Global Analysis and Discrete Mathematics
Volume 7, Issue 1, pp. 63–66
ISSN: 2476-5341 *Research Article*

Central Factor Groups of Locally Finite and Locally Nilpotent Groups

Mitra Hassanzadeh · Rasoul Hatamian · Saeed Kayvanfar

Received: 18 March 2022 / Accepted: 4 July 2022

Abstract Let $\mathcal{P}$ be a group property. A group G is called a locally $\mathcal{P}$-group if each finite subset of G is contained in a $\mathcal{P}$-subgroup of G. In this paper some relations between the central factor groups and commutator subgroups in locally nilpotent and locally finite groups are investigated.

Keywords Schur's theorem · Locally finite group · Locally nilpotent group · Isoclinism

Mathematics Subject Classification (2010) 20B07 · 20F14 · 20F50

1 Introduction

A famous theorem of Schur asserts that for a group G, if $G/Z(G)$ is finite, then G' is finite. This theorem is widely studied as follows:

$$G/Z(G) \text{ is } \chi - group \Leftrightarrow G' \text{ is } \chi - group.$$

Mann in [3] proved that if the central factor group of a group is locally finite then so is its derived subgroup. For a nilpotent group G, Hilton in [2] showed that $G/Z(G)$ is a p- group if and only if G' is a p-group.

In this paper, we investigate the converse of the mentioned theorem of Mann for the quotient group $G/Z_n(G)$ and the subgroup $\gamma_{n+1}(G)$ as follows:

M. Hassanzadeh (Corresponding Author)
Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.
E-mail: mtr.hassanzadeh@gmail.com

R. Hatamian
Department of Basic Sciences, School of Mathematical Sciences, Payame Noor University, Tehran, Iran.
E-mail: hatamianr@pnu.ac.ir

S. Kayvanfar
Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.
E-mail: skayvanf@yahoo.com

Theorem 1 *For a group G, if $\gamma_{n+1}(G)$ is locally finite and $G/Z_n(G)$ is torsion, then $G/Z_n(G)$ is locally finite.*

Furthermore, we prove a theorem of Mann's type for a group whose central factor group is a locally finite π-group.

Theorem 2 *If G is a group for which $G/Z(G)$ is a locally finite π-group, then G' is a locally finite π-group.*

In the rest of the paper a locally nilpotent group G is considered instead of a nilpotent group to verify what Hilton proved. More precisely, we show that if $G/Z(G)$ is a p-group, then G' is a p-group too. But by an example we expose that the converse does not hold in general. Thus we have to impose an extra condition to conclude the converse of the statement as follows:

Theorem 3 *For a locally nilpotent group G, if G' is a p-group and $G/Z(G)$ is torsion, then $G/Z(G)$ is a p-group.*

2 Proof of Main Theorems

To prove our main theorems the following preliminaries are needed.

Definition 1 Let G and H be two groups. An isoclinism from G to H is a pair of homomorphisms (α, β) with

$$\alpha : G/Z(G) \to H/Z(H),$$

and

$$\beta : \gamma_2(G) \to \gamma_2(H),$$

such that the following diagram is commutative:

$$\begin{array}{ccccc} \frac{G}{Z(G)} & \times & \frac{G}{Z(G)} & \to \gamma(1,G) & \gamma_2(G) \\ & \alpha^2 \downarrow & & & \beta \downarrow \\ \frac{H}{Z(H)} & \times & \frac{H}{Z(H)} & \to \gamma(1,H) & \gamma_2(H). \end{array}$$

Whenever the groups G and H are isoclinic, we write $G \sim H$.

Lemma 1 *[1, 6.1] Let G be a group. Then there exists a group T such that*

(a) *$G \sim T$,*
(b) *$Z(T) \leq \gamma_2(T)$,*
(c) *T is finitely generated, in case G is finitely generated.*

Lemma 2 *A finitely generated torsion nilpotent group is finite.*

Proof (Proof of Theorem 1) Let $H/Z_n(G)$ be a finitely generated subgroup of $G/Z_n(G)$. Consider the finitely generated torsion factor group $H/Z_n(H)$. Set $K = H/Z_{n-1}(H)$. Since $K/Z(K)$ is finitely generated, then there exists a finitely generated group S isoclinic to K. By Lemma 2, we have a finitely generated group T for which $T \sim K$ and $Z(T) \leq \gamma_2(T)$. Since $T/Z(T)$ is a finitely generated torsion group it can be seen that $\gamma_2(T)$ is finitely generated and therefore $\gamma_2(T)/\gamma_3(T)Z(T)$ is also finitely generated. The recent group is torsion abelian and so it has to be finite. To continue this process we observe that $\gamma_n(T)Z(T)$ is finitely generated and $\gamma_n(T)Z(T)/\gamma_{n+1}(T)Z(T)$ is finite. Thus $\gamma_{n+1}(T)Z(T)/Z(T)$ is finitely generated. The last is also finite for the sake of the following isomorphisms:

$$\gamma_{n+1}(\frac{T}{Z(T)}) \cong \gamma_{n+1}(\frac{H}{Z_n(H)}) \cong \frac{\gamma_{n+1}(H)}{\gamma_{n+1}(H) \cap Z_n(H)}.$$

Hence,

$$\frac{\gamma_n(T)Z(T)/Z(T)}{\gamma_{n+1}(T)Z(T)/Z(T)} \simeq \frac{\gamma_n(T)Z(T)}{\gamma_{n+1}(T)Z(T)},$$

and therefore $\gamma_n(T)Z(T)/Z(T)$ is finite. This can be continued to deduce $\gamma_2(T)/Z(T)$ is finite. This implies that $T/Z(T)$ and so $H/Z_n(H)$ are finite. As

$$\frac{H}{Z_n(H)} \cong \frac{H/Z_n(G)}{Z_n(H)/Z_n(G)},$$

and $H/Z_n(G)$ is finitely generated, then so is $Z_n(H)/Z_n(G)$. Also $Z_n(H)/Z_n(G)$ is a torsion nilpotent group which yields that $H/Z_n(G)$ is finite.

The following example shows that the assumption of being torsion for $G/Z_n(G)$ can not be omitted.

Example 1 Regard $G = \mathbf{Z}_2 \wr \mathbf{Z}$ in which $\wr$ denotes the standard wreath product of groups. It can be easily seen that G' is the direct product of countable number of the groups $\mathbf{Z}_2$ which is locally finite. Also $Z(G) = 1$, since $\mathbf{Z}$ is an infinite abelian group. Therefore $G/Z(G)$ is not torsion and so it is not locally finite.

It is known that if $G/Z(G)$ is finitely generated, then G' is not necessary finitely generated. But from the proof of Theorem 1, as an immediate consequence we have the following result.

Corollary 1 *If G is a group with finitely generated torsion central quotient, then G' is finitely generated.*

Proof (Proof of Theorem 2) Consider a finitely generated subgroup of G', say H, and assume that $H =< h_1, h_2, \ldots, h_k >$, for some $k \in \mathbf{N}$. Let

$$h_i = [x_{1_i}, y_{1_i}]^{\alpha_1} \ldots [x_{l_i}, y_{l_i}]^{\alpha_l},$$

where $\alpha_1, \ldots, \alpha_l \in \mathbf{Z}, l \in \mathbf{N}$. Set

$$K = \left\langle x_{1_i}, \ldots, x_{l_i}, y_{1_i}, \ldots, y_{l_i} \mid 1 \leq i \leq k \right\rangle.$$

If $M =< K, Z(G) >$, then $M/Z(G)$ is a finite π-group and so is $M/Z(M)$. This yields that K' is a finite π-group too. Clearly $H \leq K'$ has to be finite and this completes the proof.

Theorem 2, can be easily extended to the nth central factor group.

Theorem 4 *If $G/Z_n(G)$ is a locally finite π-group, then $\gamma_{n+1}(G)$ is a locally finite π-group.*

Corollary 2 *Suppose that G is a locally nilpotent group. If $G/Z(G)$ is a p-group, then G' is a p-group.*

Proof It is easy to check that a locally nilpotent group is locally finite if and only if it is torsion. Now use Theorem 2 to achieve the result.

Proof (Proof of Theorem 3) By contrary consider $G/Z(G)$ not to be a p-group. Since $G/Z(G)$ is torsion, then there exists an element $xZ(G) \in G/Z(G)$ such that $|xZ(G)| = q^\alpha$, for some prime $q \neq p$ and $\alpha > 0$. Therefore there is an element $y \in G$ such that $[x, y] \neq 1$. Set $H =< x, y, Z(G) >$. Then $H/Z(H)$ is a finitely generated torsion nilpotent group and hence it is finite. By [1, 7.7 and 7.8] there exists a finite group M isoclinic to H such that if $H/Z(H)$ is a π-group, then M is a π-group. It can be easily seen that M is nilpotent. Thus $M = \prod_{p\in\pi} M_p$, where M_p is the Sylow p-subgroup of M. Since $q||xZ(H)|$, $q \in \pi$, M has the q-Sylow subgroup. Since G' is a p-group, $M'_q = 1$. In this case $M_q \leq Z(M)$ which implies that $M/Z(M)$ and hence $H/Z(H)$ does not have any q-element and this is a contradiction.

Example 2 Consider G as the semidirect product of infinite prüfer group $\mathbf{Z}_{p^\infty}$ and the infinite cyclic group $\mathbf{Z}$ with the action $\theta : \mathbf{Z} \to Aut(\mathbf{Z}_{p^\infty})$ by the rule $\theta(1)(x) = x^{1+p}$. One can check that G' is a p-group and

$$Z(G) =< (0, 1/p + \mathbf{Z}) > .$$

By regarding $(1, 1/p + \mathbf{Z})Z(G)$ as an element of $G/Z(G)$, it can be seen that $G/Z(G)$ is not a torsion group. One can readily check that the semidirect product of the subgroups $\mathbf{Z}$ by $< 1/p^k + \mathbf{Z} >$ is nilpotent which implies that G is locally nilpotent.

References

1. N. S. Hekster, On the structure of n-isoclinism classes of groups, J. Pure Appl. Algebra, 40, 63–85 (1986).
2. P. Hilton, On a theorem of Schur, Int. J. Math. Sci., 28, 455–460 (2001).
3. A. Mann, The exponents of central factor and commutator groups, J. Group Theory, 10, 435–436 (2007).