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Dominated Coloring of Certain Graphs

Fatemeh Choopani · Abbas Jafarzadeh · Doost Ali Mojdeh

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Abstract A proper coloring of a graph G is called a dominated coloring whenever each color class is dominated by at least one vertex. The minimum number of colors among all dominated colorings of G is called its dominated chromatic number, denoted by $\chi_{dom}(G)$. We define a parameter related to dominated coloring, namely dominated chromatic covering. For a minimum dominated coloring of G, a set of vertices S is called a dominated chromatic covering if each color class is dominated by a vertex of S. The minimum cardinality of a dominated chromatic covering of G is called its dominated chromatic covering number, denoted by $\theta_{\chi_{dom}(G)}$. It is clear that $\theta_{\chi_{dom}}(G) \leq \chi_{dom}(G)$. In this paper, we obtain the dominated chromatic number and $\theta_{\chi_{dom}(G)}$ when G is middle and total graph of paths and cycles.

Keywords Dominated coloring · Dominated chromatic covering · Dominated chromatic covering number

Mathematics Subject Classification (2010) 05C69 · 05C15

1 Introduction

Let $G = (V, E)$ be a graph of order n with vertex set $V = V(G)$ and edge set $E = E(G)$. The *complement* of a graph G denoted by $\overline{G}$ is a graph with the

F. Choopani
Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.
E-mail: fateme.choopani@mail.um.ac.ir

A. Jafarzadeh (Corresponding Author)
Department of Mathematics, Quchan University of Technology, Quchan, Iran.
Tel.: +98-51-47017358
E-mail: jafarzadeh.a@qiet.ac.ir, abbas.jafarzadeh@gmail.com

D. A. Mojdeh
Department of Mathematics, University of Mazandaran, Babolsar, Iran.
E-mail: damojdeh@umz.ac.ir

vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = \{uv \mid u, v \in V(G)$ and $uv \notin E(G)\}$. The clique number $\omega(G)$ of a graph G is the maximum order of a complete subgraph in G (See [11]).

A subset $S \subseteq V$ is called a *dominating set* of G if every vertex in $V - S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G. A set $S \subseteq V$ is called a *total dominating set* of G if every vertex of V is adjacent to some vertex in S. The *total domination number* of a graph G is the cardinality of a smallest total dominating set, denoted by $\gamma_t(G)$, We refer to such a set as a $\gamma_t(G)$-set.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. We introduce three well-known familiar graphs obtained from the given graph G.

1. The graph $S(G)$ is a graph obtained from G by subdividing each edge exactly once. In other words, $S(G)$ is a subdivision of graph G with vertex set $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m\}$ where u_i is a vertex that subdivides the edge e_i and $e \in E(S(G))$ if and only if $e = v_l u_r$ where v_l is a vertex of e_r in G.
2. The middle graph $M(G)$ of a graph G is defined as a graph with vertex set $V \cup E$ and two vertices x and y of $M(G)$ are adjacent in $M(G)$ if either x and y are adjacent edges in G or x is a vertex in G, y is an edge of G and x is incident to y in G. In the other words, $M(G)$ is a graph with $V(M(G)) = V(S(G))$ and $E(M(G)) = E(S(G)) \cup \{e = e_i e_j \mid e_i$ and e_j have a common vertex in $G\}$.
3. The total graph $T(G)$ of a graph G is a graph with the vertex set $V \cup E$ in which two vertices x and y of $T(G)$ are adjacent in $T(G)$ if either they are adjacent vertices or adjacent edges in G or x is a vertex in G, y is an edge of G and x is incident on y in G. In other words, $V(T(G)) = V(M(G))$ and $E(T(G)) = E(M(G)) \cup E(G)$.

Without loss of generality, we can assume the set of vertices of middle and total graphs as a sequence of vertices in the form of $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m$ where u_i is a representative of edge e_i.

The k-th power of G, G^k, is a graph whose vertex set is $V(G)$ and two vertices in G^k are adjacent if their distance in G is at most k. The graph G^2 is also referred to as the square of G.

A *proper coloring* of a graph G is an assignment of colors to the vertices of G such that two adjacent vertices receive different colors. A proper coloring of G with k colors is also called a k-proper coloring of G. The minimum number of colors required for a proper coloring of G is said to be the *chromatic number* $\chi(G)$ of G. In [6], Merouane et al. defined the dominated coloring of a graph as follows. A k-*dominated coloring* of G is a proper k-coloring of G with color classes $C_1, C_2, \ldots, C_k$ such that for each i $(1 \leq i \leq k)$, there exists a vertex $u \in V$ such that $C_i \subseteq N(u)$ (i.e. vertices in C_i are dominated by vertex u); such vertex u is called a dominating vertex. The minimum number of colors among all dominated colorings of G is called its dominated chromatic number, denoted by $\chi_{dom}(G)$. Obviously, a graph has a dominated coloring if it has no isolated vertices. Hereafter, all graphs in this paper are assumed to have no isolated vertex. The k-dominated coloring has also been studied in [2].

Now, we introduce a parameter related to dominated coloring, namely dominated chromatic covering that is defined as follows.

Definition 1 Let $C_1, C_2, \ldots, C_{\chi_{dom}}$ be the color classes of a minimum dominated coloring of G. A set $S \subseteq V$ is called a *dominated chromatic covering* of graph G if every C_i is dominated by a vertex in S. The minimum cardinality of such set S is called dominated chromatic covering number of G, denoted by $\theta_{\chi_{dom}(G)}$, and the set S is called a $\theta_{\chi_{dom}}(G)$-set.

2 Dominated chromatic number and dominated chromatic covering number of $M(G)$

In this section, we study the dominated chromatic number and dominated chromatic covering number of graphs $M(P_n)$ and $M(C_n)$ of the n-path P_n and the n-cycle C_n.

Theorem 1 *For all $n \geq 2$, we have $\chi_{dom}(M(P_n)) = n$ and*

$$\theta_{\chi_{dom}}(M(P_n)) = \begin{cases} \frac{2n}{3}, & n \equiv 0 \ (mod\ 3), \\ \lceil \frac{2n}{3} \rceil, & n \equiv 1 \ (mod\ 3), \\ \lfloor \frac{2n}{3} \rfloor, & n \equiv 2 \ (mod\ 3). \end{cases}$$

Proof By the structure of $M(P_n)$, it can be seen that no three non-adjacent vertices have a common neighbor and therefore, no three vertices receive the same color and each color class has at most two vertices. See Figure 1 for $n = 8$. Since $M(P_n)$ contains $2n-1$ vertices, $\chi_{dom}(M(P_n)) \geq \lceil (2n-1)/2 \rceil = n$. Now, we need to reveal that $\chi_{dom}(M(P_n)) \leq n$. Regard the following two cases:

Case 1. $n \equiv 0, 1 \ (mod\ 3)$. In this case, we define a coloring function $C_{M(P_n)}$ as $C(M(P_n))(v_i) = i$ for all $1 \leq i \leq n$ and

$$C_{M(P_n)}(e_i) = \begin{cases} i+2, & i \equiv 1 \ (mod\ 3) \ \&\ 1 \leq i \leq n-1, \\ i-1, & i \equiv 0, 2 \ (mod\ 3) \ \&\ 1 \leq i \leq n-1. \end{cases}$$

Therefore, $M(P_n)$ is n-dominated colorable and so $\chi_{dom}(M(P_n)) \leq n$. Hence $\chi_{dom}(M(P_n)) = n$.

Case 2. $n \equiv 2 \ (mod\ 3)$. In this case, we designate a coloring function $C_{M(P_n)}$ as

$$C_{M(P_n)}(v_i) = \begin{cases} i, & 1 \leq i < n, \\ n-1, & i = n, \end{cases}$$

and

$$C_{M(P_n)}(e_i) = \begin{cases} i-1, & i \equiv 0, 2 \ (mod\ 3) \ \&\ 1 \leq i < n-1, \\ i+2, & i \equiv 1 \ (mod\ 3) \ \&\ 1 \leq i < n-1, \\ n, & i = n-1. \end{cases}$$

Hence, $\chi_{dom}M(P_n) \leq n$. So we have $\chi_{dom}(M(P_n)) = n$. Now we compute $\theta_{\chi_{dom}}(M(P_n))$. For the coloring of $M(P_n)$ which was mentioned above, the set $B_1 = \{e_1, e_2, \ldots, e_{3i+1}, e_{3i+2}, \ldots, e_{n-1}\}$ can be contemplated as a dominated chromatic covering where $n \equiv 0, 2$ (mod 3).
So, in this case $\theta_{\chi_{dom}}M(P_n) \leq |B_1| \leq \lfloor \frac{2n}{3} \rfloor$. Also

$$B_2 = \{e_1, e_2, \ldots, e_{3i+1}, e_{3i+2}, \ldots, e_{n-2}, v_n\},$$

is a dominated chromatic covering of $M(P_n)$ if $n \equiv 1$ (mod 3). So in this case $\theta_{\chi_{dom}}M(P_n) \leq |B_2| \leq \lceil \frac{2n}{3} \rceil$.

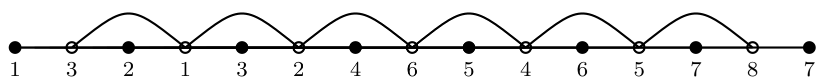

Fig. 1 Dominated coloring of $M(P_8)$. (The vertices corresponding to $V(G)$ and $E(G)$ are shown by • and ∘, respectively.)

We demonstrate that for $n \geq 2$, if $n \equiv 0, 2$ (mod 3), $\theta_{\chi_{dom}}(M(P_n)) \geq \lfloor \frac{2n}{3} \rfloor$ and if $n \equiv 1$ (mod 3), then $\theta_{\chi_{dom}}(M(P_n)) \geq \lceil \frac{2n}{3} \rceil$. As we alluded to previously, there isn't any color class of size 3 or more in the n-dominated coloring of $M(P_n)$. Now, we contrarily suppose that there are $a \geq 2$ color classes of size one. Hereon, $\chi_{dom}(M(P_n)) \geq \lceil \frac{2n-a-1}{2} \rceil + a \geq n + 1$, a contradiction. Hence we have at most one color class of size one, and since the number of vertices of $M(P_n)$ is $2n - 1$ vertices, we have exactly one color class of size one. One can conclude that no vertex covers three color classes unless the vertex belongs to a color class with only one vertex. Therefore, at least two vertices need for covering three color classes of size greater than one. We claim that any vertex v belongs to a color class of size one covers at most two color classes. If v is a vertex of degree 1 or 2, the claim is clear. If it has degree 3, since there is only one color class of size one, then this vertex dominates at most 2 color classes. Now we assume that the vertex v has degree 4. In fact, v is a corresponding vertex to an edge e_i for $2 \leq i \leq n - 2$ in path P_n. Whereby, if vertices v_i, v_{i+1}, or u_{i-1}, u_{i+1} don't belong to the same color class, then obviously v dominates less than three color classes. Now we contrary assume that vertices v_i, v_{i+1}, and also u_{i-1}, u_{i+1} are in the same color class, hence the set of vertices $\{v_1, e_1, v_2, e_2, \ldots, e_{i-2}, v_{i-1}\}$ with odd size, are partitioned into a number of color classes such that one of the color classes (except the color class v) has size one and this is a contradiction. Thus the claim is proved. According to this claim, we conclude that every three color classes are dominated by at least two vertices. Notice that if $n = 3k$, then at least $2k$ vertices, and if $n \neq 3k$, then at least $2k + 1$ vertices are needed for covering all the color classes. Therefore, if $n = 3k$, the dominated chromatic covering of the graph has at least $2n/3$ members, if $n = 3k + 1$, it has at least $\lceil 2n/3 \rceil$ members, and if $n = 3k + 2$, it has at least $\lfloor 2n/3 \rfloor$ members and the proof is complete.

The following Proposition has been proved in [6].

Proposition 1 $\chi_{dom}(G) \geq \gamma_t(G)$. *Also, if* G *is a triangle-free graph, then* $\chi_{dom}(G) = \gamma_t(G)$.

Now, we have the following Theorem.

Theorem 2 *For* $n \geq 3$, $\chi_{dom}(M(C_n)) = n$ *and*

$$\lfloor 2n/3 \rfloor \leq \theta_{\chi_{dom}}(M(C_n)) \leq \lceil 3n/4 \rceil.$$

Proof By definition, $M(C_n)$ is obtained from adding and joining one vertex to the vertices of every edge of C_n. The graph in Figure 2 shows $M(C_n)$ for $n = 10$. Conforming to the structure of $M(C_n), \gamma(M(C_n)) \geq n$. Because there must be at least one vertex of any triangle in the minimum dominating set. According to proposition 1, $n \leq \gamma(M(C_n)) \leq \gamma_t(M(C_n)) \leq \chi_{dom}(M(C_n))$. Now, let k be a natural number and regard the following four cases:
Case 1. $n = 4k$ for some natural number k. In this case, let each of the sets

$$\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{n-1}, v_n\},$$
$$\{u_1, u_3\}, \{u_2, u_4\}, \ldots, \{u_{n-3}, u_{n-1}\}, \{u_{n-2}, u_n\},$$

be a color class and it forms a dominated coloring of $M(C_n)$. Also,

$$B_0 = \{u_1, u_2, u_3, u_5, u_6, u_7, \ldots, u_{4i+1}, u_{4i+2}, u_{4i+3}, \ldots, u_{n-1}\},$$

is a dominated chromatic covering of the graph. So $\chi_{dom}(M(C_n)) = n$ and since $|B_0| = 3k = 3n/4$, therefore $\theta_{\chi_{dom}}(M(C_n)) \leq 3n/4$.
Case 2. $n = 4k + 1$ for some natural number k. In this case, let each of the sets

$$\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{n-2}, v_{n-1}\},$$
$$\{v_n, u_1\}, \{\{u_2, u_4\}, \{u_3, u_5\}, \ldots, \{u_{n-2}, u_n\},$$

be a color class and it forms a dominated coloring of $M(C_n)$. Also,

$$B_1 = \{u_1, u_3, u_4, u_5, u_7, u_8, u_9, \ldots, u_{4i}, u_{4i+1}, u_{4i+3}, \ldots, u_n\},$$

is a dominated chromatic covering related to $M(C_n)$. Hence $\chi_{dom}(M(C_n)) = n$ and since $|B_1| = 3k + 1 = \lceil 3n/4 \rceil$, we have $\theta_{\chi_{dom}}(M(C_n)) \leq \lceil 3n/4 \rceil$.
Case 3. $n = 4k + 2$ for some natural number k. Consider the sets

$$\{e_2, v_4\}, \{v_2, v_3\}, \{v_5, v_6\}, \ldots, \{v_{n-1}, v_n\},$$
$$\{v_1, e_{n-1}\}, \{e_1, e_3\}, \{e_4, e_6\}, \{e_5, e_7\}, \ldots, \{e_{n-2}, e_n\},$$

as classes of a smallest dominated coloring of $M(C_n)$ and so $\chi_{dom}(M(C_n)) = n$. Also,

$$B_2 = \{e_2, e_3, e_5, e_6, e_7, e_9, \ldots, e_{4i+1}, e_{4i+2}, e_{4i+3}, \ldots, e_n\},$$

is a dominated chromatic covering and since $|B_2| = 3k+1 = \lfloor 3n/4 \rfloor$, we have $\theta_{\chi_{dom}}(M(C_n)) \leq \lfloor 3n/4 \rfloor$.
Case 4. $n = 4k+3$ for some natural number k. The sets

$$\{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_{n-1}, v_n\},$$
$$\{e_{n-1}, v_1\}, \{e_1, e_3\}, \{e_2, e_4\}, \ldots, \{e_{n-2}, e_n\},$$

form classes of a smallest dominated coloring of $M(C_n)$ and so $\chi_{dom}(M(C_n)) = n$. Also,

$$B_3 = \{e_2, e_3, e_4, e_6, e_7, e_8, \ldots, e_{4i}, e_{4i+2}, e_{4i+3}, \ldots, e_n\},$$

is a dominated chromatic covering and since $|B_3| = 3k+2 = \lfloor 3n/4 \rfloor$, therefore $\theta_{\chi_{dom}}(M(C_n)) \leq \lfloor 3n/4 \rfloor$.

We have proved in general that $\theta_{\chi_{dom}}(M(C_n)) \leq \lceil 3n/4 \rceil$. Now we have to prove $\theta_{\chi_{dom}}(M(C_n)) \geq \lfloor 2n/3 \rfloor$. As mentioned before about $M(P_n)$, in the n-dominated coloring of $M(C_n)$, there is no color class of size 3, and since there are $2n$ vertices and n color classes, each color class will be of size 2. Also, same as proof for $M(P_n)$, in such coloring, every three color classes are covered by at least two vertices and therefore $\theta_{\chi_{dom}}(M(C_n)) \geq \lfloor 2n/3 \rfloor$.

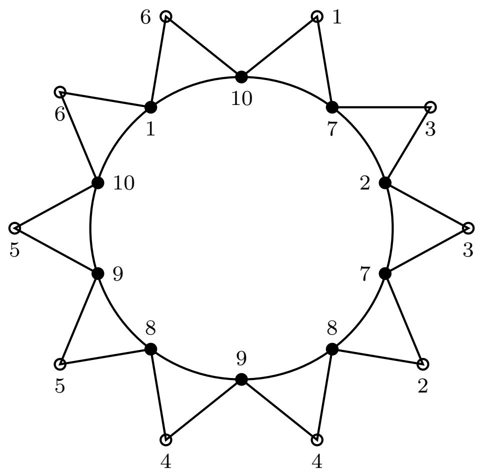

Fig. 2 Dominated coloring of $M(C_{10})$. (The vertices corresponding to $V(G)$ and $E(G)$ are shown by • and ∘, respectively.)

3 Dominated chromatic number and dominated chromatic covering number $T(G)$

In this section, we study the dominated chromatic number and dominated chromatic covering number of the $T(P_n)$ and $T(C_n)$.

Theorem 3 [3] *The total graph $T(G)$ is isomorphic to the square of the subdivision graph $S(G)$.*

Lemma 1 [2] *Let n and i be positive integers such that $n \equiv l \ (mod\ 2i)$ and $\lfloor \frac{n}{2i} \rfloor = r$. Then we have*

$$\chi_{dom}(P_n^{i-1}) = \chi_{dom}(C_n^{i-1}) = \begin{cases} ri + l, & \text{if } 0 \leq l \leq i, \\ i(r+1), & \text{if } i < l < 2i. \end{cases}$$

Using Theorem 3, we have $\chi_{dom}(T(P_n)) = \chi_{dom}(S^2(P_n)) = \chi_{dom}(P^2_{2n-1})$ and $\chi_{dom}(T(C_n)) = \chi_{dom}(S^2(C_n)) = \chi_{dom}(C^2_{2n})$. Now, using Lemma 1, we have

Theorem 4 *Let P_n and C_n be paths and cycles with n vertices, respectively. Then for all $n \geq 2$,*

$$\chi_{dom}(T(P_n)) = \begin{cases} n, & n \equiv 0, 1 \ (mod\ 3), \\ n+1, & n \equiv 2 \ (mod\ 3), \end{cases}$$

and for all $n \geq 3$,

$$\chi_{dom}(T(C_n)) = \begin{cases} n, & n \equiv 0 \ (mod\ 3), \\ n+1, & n \equiv 1, 2 \ (mod\ 3). \end{cases}$$

Now we calculate the dominated chromatic covering number of $T(P_n)$ and $T(C_n)$.

Theorem 5 *For all $n \geq 2$,*

$$\theta_{\chi_{dom}}(T(P_n)) = \begin{cases} \frac{2n}{3} - 1, & n \equiv 0 \ (mod\ 3), \\ \lfloor \frac{2n}{3} \rfloor, & n \equiv 1 \ (mod\ 3), \\ \lfloor \frac{2n}{3} \rfloor - 2, & n \equiv 2 \ (mod\ 3). \end{cases}$$

Proof By the structure of $T(P_n)$, there are no three non-adjacent vertices with a common neighbor. So there is no color class of size 3. On the other hand, in a χ_{dom}-dominated coloring of $T(P_n)$, there are at most three color classes of size one. Otherwise, we have $a \geq 4$ color classes of size one, hence $\chi_{dom}(T(P_n)) = \frac{2n-a-1}{2} + a > n + 1$, a contradiction. We know that $|V(T(P_n))| = 2n - 1$. So we have one or three color classes of size 1. If $n \equiv 0 \ (mod\ 3)$, there is an integer $i \geq 1$ such that $n = 3i$. Therefore, if there are 3 color classes of size one, then we have $\chi_{dom}(T(P_n)) = \frac{2n-1-3}{2} + 3 = 3i + 1 = n + 1 > n$ that is a contradiction. Hence we have one color class of size one. Likewise, we can illustrate if $n \equiv 1 \ (mod\ 3)$, there is one color class of size one and if $n \equiv 2 \ (mod\ 3)$, there are three color classes of size one. If $T(P_n)$ is χ_{dom}-dominated colored, no vertex dominates three color classes unless at least one of class is of size one. Now notice the following three cases:

Case 1. $n \equiv 0 \ (mod\ 3)$. In this case, we have one color class of size one. The other two classes are of size 2. The three classes are dominated by at least one

vertex and in other $n-3$ color classes every three classes are dominated by at least two vertices. So $\theta_{\chi_{dom}}(T(P_n)) \geq \frac{2(n-3)}{3}+1 \geq \frac{2n}{3}-1$.
Case 2. $n \equiv 1$ $(mod\ 3)$. Using a similar method to Case 1, we have

$$\theta_{\chi_{dom}}(T(P_n)) \geq \frac{2(n-3)}{3}+1 \geq \lfloor \frac{2n}{3} \rfloor.$$

Case 3. $n \equiv 2$ $(mod\ 3)$. In this situation, there are 3 color classes of size 1. It is clear that $\theta_{\chi_{dom}}(T(P_2)) = 1$ and we find with a simple review that $\theta_{\chi_{dom}}(T(P_5)) = 2$. Now we assume $n \geq 8$ and indeed $\chi_{dom}(T(P_n)) \geq 9$. We claim that in any χ_{dom}-coloring of $T(P_n)$, if three vertices that each of them is in a color class of size one aren't adjacent, then the corresponding dominated chromatic covering has minimum cardinality between all possible covering sets of dominated colorings of $T(P_n)$. To prove this claim, we consider three conditions:
1) All three vertices mentioned above are adjacent to each other. We denote the dominated chromatic covering number in this case by $\dot{\theta}_{\chi_{dom}}$. These three color classes can be dominated by one vertex and we need at least $\lfloor \frac{2(n+1-3)}{3}+1 \rfloor = \lfloor \frac{2n-1}{3} \rfloor$ vertices for dominated chromatic covering of $T(P_n)$. Therefore $\dot{\theta}_{\chi_{dom}} \geq \lfloor \frac{2n-1}{3} \rfloor$.
2) Two of them are adjacent. We denote dominated chromatic covering number in this case by $\ddot{\theta}_{\chi_{dom}}$. Here, six classes can be dominated by two vertices and in other $n-5$ vertices, each of the three classes can be dominated by two vertices. Then we need at least $\lfloor \frac{2(n+1-6)}{3}+2 \rfloor = \lfloor \frac{2n-4}{3} \rfloor$ vertices for the dominated chromatic covering of $T(P_n)$. Then $\ddot{\theta}_{\chi_{dom}} \geq \lfloor \frac{2n-4}{3} \rfloor$.
3) The three vertices are independent. We denote dominated chromatic covering number in this case by $\theta_{\chi_{dom}}$. Here, we need at least $\lfloor \frac{2(n+1-9)}{3}+3 \rfloor = \lfloor \frac{2n}{3} \rfloor - 2$ vertices for the dominated chromatic covering of $T(P_n)$. So $\theta_{\chi_{dom}} \geq \lfloor \frac{2n}{3} \rfloor - 2$.

Since $\dot{\theta}_{\chi_{dom}}(T(P_n)) \geq \ddot{\theta}_{\chi_{dom}}(T(P_n)) \geq \theta_{\chi_{dom}}(T(P_n))$, according to the above claim $\theta_{\chi_{dom}}(T(P_n)) \geq \lfloor \frac{2n}{3} \rfloor - 2$. On the other hand, we define $C_{T(P_n)}$ as a coloring function as follows:
If $n \equiv 0$ $(mod\ 3)$ and $n > 3$, then $C_{T(P_n)}(v_i) = i$ for $1 \leq i \leq n$ and

$$C_{T(P_n)}(e_i) = \begin{cases} i+2, & i \equiv 1\ (mod\ 3)\ \&\ 1 \leq i \leq n-1, \\ i-1, & i \equiv 0,2\ (mod\ 3)\ \&\ 1 \leq i \leq n-1. \end{cases}$$

In this situation, $B_1 = \{e_2, v_2, e_5, v_5, \ldots, e_{3i+2}, v_{3i+2}, \ldots, v_{n-4}, e_{n-4}, v_{n-1}\}$ is a dominated chromatic covering related to this coloring. Now it's clear that $\theta_{\chi_{dom}}(T(P_3)) = 1$ and so $\theta_{\chi_{dom}}(T(P_n)) \leq \frac{2n}{3} - 1$.
If $n \equiv 1$ $(mod\ 3)$, then $C_{T(P_n)}(v_i) = i$ for $1 \leq i \leq n$ and

$$C_{T(P_n)}(e_i) = \begin{cases} 3, & i = 1, \\ 1, & i = 2, \\ 2, & i = 3, \\ i-1, & i \equiv 0\ (mod\ 3)\ \&\ 4 \leq i \leq n-1, \\ i+2, & i \equiv 1,2\ (mod\ 3)\ \&\ 4 \leq i \leq n-1. \end{cases}$$

Hence, $B_2 = \{v_2, v_3, e_5, e_6, e_8, e_9, \ldots, e_{3i-1}, e_{3i}, \ldots, e_{n-1}\}$ is a dominated chromatic covering for this coloring. So $\theta_{\chi_{dom}}(T(P_n)) \leq \lfloor \frac{2n}{3} \rfloor$.
If $n \equiv 2\ (mod\ 3)$, then $C_{T(P_n)}(v_i) = i$ for $1 \leq i \leq n$ and

$$C_{T(P_n)}(e_i) = \begin{cases} i+2, & i \equiv 0\ (mod\ 3), i \equiv 2\ (mod\ 3)\ \&\ 8 < i \leq n-1, \\ i-1, & i \equiv 1\ (mod\ 3) \& i \neq 1, 4, i \equiv 2\ (mod\ 3)\ \&\ i \leq 8, \\ 3, & i = 1, \\ n+1, & i = 4. \end{cases}$$

Hence, for $n \leq 11$, the set $B_3 = \{v_2, e_4, v_7, v_8, \ldots, v_{3i+1}, v_{3i+2}, \ldots, v_{n-1}\}$ is a dominated chromatic covering according to above coloring and for $n \geq 14$, the set $\acute{B_3} = B_3 \cup \{e_{12}, e_{13}, \ldots, e_{3i}, e_{3i+1}, \ldots, e_{n-1}\}$ is a dominated chromatic covering. Therefore $\theta_{\chi_{dom}}(T(P_n)) \leq \lfloor \frac{2n}{3} \rfloor - 2$ and the proof is complete.

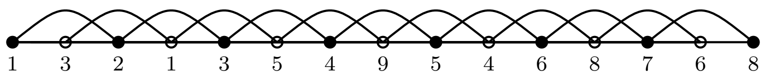

Fig. 3 Dominated coloring of $T(P_8)$. (The vertices corresponding to $V(G)$ and $E(G)$ are denoted by • and ∘, respectively.)

Theorem 6 *For all* $n \geq 3$,

$$\theta_{\chi_{dom}}(T(C_n)) = \begin{cases} \frac{2n}{3}, & n \equiv 0\ (mod\ 3), \\ \lceil \frac{2n}{3} \rceil, & n \equiv 1\ (mod\ 3), \\ \lfloor \frac{2n}{3} \rfloor - 1, & n \equiv 2\ (mod\ 3). \end{cases}$$

Proof As mentioned in Theorem 5, it is proved that each color class is of size at most 2 and also each member of the dominated chromatic covering dominates at most three color classes and if the covering vertex belongs to a color class of size 2, it dominates at most two classes. Now, let k be a natural number and consider the following three cases:
Case 1. $n = 3k$ for some positive integer k. The graph $T(C_n)$ has $2n$ vertices and n color classes. So all color classes are of size 2 and then at least $\frac{2n}{3}$ vertices are needed for covering all color classes. So $\theta_{\chi_{dom}}(T(C_n)) \geq \frac{2n}{3}$. On the other hand, considering the coloring function $C_{T(C_n)}$ as $C_{T(C_n)}(v_i) = i$ for $1 \leq i \leq n$ and

$$C_{T(C_n)}(e_i) = \begin{cases} i+2, & i \equiv 1\ (mod\ 3), \\ i-1, & i \equiv 0, 2\ (mod\ 3), \end{cases}$$

and the dominated chromatic covering $B_1 = \{v_2, v_3, \ldots, v_{3i-1}, v_{3i}, \ldots, v_n\}$ corresponding to this coloring, we have $\theta_{\chi_{dom}}(T(C_n)) \leq \frac{2n}{3}$. So,

$$\theta_{\chi_{dom}}(T(C_n)) = \frac{2n}{3}.$$

Case 2. $n = 3k + 1$ for some positive integer k. In this case, there are $2n$ vertices and $n+1$ color classes. Then there are $n-1$ color classes of size 2 and two classes of size 1. First, we claim the vertices of color classes of size one are adjacent. Suppose to the contrary they are not adjacent. Then there are 5 vertices in $T(C_n)$ which can be colored with 3 colors. Since, every color class contains at most 2 vertices and every three successive vertices are adjacent, we can say every successive three color classes contain at most 6 vertices. Also $|V(T(C_n))| = 6k+2$, so $\chi_{dom}(T(C_n)) \geq 3\lceil \frac{6k+2-5}{6} \rceil + 3 = 3k+3 = n+2$, that is a contradiction. Therefore the claim holds. Next, we claim that the vertices of color classes of size one don't dominate any color class of size 2. Assume the contrary, color class c_i contains only vertex v and dominates class c_{i+1} of size 2. Since, the vertex v is adjacent to another vertex of a color class of size 1, four vertices of $T(C_n)$ receive three colors. Since $|V(T(C_n)| = 2n = 6k+2$, hence $6k-2$ remaining vertices receives $3\lceil \frac{6k-2}{6} \rceil = 3k$. In general, the graph is colored in $3k+3$ colors which is a contradiction. Therefore the claim is proved. According to Claim 2, two color classes of size 1 are dominated by one vertex. Since every 3 color classes of size 2 are covered by two vertices, we have $\theta_{\chi_{dom}}(T(C_n)) \geq \lceil \frac{2n}{3} \rceil$. On the other hand, a coloring function $C_{T(C_n)}$ with $C_{T(C_n)}(v_i) = i$ for $1 \leq i \leq n$ and

$$C_{T(C_n)}(e_i) = \begin{cases} i+2, & i \equiv 1 \ (mod\ 3) \ \& \ i < n, \\ i-1, & i \equiv 0, 2 \ (mod\ 3) \ \& \ 1 \leq i \leq n, \\ n+1, & i = n, \end{cases}$$

and dominated chromatic covering $B_2 = \{v_2, v_3, \ldots, v_{3i-1}, v_{3i}, \ldots, v_{n-1}, v_n\}$ related to this coloring we have $\theta_{\chi_{dom}}(T(C_n)) \leq \lceil \frac{2n}{3} \rceil$ and hence we have $\theta_{\chi_{dom}}(T(C_n)) = \lceil \frac{2n}{3} \rceil$.

Case 3. $n = 3k+2$ for some positive integer k. Since there are $2n$ vertices and $n+1$ color classes, we have $n-1$ color classes of size 2 and two color classes of size one. Each vertex of a color class of size 1 dominates at most 3 color classes and in the other color classes, every three color classes are dominated by 2 vertices. therefore, we have $\theta_{\chi_{dom}}(T(C_n)) \geq \lceil \frac{2}{3}(n+1-6)+2 \rceil = \lfloor \frac{2n}{3} \rfloor - 1$. On the other hand, we have a coloring function $C_{T(C_n)}$ with $C_{T(C_n)}(v_i) = i$ for $1 \leq i \leq n$ and

$$C_{T(C_n)}(e_i) = \begin{cases} i+2, & i \equiv 0, 1 \ (mod\ 3) \text{ or } \ \& \ 8 < i < n, \\ i-1, & i \equiv 1 \ (mod\ 3) \ \& \ i \neq 1, 4, \\ i-1, & i \equiv 2 \ (mod\ 3) \ \& \ 1 \leq i \leq 8, \\ 3, & i = 1, \\ n+1, & i = 4, \\ 2, & i = n, \end{cases}$$

and if $n \leq 11$, then $B_3 = \{e_1, v_2, e_4, v_7, v_8, \ldots, v_{3i+1}, v_{3i+2}, \ldots, v_{n-1}\}$ is a dominated chromatic covering related to this coloring and also if $n \geq 14$, then $\acute{B_3} = B_3 \cup \{e_{12}, e_{13}, \ldots, e_{3i}, e_{3i+1}, \ldots, e_{n-1}\}$ is a dominated chromatic covering. Hence $\theta_{\chi_{dom}}(T(C_n)) \leq \lfloor \frac{2n}{3} \rfloor - 1$.The proof is now complete.

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