Global Analysis and Discrete Mathematics
Volume 7, Issue 1, pp. 67–73
ISSN: 2476-5341 *Research Article*

On the Relative 2-Engel Degree of A Subgroup of A Finite Group

Hanieh Golmakani · Abbas Jafarzadeh

Received: 4 October 2022 / Accepted: 10 October 2022

Abstract Let G be a finite group. The notion of n-Engel degree of G, denoted by $d_n(G)$, is the probability of two randomely chosen elements $x, y \in G$ satisfy the n-Engel condition $[y,_n x] = 1$. The case $n = 1$ is the known commutativity degree of G. The aim of this paper, is to define and investigate the relative 2-Engel degree of a subgroup H of G as the probability of two randomely chosen elements $x \in G$ and $y \in H$ satisfy the 2-Engel condition $[y,_2 x] = 1$.

Keywords Commutativity degree · n-Engel degree · Right 2-Engel subgroup

Mathematics Subject Classification (2010) 20F45 · 20P05

1 Introduction

In this article, G will always denote a finite group. Let n be a non-negative integer, G a group and $x, y \in G$. The n-Engel word $[y,_n x]$ is defined recursively by $[y,_0 x] = y$ and $[y,_{n+1} x] = [[y,_n x], x]$. Recall that an element y in a group G is said to be right n-Engel if $[y,_n x] = 1$ for all $x \in G$ and that the group G is said to be n-Engel if every element $x \in G$ is right n-Engel. A subgroup H of G is said to be a right n-Engel subgroup if all the elements of H are right n-Engel elements of G (See [4]).

We extend the notion of the relative commutativity degree of a finite group G and a subgroup H of G (See [7]) by defining the relative 2-Engel degree of H in G, which is denoted by $d_2(H, G)$. It is the probability of H to be a right

H. Golmakani
Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.
E-mail: h.golmakani@mshdiau.ac.ir

A. Jafarzadeh (Corresponding Author)
Department of Mathematics, Quchan University of Technology, Quchan, Iran.
Tel.: +98-51-47017358
E-mail: jafarzadeh.a@qiet.ac.ir, abbas.jafarzadeh@gmail.com

2-Engel subgroup of G. That is

$$d_2(H,G) = \frac{|\{(x,y) \in H \times G : [y,x,x] = 1\}|}{|H||G|}.$$

Obviously $d_2(G,G) = d_2(G)$ (See Section 2 below) and $d_2(H,G) = 1$ for every subgroup H of the center of G. In general, $d_2(H,G) = 1$ if and only if H is a right n-Engel subgroup of G. In this article, we investigate some properties and results involving general lower and upper bounds for the relative 2-Engel degree $d_2(H,G)$ for a group G and a subgroup H of G and we improve the upper bounds for $d_2(H,G)$ when G is not a 2-Engel group but belongs to a special class of groups.

2 Definitions and Known Results

The commutativity degree of a group G, denoted by $d(G)$ is defined as the probability of two randomly chosen elements of G commute, that is

$$d(G) = \frac{|\{(x,y) \in G \times G : [y,x] = 1\}|}{|G|^2}.$$

The commutativity degree of G was first introduced by P. Erdös and P. Turán (See [5]) and its generalizations are extensively studied in the literature. (See for example [1,3,7–9]). For every group G, the n-th commutativity degree $p_n(G)$ of G is the probability that the n-th power of a random element of G commutes with another random element of G. More precisely

$$p_n(G) = \frac{|\{(x,y) \in G \times G : [x^n,y] = 1\}|}{|G|^2}.$$

The n-th commutativity degree of a group has been introduced in [2]. The importance of $p_n(G)$ is due to the fact that $p_1(G) = d(G)$ is the commutativity degree of G. For a given natural number n, we define the n-Engel degree of G, denoted by $d_n(G)$ as the probability that two randomly chosen elements x, y of G satisfy the n-Engel condition $[y,_n x] = 1$, that is

$$d_n(G) = \frac{|\{(x,y) \in G \times G : [y,_n x] = 1\}|}{|G|^2}.$$

We have the following inequalites for the Engel degrees of a group G,

$$d_1(G) \leq d_2(G) \leq \cdots \leq d_n(G) \leq \cdots.$$

There are some significant results on $d_n(G)$ in [7,9]. The authors investigated some lower and upper bounds for 2-Engel degree of a group G in [6]. Recall that for every natural number n, the notations $\pi(G)$, $L_n(G)$, $L(G)$, $R_n(G)$ and $R(G)$ denote the set of all prime divisors of the order of a group G, left n-Engel elements, left Engel elements, right n-Engel elements and right Engel elements of G, respectivily. Let $k_G(X)$ be the number of the conjugacy classes of G contained in X for each normal subset X of G. Let χ be the class of all groups G such that the set $E_G(x) = \{y \in G : [y, x, x] = 1\}$ is a subgroup of G for all $x \in G$. In [6] it is proved that if $G \in \chi$ is a finite group which is not a 2-Engel group and if $p = \min \pi(G)$, then

$$d_2(G) \leq \frac{1}{p} + (1 - \frac{1}{p})\frac{|L_2(G)|}{|G|},$$

and if $L_2(G) \leq G$, then

$$d_2(G) \leq \frac{2p-1}{p^2}.$$

It is also proved that

$$d_2(G) \geq d_1(G) - (p-1)\frac{|Z(G)|}{|G|} + (p-1)\frac{k_G(L(G))}{|G|},$$

Note that, $R_2(G)$ is always a subgroup of G (See [10]), while $L_2(G)$ is not necessarily a subgroup of G. For a given group G and an element $x \in G$, $E_G(x)$ is a subgroup of G whenever $[E_G(x), x, E_G(x), x] = 1$ or $[E_G(x), x]$ is abelian.

As an example of 2-Engel degree of groups, if G is the dihedral group of order $2n$, then $d_2(G) = \frac{n+1}{2n}$ if n is odd, $d_2(G) = \frac{n+2}{2n}$ if $n = 2m$ for some odd number m and $d_2(G) = \frac{n+4}{2n}$ if $4|n$. As another example, if G is a generalized quaternion group of order $4n$, then $d_2(G) = \frac{n+1}{2n}$ if n is odd and $d_2(G) = \frac{n+2}{2n}$ if n is even.

The authors in [7] generalized the notion of the commutativity degree of groups by defining the relative commutativity degree of G and a subgroup H of G, denoted by $d(H, G)$, which is the probability that an element of H commutes with an element of G. Obviously $d(H, G) = 1$ if and only if H is contained in the center of G. They proved that the relative commutativity degree $d(H, G)$ and the commutativity degrees of G and H are compared through the following inequalities

$$d(G) \leq d(H, G) \leq d(H).$$

It is also proved in [7] that if p is the smallest prime number dividing $|G|$, then

$$\frac{|Z(G) \cap H|}{|H|} + \frac{p(|H| - |Z(G) \cap H|)}{|H||G|} \leq d(H, G) \leq \frac{|Z(G) \cap H| + |H|}{2|H|}.$$

They also proved that for a nonabelian group G and a subgroup H with $H \nsubseteq Z(G)$, if H is abelian, then $d(H, G) \leq \frac{3}{4}$ and if H is not abelian, then $d(H, G) \leq \frac{5}{8}$.

3 Main Results

In this section, we bring our main results.

Theorem 1 *If $H \leq G$, then $d_2(G) \leq d_2(H, G)$. The equality holds if and only if $H = G$.*

Proof We have

$$\begin{aligned} d_2(H,G) &= \frac{1}{|G|}\sum_{x\in H}\frac{|E_G(x)|}{|H|} \\ &\geq \frac{1}{|G|}\sum_{x\in H}\frac{|E_G(x)|}{|G|} \\ &\geq \frac{1}{|G|}\sum_{x\in G}\frac{|E_G(x)|}{|G|} \\ &= d_2(G). \end{aligned}$$

The second part is obvious. The proof is complete.

We need the following lemma to prove the next results:

Lemma 1 *If $H \leq G$ and $G \in \chi$, then $[H : E_H(x)] \leq [G : E_G(x)]$ for all $x \in G$.*

Proof Since $E_G(x) \leq G$, we have $E_H(x) = H \cap E_G(x) \leq H$. Hence

$$\frac{|H||E_G(x)|}{|E_H(x)|} = \frac{|H||E_G(x)|}{|H \cap E_G(x)|} = |HE_G(x)| \leq |G|.$$

Thus $[H : E_H(x)] \leq [G : E_G(x)]$.

First we compare the relative 2-Engel degree with 2-Engel degree of G:

Theorem 2 *If $H \leq G$ and $G \in \chi$, then $d_2(G) \leq d_2(H, G) \leq d_2(H)$.*

Proof Using Lemma 1 we have

$$\begin{aligned} d_2(H,G) &= \frac{|\{(x,y) \in H \times G : [y,x,x] = 1\}|}{|H||G|} \\ &= \frac{1}{|H||G|}\sum_{x\in H}|E_G(x)| \\ &= \frac{1}{|H|}\sum_{x\in H}\frac{|E_G(x)|}{|G|} \\ &\leq \frac{1}{|H|}\sum_{x\in H}\frac{|E_H(x)|}{|H|} \\ &= d_2(H). \end{aligned}$$

In the next two theorems, we find upper bounds for the 2-Engel degree of a pair of groups.

Theorem 3 *Let $H \leq G$ and $G \in \chi$. Then*

$$d_2(H,G) \leq \frac{1}{2}\left(1 + \frac{|L_2(H)|}{|H|}\right).$$

Proof

$$\begin{aligned}
d_2(H,G) &= \frac{1}{|H||G|}\sum_{x\in H}|E_G(x)| \\
&= \frac{1}{|H|}\sum_{x\in H}\frac{|E_G(x)|}{|G|} \\
&\leq \frac{1}{|H|}\sum_{x\in H}\frac{|E_H(x)|}{|H|} \\
&= \frac{1}{|H|}\left(\sum_{x\in L_2(H)}\frac{|E_H(x)|}{|H|} + \sum_{x\notin L_2(H)}\frac{|E_H(x)|}{|H|}\right) \\
&\leq \frac{1}{|H|}\left(|L_2(H)| + \frac{1}{2}(|H| - |L_2(H)|)\right) \\
&= \frac{1}{2}\left(1 + \frac{|L_2(H)|}{|H|}\right)
\end{aligned}$$

Theorem 4 *Let G be a finite group which is not 2-Engel. If $p = \min \pi(G)$, then*

$$d_2(H,G) \leq p([G:H]) + \frac{p-1}{p}\left(\frac{|L_2(G)\cap H|}{|H|}\right)$$

and if $L_2(H) \leq H$, then

$$d_2(H,G) \leq \frac{p^3+p-1}{p^2}([G:H])$$

Proof

$$\begin{aligned}
d_2(H,G) &= \frac{1}{|H||G|}\sum_{x\in H}|E_G(x)| \\
&= \frac{1}{|H||G|}\left(\sum_{x\in L_2(G)\cap H}|E_G(x)| + \sum_{x\in H-L_2(G)}|E_G(x)|\right) \\
&\leq \frac{1}{|H||G|}\left(|L_2(G)\cap H||G| + \frac{|G|}{p}(|G| - (|L_2(G)\cap H|)\right) \\
&= \frac{p-1}{p}\left(\frac{|L_2(G)\cap H|}{|H|}\right) + p[G:H].
\end{aligned}$$

In particular, if $L_2(H) \leq H$, then $|L_2(H)| \leq \frac{|G|}{p}$ and the results follows.

Corollary 1 *Let G be finite 2-Engel group. If $p = \min \pi(G)$, then*

$$d_2(H, G) \leq \frac{p-1}{p} + p([G : H]).$$

Proof We have $L_2(G) = G$ and by Theorem 4 the result follows.

Theorem 5 *Let G be a nonabelian group and $p = \min \pi(G)$. Then*

$$\begin{aligned} \frac{|L_2(G) \cap H|}{|H|} + p\left(\frac{|G| - |L_2(G) \cap H|}{|H||G|}\right) &\leq d_2(H, G) \\ &\leq \frac{1}{2}\left([G : H] + \frac{|L_2(G) \cap H|}{|H|}\right) \end{aligned}$$

Proof On one hand

$$\begin{aligned} d_2(H, G) &= \frac{1}{|H||G|} \sum_{x \in H} |E_G(x)| \\ &= \frac{1}{|H||G|} \left(\sum_{x \in L_2(G) \cap H} |E_G(x)| + \sum_{x \notin L_2(G) \cap H} |E_G(x)| \right) \\ &\leq \frac{1}{|H||G|} (|L_2(G) \cap H||G|) + \frac{|G|}{2} (|G| - |L_2(G) \cap H|)) \\ &= \frac{1}{2}\left([G : H] + \frac{|L_2(G) \cap H|}{|H|}\right) \end{aligned}$$

and on the other hand

$$\begin{aligned} d_2(H, G) &= \frac{1}{|H||G|} \sum_{x \in H} |E_G(x)| \\ &= \frac{1}{|H||G|} \left(\sum_{x \in L_2(G) \cap H} |E_G(x)| + \sum_{x \notin L_2(G) \cap H} |E_G(x)| \right) \\ &\geq \frac{1}{|H||G|} (|L_2(G) \cap H||G| + p(|G| - |L_2(G) \cap H|)) \\ &= \frac{|L_2(G) \cap H|}{|H|} + p\left(\frac{|G| - |L_2(G) \cap H|}{|H||G|}\right). \end{aligned}$$

Acknowledgements We would like to thank the reviewer for their careful consideration of our manuscript and their detailed suggestions. The reviewer's thoughtful comments helped us refine the manuscript in preparation for publication.

References

1. A. M. A. Alghamdi, F. G. Russo, A generalization of the probability that the commutator of two group elements is equal to a given element, Bull. Iranian Math. Soc., 38, 973–986 (2012).

2. N. M. M. Ali, N. H. Sarmin, On some problems in group theory of probabilistic nature, Menemui Matematik, 32, 35–41 (2010).
3. R. Barzgar, A. Erfanian, M. Farrokhi D. G., Finite groups with three relative commutativity degrees, Bull. Iranian Math. Soc., 39, 271–280 (2013).
4. P. G. Crosby, G. Traustason, On right n-Engel subgroups, J. Algebra, 324, 875–883 (2010).
5. P. Erdös, P. Turán, On some problems of statistical group-theory. IV, Acta Math. Acad. Sci. Hung., 19, 413–435 (1968).
6. A. Erfanian, M. Farrokhi D. G., On the probability of being a 2-Engel groups, Int. J. Group Theory, 2, 31–38 (2013).
7. A. Erfanian, R. Rezaei, P. Lescot, On the relative commutativity degree of a subgroup of a finite group, Comm. Algebra, 35, 4183–4197 (2007).
8. R. M. Guralnich, G. R. Robinson, On the commuting probability in finite groups, J. Algebra, 300, 509–528 (2006).
9. P. Lescot, Isoclinism classes and commutativity degrees of finite goups, J. Algebra, 177, 847–869 (1995).
10. W. P. Kappe, Die A–norm einer gruppe, Illinois. J. Math., 5, 187–197 (1961).