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Conformity of Fractional Volterra Integro-Differential Equation Solution with an Integral Equation of Fractional Order

Leila Torkzadeh · Somayyeh Dadsetadi

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Abstract For the feasibility of using analytical and numerical studies and findings on fractional integral equations for integro-differential of the fractional order, in this work, the equivalence of a fractional Volterra integro-differential equation of the Hammerstein type with a fractional integral equation is investigated in the Banach space. For this purpose, we use the mutual properties of the fractional order derivative and integral on each other.

Keywords Fractional integro-differential equation · Hammerstein equation · Riemann-Liouville fractional integral · Caputo derivative

Mathematics Subject Classification (2010) 34A08 · 26A33 · 45D05 · 45G10

1 Introduction

Fractional calculus is an extension of the classical calculus and has recently found many applications in various fields of science and engineering [4–6, 13, 15, 17]. For example, studies in [8] cover the latest developments in the field of fractional dynamics, and [11] concerns fractional calculus's applications in viscoelasticity dynamics. Also, there are several papers about the applications of fractional calculus in complex dynamics in biological tissues, signal processing, viscoelastic materials, temperature estimation, and financial mathematics

L. Torkzadeh (Corresponding Author)
Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, P. O. Box 35195-363, Semnan, Iran.
Tel.: +98-23-31532069
Fax: +98-23-31535700
E-mail: torkzadeh@semnan.ac.ir

S. Dadsetadi
Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, P. O. Box 35195-363, Semnan, Iran.

(for example, see [2,3,9,10,12,14,16]).
In this paper, we study the following nonlinear fractional Volterra integro-differential equation of the Hammerstein type

$$\left({}^{C}D^{\alpha}y\right)(x) = g(x) + \int_0^x k(x,t)\ G\left({}^{C}D^{\beta}y(t)\right)dt, \qquad x \in [0,a], \tag{1}$$

subject to the initial conditions

$$y^{(i)}(0) = y_i, \quad i = 0,1,...,m-1, \tag{2}$$

where a is a positive finite constant, for $m,n \in \mathbb{N}$, $m-1 < \alpha < m$, $n-1 < \beta < n$, $\beta < \alpha$, and the fractional derivatives are considered in the Caputo sense.
We show in Section 3 that the problem (1)-(2) is equivalent to an integral equation of fractional order. Therefore, with some considerations, it is possible that theorems and results obtained for each of the equations can also be generalized for other equations.

2 Preliminaries

In this section we give the preliminary concepts [7,15], that are used in Section 3. The following lemma is a result of Lemma 1.3 in [7] which characterizes the space $C^n[0,a]$.

Lemma 1 *Let $n \in \mathbb{N}_0$. The space $C^n[0,a]$ consists of those and only those functions f which are represented in the form*

$$f(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\varphi(t)dt + \sum_{i=0}^{n-1} c_i x^i,$$

where $\varphi(t) \in C[0,a]$ and c_i $(i = 0,1,...,n-1)$ are appropriate constants, moreover,

$$\varphi(t) = f^{(n)}(t), \quad c_i = \frac{f^{(i)}(0)}{i!} \quad (i = 0,1,...,n-1).$$

Definition 1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y(x)$, is defined as

$$\left(I^{\alpha}y\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\, y(\tau)d\tau, \qquad x > 0, \tag{3}$$

where Γ is the Gamma function.

For $y(x) = sin(2x)$, Figures 1 and 2 show the Riemann-Liouville fractional integral of orders $\alpha = 0.25, 0.5, 0.75, 0.95$ and $\alpha = 1.25, 1.5, 1.75, 1.95$, respectively.

The Riemann-Liouville fractional integral has the following properties:

$$\left(I^{\alpha}I^{\beta}y\right)(x) = \left(I^{\alpha+\beta}y\right)(x), \tag{4}$$

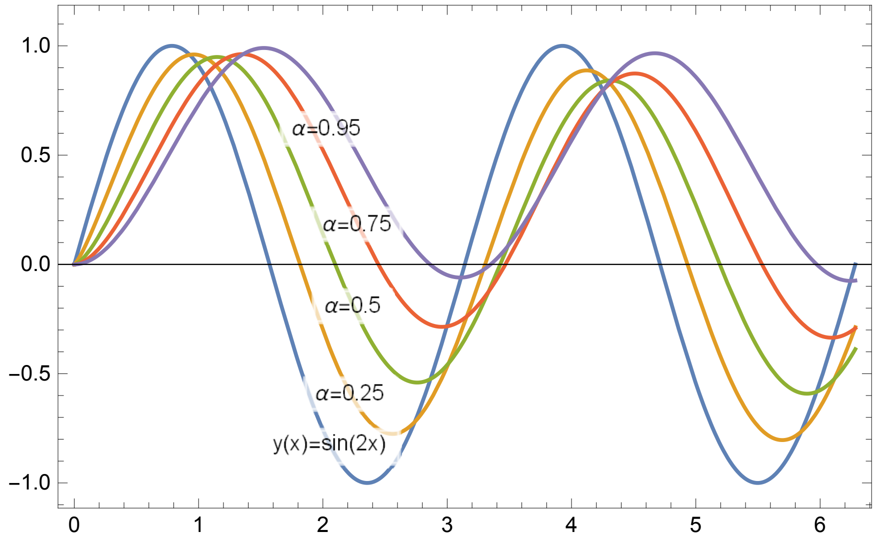

Fig. 1 Function $y(x) = sin(2x)$ and its Riemann-Liouville fractional integrals with $\alpha = 0.25, 0.5, 0.75, 0.95$.

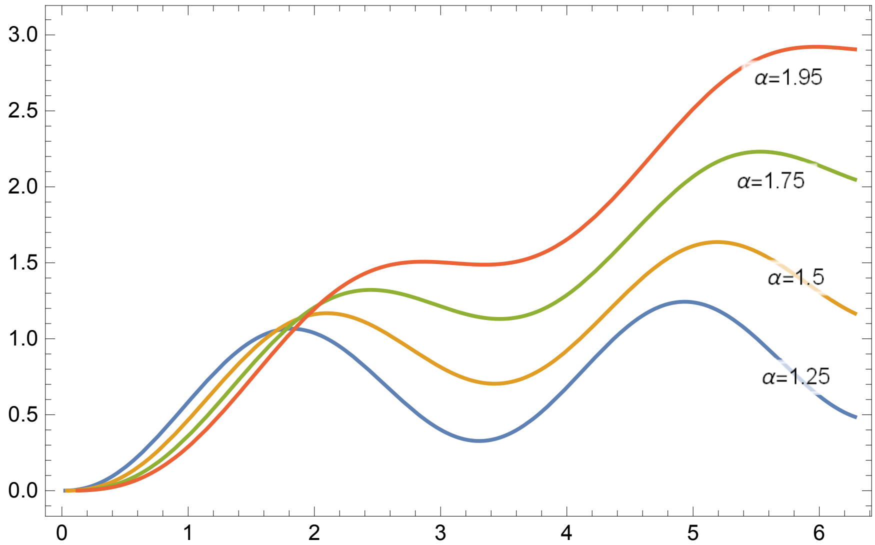

Fig. 2 The Riemann-Liouville fractional integrals of $sin2x$ with $\alpha = 1.25, 1.5, 1.75, 1.95$.

$$\left(I^{\alpha}\left(t-a\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(x-a\right)^{\beta+\alpha-1}, \tag{5}$$

where $\beta > 0$.

Definition 2 The Caputo derivative of fractional order $\alpha \geq 0$ on $[0, a]$ for a function $y(x)$ is defined by

$$\left({}^{C}D^{\alpha}y\right)(x) = \frac{1}{\Gamma(r-\alpha)}\int_0^x (x-\tau)^{r-\alpha-1}\, y^{(r)}(\tau)d\tau,$$

where

$$r = [\alpha] + 1 \quad \text{for} \quad \alpha \notin \mathbb{N}_0; \quad r = \alpha \quad \text{for} \quad \alpha \in \mathbb{N}_0, \tag{6}$$

which $[\alpha]$ means the integer part of the real number α.

The behavior of the Caputo derivatives of function $y(x) = sin(2x)$ for fractional orders $\alpha = 0.25, 0.5, 0.75, 0.95$ and $\alpha = 1.25, 1.5, 1.75, 1.95$, are shown in Figures 3 and 4, respectively. The Caputo derivative has the following

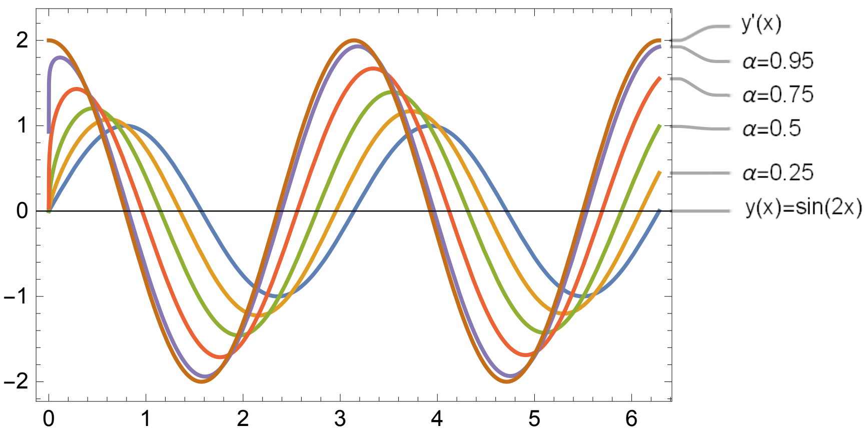

Fig. 3 Function $y(x) = sin(2x)$ and its Caputo derivatives with $\alpha = 0.25, 0.5, 0.75, 0.95$.

properties:

$$\left({}^{C}D^{\alpha}\,(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\,(x-a)^{\beta-\alpha-1}, \qquad \beta > r, \tag{7}$$

$$\left({}^{C}D^{\alpha}\,(t-a)^{i}\right)(x) = 0, \quad i = 0, 1, ..., r-1, \tag{8}$$

where $\alpha, \beta > 0$ and r is given by relation (6).

Proposition 1 *Let $\alpha > 0$ and r is given by relation* (6)*. If $y(x) \in C^r[0, a]$, then*

(i) $\left(I^{\alpha}\ {}^{C}D^{\alpha}y\right)(x) = y(x) - \sum_{i=0}^{r-1} \dfrac{y^{(i)}(0)}{i!}x^i,$

(ii) $\left({}^{C}D^{\alpha}\ I^{\alpha}y\right)(x) = y(x).$

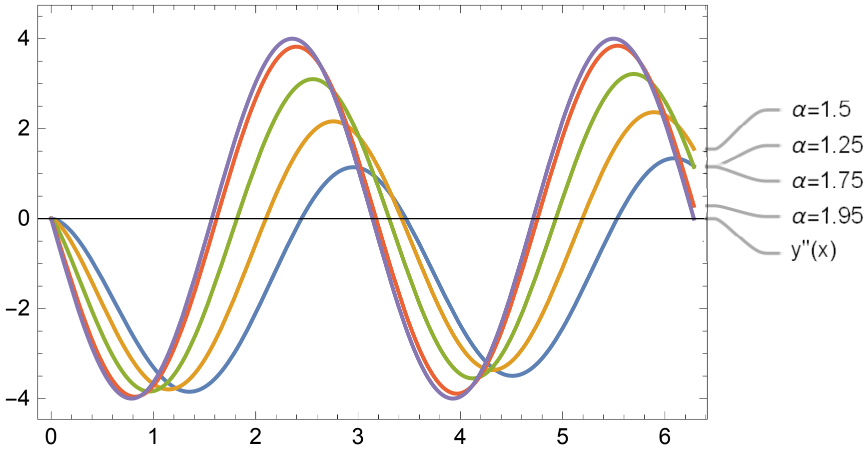

Fig. 4 The Caputo derivatives of $sin 2x$ with $\alpha = 1.25, 1.5, 1.75, 1.95$.

Lemma 2 *([1]). Let $m-1 < \alpha < m$, $n-1 < \beta < n$ and $\beta < \alpha$. For $x \in [0, a]$,*
(i) if $y(x) \in C[0, a]$, then

$$\left({}^{C}D^{\beta} I^{\alpha} y\right)(x) = \left(I^{\alpha-\beta} y\right)(x),$$

(ii) if $y \in C^{m-1}[0, a]$ and $\left({}^{C}D^{\alpha} y\right)(x) \in C[0, a]$, then $\left({}^{C}D^{\beta} y\right)(x) \in C[0, a]$.

3 Equivalence of equations

In the following theorem, we show that problem (1) - (2) is equivalent to a fractional integral equation.

Theorem 1 *Let g, k and G be continuous functions and $m-1 < \alpha < m$, $n-1 < \beta < n$ and $\beta < \alpha$. Then a function $y \in C^{m-1}[0, a]$ with $\left({}^{C}D^{\alpha} y\right)(x) \in C[0, a]$ is a solution of fractional integro-differential equation* (1) *if and only if*

$$y(x) = \sum_{i=0}^{n-1} \frac{y_i}{i!} x^i + \frac{1}{\Gamma(\beta)} \int_0^x \frac{u(s)}{(x-s)^{1-\beta}} ds, \tag{9}$$

where for $n \leq m$, $u \in C[0, a]$ satisfies the integral equation

$$u(x) = \sum_{i=n}^{m-1} \frac{y_i}{\Gamma(i-\beta+1)} x^{i-\beta} + I^{\alpha-\beta} g(x) + I^{\alpha-\beta} \int_0^x k(x,t)\ G(u(t)) dt. \tag{10}$$

Proof Let $y \in C^{m-1}[0, a]$ be a solution of (1) which $\left({}^{C}D^{\alpha} y\right)(x) \in C[0, a]$. Using Lemma 2, we conclude $\left({}^{C}D^{\beta} y\right)(x) \in C[0, a]$. Since g, k, G and $\left({}^{C}D^{\beta} y\right)(x)$ are continuous, we can apply the operator I^{α} to both sides of Eq. (1). Thus using Proposition 1, we obtain

$$y(x) = \sum_{i=0}^{m-1} \frac{y^{(i)}(0)}{i!} x^i + I^{\alpha} g(x) + I^{\alpha} \left(\int_0^x k(x,t)\ G\left({}^{C}D^{\beta} y(t)\right) dt \right). \tag{11}$$

Putting $\left({}^{C}D^{\beta}y\right)(x) := u(x)$, so $u \in C\,[0,a]$, and we can apply the operator I^{β} to both sides of this relation and using Proposition 1, we get

$$y(x) = \sum_{i=0}^{n-1} \frac{y_i}{i!}x^i + \frac{1}{\Gamma(\beta)}\int_0^x \frac{u(s)}{(x-s)^{1-\beta}}ds.$$

From (11) and Lemma 2, we have

$${}^{C}D^{\beta}y(x) = {}^{C}D^{\beta}\left(\sum_{i=0}^{m-1}\frac{y_i}{i!}x^i\right) + I^{\alpha-\beta}g(x) + I^{\alpha-\beta}\left(\int_0^x k(x,t)\ G\left({}^{C}D^{\beta}y(t)\right)dt\right). \tag{12}$$

Using relations (7) and (8), we get

$$u(x) = \sum_{i=n}^{m-1}\frac{y_i}{\Gamma(i-\beta+1)}x^{i-\beta} + I^{\alpha-\beta}g(x) + I^{\alpha-\beta}\int_0^x k(x,t)\ G(u(t))dt,$$

for $n = m$, the first term of the right hand of above relation is equal to zero.Conversely, assume that $u \in C\,[0,a]$ is a solution of Eq. (10), we show that the function $y(x)$, defined by relation (9), satisfies in Eq. (1). Since $u \in C\,[0,a]$, we can apply the operator ${}^{C}D^{\beta}$ on both sides of Eq. (9), then from Proposition 1 and Eq. (8), we obtain

$$\left({}^{C}D^{\beta}y\right)(x) = u(x),$$

and hence $\left({}^{C}D^{\beta}y\right)(x) \in C\,[0,a]$. Applying I^{β} on both sides of Eq. (10) and using Eqs. (4), (5) and Proposition 1, we get

$$y(x) = \sum_{i=0}^{m-1}\frac{y_i}{i!}x^i + I^{\alpha}g(x) + I^{\alpha}\left(\int_0^x k(x,t)\ G\left({}^{C}D^{\beta}y(t)\right)dt\right). \tag{13}$$

Also, from the continuity of $\left({}^{C}D^{\beta}y\right)(x)$, g, k and G, Eq. (8) and applying the operator ${}^{C}D^{\alpha}$ on both sides of Eq. (13), we have

$$\left({}^{C}D^{\alpha}y\right)(x) = g(x) + \int_0^x k(x,t)\ G\left({}^{C}D^{\beta}y(t)\right)dt,$$

and consequently $\left({}^{C}D^{\alpha}y\right)(x) \in C\,[0,a]$.

Now we show that $y^{(i)}(0) = y_i$ $(i = 0,1,...,m-1)$. First using the property of the fractional calculus, we obtain

$$\begin{aligned} |\left(I^{\alpha}y\right)^{(i)}(x)| = |\left(D^{i}I^{\alpha}y\right)(x)| &= |\left(I^{\alpha-i}y\right)(x)| \\ &= \left|\frac{1}{\Gamma(\alpha-i)}\int_0^x (x-s)^{\alpha-i-1}y(s)ds\right| \\ &\le \frac{\|y\|_C}{\Gamma(\alpha-i+1)}x^{\alpha-i}, \end{aligned}$$

for $i = 0, 1, ..., m-1$, thus

$$(I^{\alpha} y)^{(i)}(0) = 0, \quad i = 0, 1, ..., m-1. \tag{14}$$

Later on, for $m = 1$ according to Eq. (13), we have

$$y(x) = y_0 + I^{\alpha} g(x) + I^{\alpha} \left(\int_0^x k(x,t)\ G\left({}^C D^{\beta} y(t)\right) dt \right).$$

Using the continuity of the operator I^{α} on $C[0,a]$ and Eq. (14), we find $y(x) \in C[0,a]$ and $y(0) = y_0$ $(y^{(0)}(x) = y(x))$.
Now, for $m \geq 2$, by Eqs. (4) and (13), we have

$$\begin{aligned} y(x) = \sum_{i=0}^{m-2} \frac{y_i}{i!} x^i + I^{m-1} \Big[& y_{m-1} + I^{\alpha-m+1} g(x) \\ & + I^{\alpha-m+1} \left(\int_0^x k(x,t)\ G\left({}^C D^{\beta} y(t)\right) dt \right) \Big]. \end{aligned}$$

Thus from Lemma 1, we have $y(x) \in C^{m-1}[0,a]$, $y^{(i)}(0) = y_i$ for $i = 0, 1, ..., m-2$ and

$$y^{(m-1)}(x) = y_{m-1} + I^{\alpha-m+1} g(x) + I^{\alpha-m+1} \left(\int_0^x k(x,t)\ G\left({}^C D^{\beta} y(t)\right) dt \right).$$

In the same way of Eq. (14), we can show that

$$\left[I^{\alpha-m+1} \left(\int_0^x k(x,t)\ G\left({}^C D^{\beta} y(t)\right) dt \right) \right](0) = 0, \qquad I^{\alpha-m+1} g(0) = 0.$$

Therefore, $y^{(m-1)}(0) = y_{m-1}$ and the proof is complete.

4 Conclusion

This paper proved that the nonlinear fractional Volterra integro-differential equation of the Hammerstein type is equivalent to an integral equation. For this purpose, we used the relations governing the Caputo fractional derivatives and the Riemann-Liouville fractional integral. Therefore, in future works, it is possible to analyze the integro-differential equations, such as the existence and uniqueness of the solution, using its corresponding integral equation.

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