 0, \end{cases} \tag{5}$$

where

$$f(\xi) = 2\xi + 20\xi \cos^2(\frac{\pi}{2} e^{\xi}) - 10\pi \xi^2 e^{\xi} \cos(\frac{\pi}{2} e^{\xi}) \sin(\frac{\pi}{2} e^{\xi}),$$

for every $\xi \in \mathbb{R}$. By the expression of f, we have

$$F(\xi) = \xi^2 \left(1 + 10 \cos^2(\frac{\pi}{2} e^{\xi})\right),$$

for every $\xi \in \mathbb{R}$. Direct calculations give $\rho = \frac{64}{5}$. By simple calculations, we see that

$$\liminf_{\xi \to +\infty} \frac{\sup_{|x| \le \xi} F(x)}{\xi^2} = 1,$$

and

$$\limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 11.$$

We clearly see that all assumptions of Theorem 2 are satisfied. Then, for every $\lambda \in (\frac{3}{11}, \frac{32}{5})$, the problem (5) admits a sequence of solutions which is unbounded in

$$\{u : [0,6]_Z \to \mathbb{R},\ u(1) = \Delta u(0) = \Delta u(4) = \Delta^3 u(0) = \Delta^3 u(3) = 0\}.$$

Remark 1 Under the conditions $A = 0$ and $B = +\infty$, Theorem 2 concludes that for every $\lambda > 0$, the problem (P_λ^f) admits infinitely many solutions in E.

Remark 2 Put $\hat{\lambda}_1 = \lambda_1$ and

$$\hat{\lambda}_2 = \frac{1}{\lim_{n\to+\infty} \dfrac{\sum_{k=2}^{T} \max_{|t| \le c_n} F(k,t) - \sum_{k=2}^{T} F(k, b_n)}{\dfrac{\rho}{2} c_n^2 - \dfrac{1}{2} \left(2 + \alpha + (T-1)\beta\right) b_n^2}}.$$

We explicitly observe that the assumption (A_2) in Theorem 2 could be replaced by the following more general condition
(A_3) there exist two sequence $\{c_n\}$ with $\{b_n\}$ for all $n \in \mathbb{N}$ and

$$b_n^p < \frac{\rho}{2 + \alpha + (T-1)\beta} c_n^p,$$

for every $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} c_n = +\infty$ such that

$$\lim_{n\to+\infty} \frac{\sum_{k=2}^{T} \max_{|t| \le c_n} F(k,t) - \int_0^{\sigma(T)} F(t, b_n) \Delta t}{\dfrac{\rho}{2} c_n^2 - \dfrac{1}{2} \left(2 + \alpha + (T-1)\beta\right) b_n^2}$$
$$< \frac{2}{2 + \alpha + (T-1)\beta} \limsup_{n\to+\infty} \frac{\sum_{k=2}^{T} F(k, \eta_n)}{\eta_n^2}.$$

Obviously, from (A_3) we obtain (A_2), by choosing $b_n = 0$ for all $n \in \mathbb{N}$. Moreover, if we assume (A_3) instead of (A_2) and set

$$r_n = \frac{\rho}{2} c_n^2,$$

for all $n \in \mathbb{N}$, by the same arguing as inside in Theorem 2, we obtain

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r_n)} \Psi(u)) - \Psi(u)}{r_n - \Phi(u)} \\ &\le \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_n)} \Psi(u) - \lambda\Psi(w_n)}{r_n - \Phi(u)} \\ &\le \frac{\sum_{k=2}^{T} \max_{|t|\le c_n} F(k,t) - \sum_{k=2}^{T} F(k,b_n)}{\frac{\rho}{2}c_n^2 - \frac{1}{2}\left(2+\alpha+(T-1)\beta\right) b_n^p}.\end{aligned}$$

We have the same conclusion as in Theorem 2 with Λ replaced by $\Lambda' :=]\hat{\lambda}_2, \hat{\lambda}_2[$.

Here we point out the following consequence of Theorem 2.

Corollary 1 *Assume that* (A_1) *holds and*

(A_4) $\liminf_{\xi\to+\infty} \dfrac{\sum_{k=2}^{T} \max_{|t|\le\xi} F(k,t)}{\xi^2} < \dfrac{\rho}{2}$,

(A_5) $\limsup_{\xi\to+\infty} \dfrac{\sum_{k=2}^{T} F(k,\xi)}{\xi^2} > \dfrac{2+\alpha+(T-1)\beta}{2}$.

Then, the problem

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = f(k,u(k)), \quad k \in [2,T]_{\mathbb{Z}}, \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0, \end{cases} \qquad (P_1^f)$$

has an unbounded sequence of solutions in E.

In the same way as in the proof of Theorem 2 but using conclusion (c) of Theorem 1 instead of (b), we will obtain the following result.

Theorem 3 *Assume that all the hypotheses of Theorem 2 hold except for Assumption* (A_2). *Suppose that*

(B_1)

$$\bar{A} < \frac{\rho}{2}\bar{B},$$

where

$$\bar{A} = \liminf_{\xi\to 0^+} \frac{\sum\limits_{k=2}^{T} \max\limits_{|t|\le\xi} F(k,t)}{\xi^2},$$

and

$$\bar{B} = \frac{2}{2+\alpha+(T-1)\beta} \limsup_{\xi \to 0^+} \frac{\sum_{k=2}^{T} F(k,\xi)}{\xi^2}.$$

Then, for each $\lambda \in]\lambda_3, \lambda_4[$ *where*

$$\lambda_3 := \frac{1}{\bar{B}},$$

and

$$\lambda_4 := \frac{\rho}{2\bar{A}},$$

the problem (P_λ^f) *has a sequence of pairwise distinct solutions which strongly converges to* 0 *in* E.

Proof We take Φ and Ψ as in the proof of Theorem 2 and put

$$I_{\bar{\lambda}}(u) = \Phi(u) - \bar{\lambda}\Psi(u),$$

for every $u \in E$. We verify that $\delta < +\infty$. For this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to 0^+$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} \frac{\sum_{k=2}^{T} \max_{|t|\le \xi_n} F(k,t)}{\xi_n^2} < +\infty.$$

Put

$$\bar{A} = \lim_{n\to+\infty} \frac{\sum_{k=2}^{T} \max_{|t|\le \xi_n} F(k,t)}{\xi_n^2},$$

and

$$r_n = \frac{\rho}{2}\xi_n^2,$$

for each $n \in \mathbb{N}$. Therefore, from assumption (B_1), one has

$$\delta \le \liminf_{n\to+\infty} \varphi(r_n) \le \frac{2}{\rho}\bar{A} < +\infty.$$

Let us show that the functional $I_{\bar{\lambda}}$ does not have a local minimum at zero. For this, let $\{\eta_n\}$ be a sequence of positive such that $\eta_n \to 0^+$ as $n \to +\infty$. Put

$$\bar{B} = \frac{2}{2+\alpha+(T-1)\beta} \lim_{n\to 0^+} \frac{\sum_{k=2}^{T} F(k,\eta_n)}{\eta_n^2}. \tag{6}$$

Let $\{w_n\}$ be a sequence in E with $w_n(k) = \eta_n$ for all $k \in [2,T]_{\mathbb{Z}}$. Moreover, from the assumption (A_1) we obtain

$$\Psi(w_n) = \sum_{k=2}^{T} F(k,\eta_n).$$

Then,

$$\begin{aligned} I_{\bar\lambda}(w_n) &= \Phi(w_n) - \bar\lambda\Psi(w_n) \\ &\leq \frac{2+\alpha+(T-1)\beta}{2}\eta_n^2 - \bar\lambda\sum_{k=2}^{T}F(k,\eta_n). \end{aligned}$$

Consider the following cases.
If $\bar B < +\infty$, let $\varepsilon \in]0, \bar B - \frac{1}{\bar\lambda}[$. By (6), there exists ν_ε such that

$$\sum_{k=2}^{T}F(k,\eta_n) > (\bar B - \varepsilon)\frac{2+\alpha+(T-1)\beta}{2}\eta_n^2,$$

for all $n > \nu_\varepsilon$, hence

$$\begin{aligned} I_{\lambda}(w_n) <& \frac{2+\alpha+(T-1)\beta}{2}\eta_n^2 - \bar\lambda(\bar B-\varepsilon)\sum_{k=2}^{T}F(k,\eta_n) \\ =& \frac{2+\alpha+(T-1)\beta}{2}\eta_n^2(1-\bar\lambda(\bar B-\varepsilon)). \end{aligned}$$

Since $1 - \bar\lambda(\bar B - \varepsilon) < 0$, and by considering (4), one has

$$\lim_{n\to+\infty} I_{\bar\lambda}(w_n) = 0.$$

If $\bar B = +\infty$, fix $N_0 > \frac{1}{\bar\lambda}$. There exists ν_{N_0} such that

$$\sum_{k=2}^{T}F(k,\eta_n) > N_0\frac{2+\alpha+(T-1)\beta}{2}\eta_n^2,$$

for all $n > \nu_{N_0}$, and moreover,

$$I_{\bar\lambda}(w_n) < \frac{2+\alpha+(T-1)\beta}{2}\eta_n^2(1-\bar\lambda N_0).$$

Since $1 - \bar\lambda N_0 < 0$, and as above, we can say

$$\lim_{n\to+\infty} I_{\bar\lambda}(w_n) = 0.$$

Since $I_{\bar\lambda} = 0$, this implies that the functional $I_{\bar\lambda}$ does not have a local minimum at zero. Hence, part (c) of Theorem 1 ensures that there exists a sequence $\{u_n\}$ in E of critical points of $I_{\bar\lambda}$ such that $\|u_n\| \to 0$ as $n \to \infty$, and the proof is complete.

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