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Nonsplit Domination Vertex Critical Graph

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Abstract A *dominating set* D of a graph $G = (V, E)$ is a *nonsplit dominating set* if the induced graph $\langle V - D\rangle$ is connected. The *nonsplit domination number* $\gamma_{ns}(G)$ is the minimum cardinality of a *nonsplit domination set.* The purpose of this paper is to initiate the investigation of those graphs which are critical in the following sense: A graph G is called *vertex domination critical* if $\gamma(G-v) < \gamma(G)$ for every vertex v in G. A graph G is called *vertex nonsplit critical* if $\gamma_{ns}(G - v) < \gamma_{ns}(G)$ for every vertex v in G. Initially we test whether some particular classes of graph are γ_{ns}-critical or not and then we have shown that there is no existence of 2-γ_{ns}*-critical* graph. Then 3-γ_{ns}-critical graphs are characterized.

Keywords Domination · Non split domination · Critical graph

Mathematics Subject Classification (2010) 05C69

1 Introduction

Throughout this paper all our graphs will be finite, undirected, connected and without loops or multiple edges such that $G - v, v \in V(G)$ is not a null graph. Terminology not defined here will conform to that in [2].

A *end vertex* in a graph G is a vertex of *degree one* and *support vertex* is a vertex which is adjacent to an *end vertex.* The diameter of the graph G is denoted by $dia(G)$.

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A set of vertices S is said to *dominate* the graph G if for each $v \notin S$, there is a vertex $u \in S$ such that v is adjacent to u. The minimum cardinality of any *dominating set* is called the *domination number* of G and is denoted by $\gamma(G)$.

The concept of nonsplit domination number was introduced by V.R. Kulli and B. Janakiram[3]. A *dominating set* D of a graph $G = (V, E)$ is a *nonsplit dominating set* if the induced graph $\langle V - D \rangle$ is *connected*. The *nonsplit domination number* $\gamma_{ns}(G)$ is the minimum cardinality of a *nonsplit domination set*. The concept of γ-*critical* graphs has been studied by Sumner and Blitch [1] and Sumner [6]. Furthermore the concept of various parameter of critical domination has been studied by [4,7].

In this paper, we study the *nonsplit domination critical* graphs. A graph G is called *vertex nonsplit critical* if $\gamma_{ns}(G - v) < \gamma_{ns}(G)$ for every vertex v in G. Thus, G is k-γ_{ns}-*critical* if $\gamma_{ns}(G) = k$, for each vertex $v \in V(G)$, $\gamma_{ns}(G - v) < k$.

First we discuss whether some particular classes of graphs are γ_{ns}-*critical* or not and then we have shown that there is no existence of 2-γ_{ns}-*critical* graph. Then 3-γ_{ns}-*critical* graphs are characterized.

2 We require the following theorems to prove our later results

In [3] they had proved the following theorems.

Theorem 1 *For any cycle* C_n, $\gamma_{ns}(C_n) = n - 2, n \geq 3$.

Theorem 2 *For any path* P_n, $\gamma_{ns}(P_n) = n - 2, n > 3$, *otherwise* $\gamma_{ns}(P_n) = n - 1, n \leq 3$.

Theorem 3 *For any complete graph* K_n, $\gamma_{ns}(K_n) = 1, n \geq 2$.

Theorem 4 *For any bipartite graph* $K_{m,n}$, $\gamma_{ns}(K_{m,n}) = 2$ *for* $2 \leq m \leq n$.

Theorem 5 *For any star graph* $K_{1,n-1}$, $\gamma_{ns}(K_{1,n-1}) = n - 1$ *for* $n \geq 1$.

3 Main results

Theorem 6 *Let G be a connected graph. For any vertex $v \in V(G)$,*

$$\gamma_{ns}(G) - 1 \leq \gamma_{ns}(G - v) \leq \gamma_{ns}(G) + n - 3, n \geq 3.$$

Proof Let G be a connected graph and $v \in V(G)$. Let D be a γ_{ns}-set of G. Since removal of a vertex can increase the domination number by more than one and decrease by at most one, $\gamma_{ns}(G) - 1 \leq \gamma_{ns}(G - v)$. For the upper bound, let us assume that $\gamma_{ns}(G - v) > \gamma_{ns}(G) + n - 3$. We consider the following two cases.

Case 1. If $\gamma_{ns}(G) = 1$, then there exists a vertex say, $v_1 = v \in D \cap V(G)$ such that $v_1 \in N(V(G) - v_1)$ and since $\langle G - v_1 \rangle$ has to be connected, therefore the

graph $G - v_1 = K_n$ and by Theorem 3, $\gamma_{ns}(G - v_1) = 1 \leq \gamma_{ns}(G) + n - 3$, a contradiction. Hence $\gamma_{ns}(G - v) \leq \gamma_{ns}(G) + n - 3$.
Case 2. If $\gamma_{ns}(G) \geq 2$.
Now, suppose $\gamma_{ns}(G - v) > \gamma_{ns}(G) + n - 3 > 2 + n - 3 = n - 1$ and since $\gamma_{ns}(G) \leq n - 1$, which is a contradiction. Hence $\gamma_{ns}(G - v) \leq \gamma_{ns}(G) + n - 3$.

Theorem 7 *The graph $G = C_n$ is γ_{ns}-critical for $n \geq 5$ and not γ_{ns}-critical for $n < 5$.*

Proof **Case 1.** For $n = 3$.
By Theorem 1, $\gamma_{ns}(G) = 1$ and $G - v$ will be K_2 for any vertex $v \in V(G)$ and by Theorem 3, $\gamma_{ns}(K_2) = 1$. Thus $\gamma_{ns}(G) = \gamma_{ns}(G - v)$. Hence $\gamma_{ns}(G)$ is not critical for $n = 3$.
Case 2. For $n = 4$.
By Theorem 1, $\gamma_{ns}(G) = 2$ and $G - v$ will be $K_{1,2}$ for any vertex $v \in V(G)$ and by Theorem 5, $\gamma_{ns}(K_{1,2}) = 2$. Thus $\gamma_{ns}(G) = \gamma_{ns}(G - v)$. Hence $\gamma_{ns}(G)$ is not critical for $n = 4$.
Case 3. For $n \geq 5$.
By Theorem 1, $\gamma_{ns}(G) = n - 2$ and $G - v$ will be path with $n - 1$ vertices and by Theorem 2, $\gamma_{ns}(P_{n-1}) = n - 3$. Since $n - 3 < n - 2$, $\gamma_{ns}(G - v) < \gamma_{ns}(G)$. Hence $\gamma_{ns}(G)$ is critical for $n \geq 5$.
The result follows from the cases above.

Theorem 8 *The graph P_n is not γ_{ns}-critical for $4 \leq n < 8$ and γ_{ns}-critical for $n \geq 8$.*

Proof We consider the following three cases.
Case 1. For $n = 4$.
By Theorem 2, $\gamma_{ns}(P_n)$=2. If v is an end vertex, then $P_n - v$ will be path P_3 and by Theorem 2, $\gamma_{ns}(P_3) = 2$, Hence $\gamma_{ns}(P_n - v) = \gamma_{ns}(P_n)$. Otherwise if v is a support vertex then, $P_n - v$ will be disconnected into two components $G_1 = v_k, d(v_k) = 0$ and $G_2 = P_2$. Let G_1 and G_2 are the graphs with $V(G_1) = n_1$ and $V(G_2) = n_2$ with $n_1 + n_2 + 1 = n$. Thus $\gamma_{ns}(P_n - v) = \gamma_{ns}(G_1) + \gamma_{ns}(G_2) = (n_1 + n_2 - 1) = 1 + 2 - 1 = 2$. Hence P_n is not γ_{ns}-critical for $n = 3$.
Case 2. For $n > 4$ and $n < 8$.
By Theorem 2, $\gamma_{ns}(P_n) = n - 2$. If v is an end vertex, then $P_n - v$ will be a path with $n - 1$ vertices and by Theorem 2. $\gamma_{ns}(P_{n-1}) = n - 3$. Since $n - 2 < n - 3$, thus $\gamma_{ns}(P_n - v) < \gamma_{ns}(P_n)$. Otherwise if v is an not an end vertex, then the graph $P_n - v$ is disconnected into components say G_1 and G_2. Let V_1 and V_2 be the vertex set of G_1 and G_2 with $|V(G_1)| \leq |V(G_2)|$ and $n_1 + n_2 + 1 = n$. If G_1 and G_2 both contains the number of vertices ≤ 3, then $\gamma_{ns}(P_n - v) = n_1 + n_2 - 1 = n - 2$. Hence, $\gamma_{ns}(P_n - v) = \gamma_{ns}(P_n)$. Otherwise there exists atleast one of G_1 or G_2 contain the number of vertices > 3, then $\gamma_{ns}(P_n - v) = n_1 + n_2 - 2 = n - 3$. Thus $\gamma_{ns}(P_n - v) < \gamma_{ns}(P_n)$. Hence P_n is not γ_{ns}-critical for $n < 8$ and $n > 4$.
Case 3. For $n \geq 8$.
By Theorem 2, $\gamma_{ns}(P_n) = n - 2$. If v is an end vertex, then $P_n - v$ will be

a path with $n-1$ vertices and by Theorem 2. $\gamma_{ns}(P_n - v) = n-3$. Hence $\gamma_{ns}(P_n - v) < \gamma_{ns}(P_n)$. Otherwise if v is an not an end vertex, then the graph $P_n - v$ is disconnected into two components say G_1 and G_2. Let V_1 and V_2 be the vertex set of G_1 and G_2 with $|V(G_1)| \leq |V(G_2)|$ and $n_1 + n_2 + 1 = n$ with either one $|V(G_1)|$ or $|V(G_2)| \geq 4$. Hence $\gamma_{ns}(P_n - v) = n_1 + n_2 - 2 = n-3$. Thus $\gamma_{ns}(P_n - v) < \gamma_{ns}(P_n)$. Hence P_n is γ_{ns}-critical for $n \geq 8$.
This completes the proof.

Lemma 1 *K_n is not γ_{ns} critical for $n \geq 2$.*

Lemma 2 *$K_{m,n}$ is not γ_{ns} critical for $m, n \geq 1$.*

Lemma 3 *W_n is not γ_{ns} critical for $n \geq 4$.*

Theorem 9 *If T is a tree which in not a path, then T is not γ_{ns}-vertex critical.*

Proof Let us consider the tree T which is not a path. Let A be the vertex set of a tree T. Let $B = \{v_i \in V(G)/deg(v_i) = 1\} \subseteq A$ and $C = A - B$. We consider the following two cases.
Case 1. If $v_i \in B$. Then $\gamma_{ns}(T - v_i) = \gamma_{ns}(T) - |v_i|$. Hence,

$$\gamma_{ns}(T - v_i) < \gamma_{ns}(T).$$

Case 2. If v_i is a non-end vertex $\in C$, then $T - v_i$ is disconnected into components say $T_{11}, T_{12}, T_{13}, T_{14}, \ldots, T_{1n}$. Let $D_1, D_2, D_3, \ldots, D_n$ be a γ_{ns}-set of $T_{11}, T_{12}, T_{13}, T_{14}, \ldots, T_{1n}$ and let $n_1, n_2, n_3, \ldots, n_n$ be the number of vertices in $T_{11}, T_{12}, T_{13}, T_{14}, \ldots, T_{1n}$ such that $n_1 + n_2 + n_3 + \cdots + n_n + 1 = n$. Let T_{11} be the subtree corresponding to max of $(n_i - D_i), i = 1, 2, 3, \ldots, n$. Then,

$$\begin{aligned}
\gamma_{ns}(T - v_i) &= \gamma_{ns}(T_{11}) + n_1 + n_2 + \cdots + n_n \\
&= |D_{11}| + n_1 + n_2 + \cdots + n_n \\
&\geq \gamma_{ns}(T_{11}) + \gamma_{ns}(T_{12}) + \gamma_{ns}(T_{13}) + \cdots + \gamma_{ns}(T_{1n}) \\
&\geq \gamma_{ns}(T).
\end{aligned}$$

Hence, T which in not a path is not γ_{ns}-vertex critical.
This completes the proof.

Corollary 1 *If in a graph G, every vertex in adjacent to an end vertex, then the graph G is not γ_{ns}-critical.*

Theorem 10 *There exists no 2-γ_{ns}-critical graph with respect to vertex non-split domination.*

Proof For the existence of 2-$\gamma_{ns}-$critical graph with n vertices, we need at most $n-1$ vertices of degree $n-2$ and removal of any vertex should make the degree of each vertex of equal to $n-2$ which is impossible. Therefore, there is no existence of 2-$\gamma_{ns}-$critical graph with respect to vertex nonsplit domination.

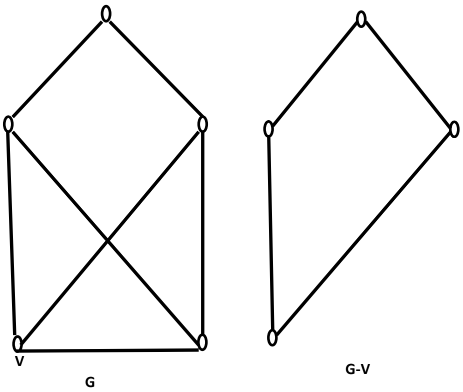

Fig. 1 Example for no existence of nonsplit domiantion vertex critical graph

Theorem 11 *If a graph is* 3-γ_{ns}*-critical, then* $dia(G) < 3$.

Proof Let G be a connected 3-γ_{ns}-critical graph and suppose that G has a $dia(G) = 3$. Assume that $v_1, v_2, \ldots, v_d$ be the longest diametrical path with the distance equal to the diameter of G. Since G is 3-γ_{ns}-critical graph, then $\gamma_{ns}(G - v) = 2$ for any vertex $v \in G$. Let D be a γ_{ns} set of G and D_1 be a γ_{ns} set of $G - v$. Since $|D_1| = 2$ and if suppose $v_1, v_d \in D_1$, then removal of v_1, v_d from $G - v$ will make the graph $G - v$ disconnected, which is a contradiction to the definition of the nonsplit domination of the graph. Otherwise there exists at least one vertex v_k in the graph $G - v$ which is not dominated by any of the vertices of D_1 which is a contradiction. Hence the proof.

Theorem 12 *If* $G \neq P_n$ *is* γ_{ns}*-critical, then there is no support vertex* $v \in G$.

Proof Suppose v is a support vertex which is adjacent to an end vertex, say x of a graph $G \neq P_n$. Let D be a γ_{ns}-set of G and D^1 be a γ_{ns}-set of $G - v$. Since x is vertex of degree one, then $x \in D$ in G and in $G - v$, x is an isolated vertex and thus $x \in D$.

Case 1. $d(v) = 2$.

If $v \in D$ and $A = \{v_k / v_k \in N(v) - \{x\}\}$ and $v_k \in D$, then there exists at least one vertex $v_j \in V(G) - D, v_j \notin D, v_j \neq x$, then we can remove $v_l, v_l \in N(v_j) \cap D$, removal of v_l from D, there exists at least one vertex which is not covered by any vertex of $D - v_l$ or $G - v_l$ is disconnected. Thus $\gamma_{ns}(G - v) = \gamma_{ns}(G)$. Otherwise if at least one vertex of $v_k \notin D$, then v_k is not covered by any vertex of $D - \{v\}$. Hence $\gamma_{ns}(G - v) = |D| - |v| + |v_k| = |D| = \gamma_{ns}(G)$. Otherwise, If $v \notin D$, then $G - v$ will be disconnected into two components G_1 and G_2 and $\gamma_{ns}(G_1) + \gamma_{ns}(G_2) = \gamma_{ns}(G)$. Hence $\gamma_{ns}(G - v) = \gamma_{ns}(G_1) + \gamma_{ns}(G_2) = \gamma_{ns}(G)$. Hence it is not γ_{ns}-critical.

Case 2. $d(v) \neq 2$.

If $v \in D$ and $A = \{v_k / v_k \in N(v) - \{x\}\}$ and $v_k \in D$, then there exists at least one vertex $v_j \in (V(G) - D), v_j \notin D, v_j \neq x$, then we can remove $v_l, v_l \in N(v_j) \cap D$, removal of v_l from D, there exists at least one vertex which is not covered by any vertex of $D - v_l$ or $G - v_l$ is disconnected. Thus,

$$\gamma_{ns}(G - v) = \gamma_{ns}(G).$$

Otherwise, if at least one vertex say $v_r \in A, v_r \notin D$, then v_r is not covered by any vertex of $D - \{v\}$. Hence,

$$\gamma_{ns}(G - v) \geq |D| - |v| + |v_r| = |D| = \gamma_{ns}(G).$$

Otherwise, If $v \notin D$, then $G - v$ will be disconnected into components say $G_1, G_2, \ldots, G_n$ and $\gamma_{ns}(G_1) + \gamma_{ns}(G_2) + \cdots + \gamma_{ns}(G_n) \geq \gamma_{ns}(G)$. Hence,

$$\gamma_{ns}(G - v) \geq \gamma_{ns}(G_1) + \gamma_{ns}(G_2) + \cdots + \gamma_{ns}(G_n) = \gamma_{ns}(G).$$

Hence it is not γ_{ns}-critical.
The results follows from cases above.

Corollary 2 *If $G \neq P_n$ is γ_{ns}-critical, then no two support vertices are adjacent.*

Theorem 13 *For any graph $G \neq P_n$ and does not contain support vertex, if $\kappa(G) = 1$, then the graph is not γ_{ns}- vertex critical.*

Proof Let $G \neq P_n$ has the vertex connectivity one and let D be a γ_{ns}-set of G. Let v be the cut-vertex which disconnects G into two components G_1 and G_2 and let D_1 and D_2 are γ_{ns}-set of G_1 and G_2. Let n_1 and n_2 be the number of vertices in G_1 and G_2. In the graph G, v is a cut-vertex $v \notin D$. Then,

$$\gamma_{ns}(G - v) = min\{|D_1| + n_2, |D_2| + n_1\} \geq |D_1| + |D_2|.$$

Since, $|D_1| + |D_2| \geq |D|$., hence, the graph G is not γ_{ns}-critical.
This completes the proof.

4 Application

Let us consider the two Military groups say A and B which are interconnected with each other with group members as the vertices and the edges as the communication with them. Among these two groups, there are minimum number of people who had the communication with all the members of the two group which are called domination members, among them few are having communication between two groups also. Since the two military groups are connected with each other, they will form a strong military base.

Suppose if the terrorist people thinks to make the military base to become inactive, they may think to destroy the domination members in such a way that the two groups gets separated so that there is no communication between the groups and the also between the members of the groups. In such a case, military people can take a precautionary measure in such a way that even if they destroy the domination members, still they can have a communication with the groups and also within the group. This is the purpose for studying the nonsplit domination.

Now, if we can build the network in such way that if it is critical ie., any one of the members in a network is not available in a group for some reasons, then the domination members can be reduce by atleast one.

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