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Some Result on Weak-Tenacity of A Graph

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Abstract Connectivity has been used in the past to describe the stability of graphs. If two graphs, have the same connectivity, then it dose not distinguish between these graphs. That is, the connectivity is not a good measure of graph stability. Then we need other graph parameters to describe the stability. Suppose that two graphs have the same connectivity and the order (the number of vertices or edges) of the largest components of these graphs are not equal. Hence, we say that these graphs must be different in respect to stability and so we can define a new measure which distinguishes these graphs. In this paper, the Weak-Tenacity of graph G is introduced as a new measure of stability in this sense and it is defined as

$$T_w(G) = \min_{S \subseteq V(G)} \left\{ \frac{|S| + m_e(G-S)}{\omega(G-S)} \ : \ \omega(G-S) > 1 \right\},$$

where $m_e(G-S)$ denotes the number of, edges of the largest component of $G-S$. At last, We give the Weak-Tenacity of graphs obtained via various operations.

Keywords Connecctivity · Tenacity · Weak tenacity · Vulnerability

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1 Introduction

The stability of a (computer, or communication, or transportation) network composed of (processing) nodes and (communication or transportation) links is of prime importance to network designers. One way of measuring the stability of a network is through the cost of disrupting the network. In an analysis of the

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stability of a network against disruption, we have two fundamental questions (there may be others):

(1) What is the size of the largest remaining group within which mutual communication can still occur?
(2) How difficult is it to reconnect the network?

Let S be a set of edge or vertices of a graph G. Question (1) is sometimes analyzed by considering the number of vertices or the number of edges of the largest component of $G-S$. Question (2) is sometimes analyzed by considering the number of components of $G-S$ [3].The connectivity is the minimum number of vertices whose removal disconnects the graph and edges connectivity is the minimum number of edges whose removal disconnects the graph. The difficulty with these two parameters is that they do not take into account what remains after the graph has been disconnected. To avoid this difficulty, other parameters have been proposed, including Integrity, which takes into account the size of the largest component that remains after disconnection of the graph, toughness which takes into account the number of components once the graph has been disconnected, and tenacity which includes both the size of the largest component and the number of components remaining after disconnection of the graph [1–3] i.e. the integrity family is a measure which deal with question (1) and so the integrity, edge integrity and pure edge integrity were introduced as a measure of stability in this sense. For convenience, we recall some parameters of [3]. Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$ Formally, the vertex-tenacity (frequently called just tenacity) is

$$T(G) = \min_{S \subseteq V(G)} \left\{ \frac{|S| + m(G-S)}{\omega(G-S)} \ : \ \omega(G-S) > 1 \right\},$$

where $\omega(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component of $G-S$. A set $S \subseteq V(G)$ is a *cutSet* of G, if either $G-S$ is disconnected or $G-S$ has only one vertex. We shall use $\lfloor x \rfloor$ for the largest integer not larger than x. A subset S of $V(G)$ is called an independent set of G if no two vertices of S are adjacent in G. An independent set S is a maximum if G has no independent set S' with $|S'| > |S|$. The *independence number* of G, $\beta(G)$, is the number of vertices in a maximum independent set of G. A subset S of $V(G)$ is called a covering of G if every edge of G has at least one end in S. A covering S is a minimum covering if G has no covering S' with $|S'| > |S|$. The *covering number*, $\alpha(G)$, is the number of vertices in a minimum covering of G.We use Bondy and Murty [2] for terminology and notations not defined here. For comparing, the following graph parameters are listed. The connectivity is a parameter defined based on Quantity (1). The connectivity of an incomplete graph G is defined by

$$\kappa(G) = \min\left\{ |S| \ : \ S \subset V(G), \omega(G-S) > 1 \right\},$$

and that of the complete graph K_n is defined as $n-1$. The toughness of an incomplete connected graph G is defined by

$$t(G) = \min\left\{ \frac{|S|}{\omega(G-S)} \ : \ S \subset V(G),\ \omega(G-S) > 1 \right\}.$$

The integrity is defined based on Quantities (1) and (2). The integrity of a graph G is defined by

$$I(G) = \min\left\{ |S| + m(G-S) \ : \ S \subset V(G) \right\}.$$

The corresponding edge analogues of these concepts are defined similarly. Among the above parameters, the tenacity is a reasonable parameter can be used for measuring the vulnerability of networks.
In this paper, we consider the problem of computing the weak-tenacity of graphs. In Section 2, we give some results on the weak-tenacity some graphs. After that, in Section 3, we compute the weak-tenacity of some special operations on some graphs. Finally, the relationships between the weak-tenacity and some other vulnerability parameters, namely the tenacity, toughness, and integrity are established in Section 4.

2 Weak-Tenacity of a graph

In this section, we introduce a new stability measure like as Tenacity and deal with the above questions. The Weak-Tenacity of a graph G is defined as

$$T_w(G) = \min_{S \subseteq V(G)} \left\{ \frac{|S| + m_e(G-S)}{\omega(G-S)} \ : \ \omega(G-S) > 1 \right\},$$

where the minimum is taken over all cut-sets S of G and $m_e(G-S)$ denotes the number of edges of the largest component of $G-S$.We see that $T_w(K_n) = n-1$,for every $n \geq 1$. Since by removal any subset $S \subseteq V(G)$ of complete graph K_n result is a complete graph. Thus, the largest component of $K_n - S$ must be a complete subgraph of K_n. So,

$$m_e(K_n - S) = \binom{n-|S|}{2}.$$

If set $|S| = x$, then

$$T_w(K_n) = \frac{x + \binom{n-x}{2}}{\omega(K_n - S)} = f(x),$$

but $\omega(K_n - S) = 1$. Thus minimum $f(x)$ by calculus will be $n-1$. A set S is said to be a T_w-set of G if

$$T_w(G) = \frac{|S| + m_e(G-S)}{\omega(G-S)}.$$

Also, note that if G is disconnected the set S may be empty. There are many example of graphs which suggest that T_w is a suitable measure of stability in that it is able distinguish among graphs.

Next, we will give some basic results about the weak-tenacity.

Proposition 1 *If G is spanning subgraph of H then $T_w(G) \leq T_w(H)$.*

Proof Since G is spanning subgraph of H,we have $\omega(G-S) \leq \omega(H-S)$ and $m_e(G-S) \leq m_e(H-S)$ for any subset $S \subseteq V(G)$. Thus we have

$$T_w(G) \leq T_w(H).$$

Proposition 2 *For any graph G, $T_w(G) \geq \frac{\kappa(G)}{\beta(G)}$ where $\beta(G)$ is the independent number of G.*

Proof For any cut set$S$ of G, we know that $|S| \geq \kappa(G)$. Thus we can see that the number of components of $G-S$ is at most $\beta(G)$. Therefore

$$\frac{|S| + m_e(G-S)}{\omega(G-S)} \geq \frac{\kappa(G)}{\beta(G)}.$$

Proposition 3 *If G is not complete, then*

$$T_w(G) \leq \frac{n-\beta(G)}{\beta(G)}.$$

Proof Let X be a largest independent set of vertices in G, define $S = V(G) - X$. Then $|S| = n - \beta(G)$, $me(G-S) = 0$, and $\beta(G) = \omega(G-S)$. Hence the result follows.

Proposition 4 *If $m \leq n$. Then $T_w(K_{m,n}) = \frac{m}{n}$.*

Proof If $G = K_{m,n}$ with $m \leq n$. Then $\kappa(G) = m$ and $\beta(G) = n$. Combining propositions 2 and 3. We obtain

$$\frac{\kappa(G)}{\beta(G)} \leq T_w(G) \leq \frac{m+n-\beta(G)}{\beta(G)}.$$

Hence $T_w(K_{m,n}) = \frac{m}{n}$.

Theorem 1 *For any graph G, $T_w(G) \geq t(G)$, where $t(G)$ is toughness of G.*

Proof Let$A \subseteq V(G)$, be a t-set and $B \subseteq V(G)$, be a T_w-set. Then

$$T_w(G) = \frac{|B| + m_(G-B)}{\omega(G-B)} \geq \frac{|B|}{\omega(G-B)} \geq \frac{|A|}{\omega(G-A)} = t(G).$$

This result gives us a number of corollaries.

Corollary 1 *For any graph G, $T_w(G^2) > \kappa(G)$.*

Proof In [2], Chavatàl obtained the result $t(G^2) > \kappa(G)$.

Corollary 2 *Let G be a non-empty graph and let m be largest integer such that $K_{1,m}$ is an induced sub-graph of G. Then $T_w(G) \geq \frac{\kappa(G)}{m}$.*

Proof In [4] Goddard and Swart proved that under these conditions we have $t(G) \geq \frac{\kappa(G)}{m}$.

3 Weak-Tenacity and Operation on Graphs

Theorem 2 *For any graph G we have*

$T(G) \leq T_w(G) + \frac{1}{2}$.

Proof Let S be a T_w-set G. Since

$$T(G) \leq \frac{|S| + m(G-S)}{\omega(G-S)},$$

and

$$m(G-S) \leq m_e(G-S) + 1,$$

for any set $S \subseteq V(G)$. Thus

$$T(G) \leq \frac{|S| + m(G-S)}{\omega(G-S)} \leq \frac{|S| + m_e(G-S) + 1}{\omega(G-S)}.$$

So, $T(G) \leq T_w(G) + \frac{1}{\omega(G-S)} \leq T_w(G) + \frac{1}{2}$.

Theorem 3 *If a graph G of order n is isomorphic to a cycle graph or a tree then, $T_w(G) = T(G) - 1$.*

Proof Let S be a subset of $V(G)$, such that $T(G) = \frac{|S|+m(G-S)}{\omega(G-S)}$. If we remove the vertices in S, then each of the components of $G - S$ is a tree or isolated vertex and $m(G-S) = m_e(G-S) + 1$. Hence,

$$\begin{aligned} T_w(G) &\leq \frac{|S| + m_e(G-S)}{\omega(G-S)} \\ &= \frac{|S| + m(G-S) - 1}{\omega(G-S)} \\ &= T(G) - \frac{1}{\omega(G-S)} \\ &\leq T(G) - \frac{1}{\beta(G)}. \end{aligned} \tag{1}$$

So, the proof is completed by (1) and Theorem 2.

Proposition 5 *Let G be any non trivial, non complete graph with n vertices. For any vertex v, we have, $T_w(G-v) \geq T_w(G) - \frac{1}{2}$. Hence,*

$$T_w(G-v) \geq T_w(G) - 1.$$

Proof Let $G' = G - v$. If $G' = K_{1,n}$, then $T_w(G') = n - 1$ and by proposition 2.3 , $T_w(G) \leq \frac{n-1}{2}$ Thus, the theorem holds. Hence, assume $G' \neq K_{n-1}$. Let A' be a T_w -set for G' , then $T_w(G') = \frac{|A'|+m_e(G'-A')}{\omega(G'-A')}$. Now define $A = A' \cup \{v\}$, clearly A is a cut set for G and so $T_w(G) \leq \frac{|A|+m_e(G-A)}{\omega(G-A)}$. But $|A| = |A'| + 1$ and $G - A = G' - A'$, so

$$T_w(G) \leq \frac{|A'| + m_e(G' - A') + 1}{\omega(G' - A')}$$
$$= \frac{|A'| + m_e(G' - A')}{\omega(G' - A')} + \frac{1}{\omega(G' - A')} \leq T_w(G') + \frac{1}{2}.$$

Theorem 4 *If G is a bipartite, r-regular and r-connected graph on n vertices, then $T_w(G)$=1.*

Proof From [4], we know that $t(G) \geq 1$, and so by Theorem 2.5 we have $T_w(G) \geq 1$. Let A be one of partite sets. Then, since G is r -regular, $|A| = \frac{n}{2}$, $m_e(G - A) = 0$, and $\omega(G - A) = \frac{n}{2}$. Therefore,

$$T_w(G) \leq \frac{|A| + m_e(G - A)}{\omega(G - A)} = \frac{\frac{n}{2} + 0}{\frac{n}{2}} = 1.$$

Hence, $T_w(G) = 1$.

This result gives several interesting corollaries.

Corollary 3 *If G_1 is a bipartite, n-regular, n-connected graph and G_2 is a bipartite, m-connected, m-regular graph. Then $T_w(G_1 \times G_2) = 1$.*

Proof It is well-known that $G_1 \times G_2$ is bipartite, $(m + n)$-regular, $(m + n)$-connected.

Corollary 4 *For any integer n, we have $T_w(Q_n) = 1$.*

Corollary 5 *For any even integers n and m we have $T_w(C_n \times C_m) = 1$.*

Corollary 6 *For any even integer n, $T_w(C_n \times K_2) = 1$.*

We next obtain some bounds on the Weak-Tenacity of products of graphs. Note that the first inequality in the Theorem 3.

Theorem 5 *If $n \geq m$, then*

$$\frac{m + n - 2}{2} \leq T_w(K_n \times K_m) \leq \frac{1}{2n} \left\lceil \frac{n}{m} \right\rceil (\left\lceil \frac{n}{m} \right\rceil - 1) + (m - 1).$$

Proof In [2] Chavatal proved that, if $m, n \geq 2$ then, $t(K_m \times K_n) = \frac{m+n-2}{2}$. Let $V(K_n) = \{1, 2, \ldots, n\}$ and $V(K_m) = \{1, 2, \ldots, m\}$. Then

$$V(K_n \times K_m) = \{(i, j)|1 \leq i \leq m, 1 \leq j \leq n\}.$$

Also, let $n = am + b$, for $0 \leq b \leq m - 1$, so if $b = 0$, then, $a = \lceil \frac{n}{m} \rceil = \frac{n}{m}$, and otherwise $\lceil \frac{n}{m} \rceil = a + 1$.Now, if $b = 0$, then define the sets W_i as follows.

$$W_i = \{(i, ia), (i, ia + 1), \ldots, (i, ia - a + 1)\},$$

for each i, $1 \le i \le m$. Otherwise define the sets W_i as follows.

$$W_i = \begin{cases} \Big\{(i, ia+i), (i, ia+i+1), \ldots, (i, ia+i-a)\Big\}, & 1 \le i \le b, \\ \\ \Big\{(i, ia+b), (i, ia+b+1), \ldots, (i, ia+b-a+1)\Big\}, & 1+b \le j \le m, \end{cases}$$

and let, $W = \bigcup\limits_{i=1}^{m} W_i$, define $A = V(G) - W$ and so $|A| = mn - n$.

It is easy to see that, the $W_i, 1 \le i \le m$, are components of $G - A$ and so $m_e(G-A) = (\lceil \frac{n}{m} \rceil - 1) \lceil \frac{n}{m} \rceil \times \frac{1}{2}$ and $\omega(G-A) = n$, thus,

$$\frac{m+n-2}{2} \le T_w(K_m \times K_n) \le \frac{1}{2n} \left\lceil \frac{n}{m} \right\rceil (\left\lceil \frac{n}{m} \right\rceil - 1) + (m-1).$$

Thus, the result follows.

Corollary 7 *For any integer $n, T_w(K_n \times K_n) = n - 1$.*

Lemma 1 *If A is a minimal T_w-set for the graph G. Then for each vertex v of A, the induced subgraph $< V(G) - (A - v) >$ has fewer components than dose $G - A$.*

Proof Let $A' = A - v$. Thus, $|A'| = |A| - 1$ and $m_e(G-A) + 1$. Now assume that $\omega(G - A') \ge \omega(G - A) + 1$. Thus,

$$\frac{|A'| + m_e(G-A')}{\omega(G-A')} \le \frac{|A| + m_e(G-A)}{\omega(G-A)} = T_w(G).$$

Contrary to our choice of A.

Proposition 6 *If G is connected, then $T_w(G) \ge \frac{1}{\Delta(G)}$.*

Proof If A is a cut set of size n, thus $\omega(G) \le n\Delta(G)$ and $m_e(G-S) \ge 0$. Hence,

$$T_w(G) = \min_{S \subset V(G)} \frac{|S| + m_e(G-S)}{\omega(G-S)} \ge \frac{n+0}{n\Delta(G)} = \frac{1}{\Delta(G)}.$$

Proposition 7 *Let G be a graph with n vertices and $G \ne K_n$, then*

$$T_w(G) + T_w(\overline{G}) \ge \frac{1}{\Delta(G)} \ge \frac{1}{n-1}.$$

Proof We observe that at least one of G or $\overline{G}$ is connected. Suppose $\overline{G}$ is not connected. we proved that $T_w(G) \ge \frac{1}{\Delta(G)} \ge \frac{1}{n-1}$ Now, suppose G is not connected but $\overline{G}$ is connected. again by previous proposition we have $T_w(\overline{G}) \ge \frac{1}{n-1}$. Therefore in each case we have $T_w(G) + T_w(\overline{G}) \ge \frac{1}{n-1}$.

4 Weak-Tenacity and other vulnerability parameters

In this section, the relationships between the weak-tenacity and some vulnerability parameters, namely the tenacity, toughness and integrity are established.

Theorem 6 *Let G be a non-complete connected graph of order $n \geq 2$. Then* $T_w(G) \geq \frac{\delta(G)}{n-\delta(G)}$.

Proof Let X be an arbitrary vertex cut of G. Denote the components of $G-X$ by $G_1, G_2, ..., G_\omega$ and set $|V(G_i)| = n_i$ $(1 \leq i \leq \omega)$. Then $\sum_{i=1}^{\omega} n_i = n - |X|$. Clearly, we have $2 \leq \omega \leq n - |X|$ and $m_e(G-X) \geq \frac{n-|X|}{\omega} - 1$. Therefore

$$\frac{|X| + m_e(G-X)}{\omega(G-X)} \geq \frac{|X| + \frac{n-|X|}{\omega} - 1}{\omega} = \frac{(|X|-1)\omega + (n-|X|)}{\omega^2}. \qquad (2)$$

Set $f(\omega) = \frac{(|X|-1)\omega+(n-|X|)}{\omega^2}$.
It is easy to check that $f(\omega)$ is a decreasing function when $\omega \geq 2$. We have two case
Case I : $|X| \geq \delta$.
It follows from (2), and the fact that $f(\omega)$ is a decreasing function when $\omega \geq 2$, that

$$\frac{|X| + m_e(G-X)}{\omega(G-X)} \geq f(n-|X|) = \frac{|X|}{n-|X|} \geq \frac{\delta}{n-\delta}. \qquad (3)$$

Case II : $|X| \leq \delta - 1$. In this case, every component of $G-X$ has at least two vertex. Otherwise, suppose that there exists a component G_k with $n_k = 1$. Denote the unique vertex of G_k by u, then $deg(u) \leq |X| < \delta$, a contraction. Since, $(n_i - 1) + |X| \geq \delta$, for every $1 \leq i \leq \omega$. We have

$$\omega\delta \leq \sum_{i=1}^{\omega} ((n_i - 1) + |X|) = \sum_{i=1}^{\omega} n_i + \omega(|X| - 1) = (n - |X|) + \omega(|X| - 1).$$

Therefore, $2 \leq \omega \leq \frac{n-|X|}{\delta-|X|+1}$. So, from (2) and the fact that $f(\omega)$ is a decreasing function when $\omega \geq 2$, we have

$$\frac{|X| + m_e(G-X)}{\omega(G-X)} \geq f(\frac{n-|X|}{\delta-|X|+1}) = \delta(\frac{n-|X|}{\delta-|X|+1}).$$

Now, set $g(x) = \delta(\frac{n-|X|}{\delta-|X|+1})$. It is easy to check that g(x) is a decreasing function when $x \leq \delta - 1$. Thus

$$\frac{|X| + m_e(G-X)}{\omega(G-X)} \geq g(\delta-1) = \frac{2\delta}{n-\delta+1} \geq \frac{\delta}{n-\delta}. \qquad (4)$$

Hence, by the definition of weak-tenacity and the choice of X, from (3) and (4) we have $T_w(G) \geq \frac{\delta(G)}{n-\delta(G)}$. Since $\delta(G) \geq \kappa(G)$ for any graph G, the following result is immediate.

Corollary 8 *Let G be a non-complete connected graph of order n. Then*

$$T_w(G) \geq \frac{\kappa(G)}{n-\kappa(G)}.$$

Theorem 7 *Let G be a non-complete connected graph of size q. Then*

$$t(G) \geq \frac{\kappa(G)(1+T_w(G))}{1+q}.$$

Proof Denote $\kappa(G)$, $t(G)$ and $T_w(G)$ by κ, t and T_w respectively. Let X be an arbitrary vertex cut of G and denote $\omega(G-X)$ by ω. From the definition of weak-tenacity we have $\omega T_w \leq |X| + m_e(G-X)$. furthermore,

$$m_e(G-X) + \omega + |X| - 1 \leq q.$$

Theorem 8 *Let G be a non-complete connected (p,q)-graph. Then $I_w(G) \geq 2T_w(G)$.*

Proof Let X be an arbitrary vertex cut of G. As in the proof of theorem 7, we have $2 \leq \omega(G-X) \leq \frac{1+q}{1+T_w}$. Thus,

$$\frac{|X| + m_e(G-X)}{\omega(G-X)} \geq \frac{(1+T_w)(|X| + m_e(G-X))}{1+q} \geq \frac{I_w(1+T_w)}{1+q}.$$

By the definition of weak-tenacity and the choice of X, we have

$$T_w(G) \geq \frac{I_w(G)(1+T_w(G))}{1+q},$$

i.e. $I_w \leq \frac{1+q}{1+T_w} T_w$.

5 Conclusion

If a system such as a communication network is modeled by a graph G, there are many graph theoretical parameters used to describe the vulnerability of communication networks including connectivity, integrity, toughness, binding number, tenacity and rupture degree. Two ways of measuring the vulnerability of a network is through the ease with which one can disrupt the network, and the cost of a disruption. Connectivity has the least cost as far as disrupting the network, but it does not take into account what remains after disruption. One can associate the cost with the number of the vertices destroyed to get small components and the reward with the number of the components remaining after destruction. The tenacity measure is compromise between the cost and the reward by minimizing the cost: reward ratio. Thus, a network with a large tenacity performs better under external attack. In this paper, we have obtained the exact values or bounds for the tenacity of some special graphs.

References

1. C. A. Barefoot, R. Entringer, H. C. Swart, Integrity of tree and powers of cycles, Congr. Numer., 58, 103–114 (1987).
2. J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, Macmillan London and Elsevier, New York, (1976).
3. V. Chvatal, Tough graphs and Hamiltonian Circuits. Discrete Math., 5, 111–119 (1973).
4. M. D. Cozzens, D. Moazzami, S. Stueckle, The Tenacity of a Graph, Graph Theory, Combinatorics, and Algorithems (Yousef Alavi and Allen Schwenk eds.) wiley, New Yourk, 1111–1121 (1995).
5. W. D. Godard, H. C. Swart, On the toughness of a graph. Quaestions Math., 13, 217–232 (1990).
6. A. Kirlangic, On the weak-integrity of graphs, J. Mathematical Modeling and Algorithems, 2, 81–95 (2003).
7. A. Kirlangic, On the weak-integrity of trees, Turk J. math., 27, 375–388 (2003).
8. D. Moazami, Stability Measure of a Graph-a Survey, J. Utilitas Mathematica, 57, 171–191 (2000).
9. D. R. Woodall, The binding number of a graph and its Anderson number, J. Combin. Theory Ser. B., 15, 225–255 (1973).