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Legendre Spectral Tau Method for Solving the Fractional Integro–Differential Equations with A Weakly Singular Kernel

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Abstract In this paper, the Tau method based on shifted Legendre polynomials has been introduced to approximate the numerical solutions of a class of fractional integro-differential equations with a weakly singular kernel. By using operational matrices we reduce the problem to a set of algebraic equations. Also, the upper bound of the error of the shifted Legendre expansion is investigated. Finally, several numerical examples are given to illustrate the high accuracy of the method.

Keywords Shifted Legendre Tau method · Weakly singular kernel · Fractional integro-differential equation

Mathematics Subject Classification (2010) 65R20 · 45D05

1 Introduction

In recent years scientist have been focused on the study of fractional differential equations due to their important application in various branches of science and engineering. Fractional integral equations are used to model alot of physical problems [1, 4, 11, 22].

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The application of fractional integro-differential equations with a weakly singular kernel create a wide area of research for many researches such as in the elasticity and fracture mechanics [27], radiative equilibrium [12] and so on.

Since solving the fractional integral equations and integro-differential equations is very difficult analytically, therefore several researchers have been solved these problems numerically. Many published papers have been devoted to solve some physical phenomenon both analytically and numerically modeled by fractional differential equations and fractional partial differential equations [7, 21, 20].

Some analytical methods like the homotopy analysis method [3], the fractional differential transform method [2], the Adomian decomposition [16, 17] and Laplace transform method [19] were investigated by the authors. Also, there have been several numerical methods for the fractional integro-differential. For instance, the second kind Chebyshev wavelet method [26], the Taylor expansion method [9], the cubic B-spline wavelet method [15], the hybrid collection method [13]. In [23], the Nystrom method has been used for solving fractional Volterra-Fredholm integro-differential equations with mixed conditions.
However, only a few methods are proposed to solve the fractional integro-differential equations with a weakly singular kernel [25, 24].

In this work, we applied the shifted Legendre Tau (SLT) method for solving fractional integro-differential equations with weakly singular kernel and we compared our numerical results with the results in [25, 24].

Let us consider the general form of a class of fractional integro-differential equation with weakly singular kernel.

$$D^{\alpha}y(t) = f(t) + \lambda_1 \int_0^t \frac{y(\tau)}{(t-\tau)^{\beta}} d\tau + \lambda_2 \int_0^1 k(t,\tau) y(\tau) d\tau, \quad t \geq 0, \tag{1}$$

subject to initial conditions

$$y^{(s)}(0) = d_j, \quad s = 0, 1, \dots, r-1, \quad j = 0, 1, \dots, r-1, \quad r-1 < \alpha \leq r, \tag{2}$$

where $r \in \mathbb{N}$ and $y^{(s)}(t)$ stands for the s-th order derivative of $y(t)$, $D^{\alpha}(.)$ denotes the Caputo fractional order derivative of order α and $y(t)$ is the output response. L_2-functions $k(t,\tau)$ and $f(t)$ are known. Here λ_1, λ_2 are real constants and $0 < \beta < 1$.

For $\alpha = 1$ and $\beta = 0$, equation (1) becomes the linear integro-differential equation. Especially if $\alpha = 0$ and $\lambda_2 = 0$, equation (1) reduces to the Abel's equation

$$y(t) = f(t) + \lambda_1 \int_0^t \frac{y(\tau)}{(t-\tau)^{\beta}} d\tau$$

which occurs in many branches of science such as microscopy, seismology, radio astronomy, and atomic scattering [10, 14, 8, 5, 6].

The main goal of this paper is to present an efficient numerical algorithm for the solution of equation (1). Our method is devoted to reducing the problem to a set of algebraic equations by expanding the approximate solution $y(t)$

as shifted Legendre polynomials with unknown coefficients. The operational matrices of integral and differential parts appearing in equation are given. These matrices are utilized to evaluate the unknown coefficients of shifted Legendre polynomials.

The organization of the rest of this article is as follows. In the next section, we introduce the properties of shifted Legendre polynomials and function approximation. We express definitions and properties of fractional operatores in section 3. In section 4 after constructing the operational matrices of shifted Legendre polynomials, we summarize the process of solving the fractional integro-differential equations with a weakly singular kernel based on the shifted Legendre Tau method. The upper bound of the error of the shifted Legendre expansion is proposed in section 5. In section 6, we show the numerical results to illustrate the performance of the method and compare with the results in [24] and [25]. Finally, the conclusion is given in section 7.

2 Properties of shifted Legendre polynomials

The classical Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formulae

$$L_0(t)=1,\ L_1(t)=t,$$
$$L_{i+1}(t)=\frac{2i+1}{i+1}t\ L_i(t)-\frac{i}{i+1}L_{i-1}(t),\quad i=1,2,\dots .$$

Assume that $t\in[a,b]$ and let $\bar{t}=\frac{2t-a-b}{b-a}$. Then $\{L_i(\bar{t})\}$ are called the shifted Legendre polynomials on $[a,b]$. In this paper, we mainly consider the shifted Legendre polynomials defined on $[0,l]$.

For $t\in[0,l]$, let $L_{l,i}(t)=L_i(\frac{2t-l}{l})$, $i=0,1,2,\dots$. Then the shifted Legendre polynomials $\{L_{l,i}(t)\}$ are defined by

$$\begin{aligned}
&L_{l,0}(t)=1,\\
&L_{l,1}(t)=\frac{2t-l}{l},\\
&L_{l,i+1}(t)=\frac{(2i+1)(2t-l)}{(i+1)l}L_{l,i}(t)-\frac{i}{i+1}L_{l,i-1}(t),\quad i=1,2,\dots .
\end{aligned}$$

If $\Phi_{l,m}(t)$ is a vector function of shifted Legendre polynomials on the interval $[0,l]$, as

$$\Phi_{l,m}(t)=\left[L_{l,0},L_{l,1},\dots,L_{l,m}\right]^T, \tag{3}$$

then the set of $L_{l,i}(t)$ is a complete $L^2(0,l)$-orthogonal system, namely

$$\int_0^l L_{l,i}(t)L_{l,j}(t)dt=\begin{cases}\dfrac{l}{2i+1}, & i=j,\\ 0, & i\neq j.\end{cases}$$

So we define $\Pi_m = \text{span}\{L_{l,0}, L_{l,1}, \ldots, L_{l,m}\}$. For any $y(t) \in L^2(0,l)$, we write

$$y(t) = \sum_{j=0}^{\infty} c_j L_{l,j}(t),$$

where the coefficients c_j are given by

$$c_j = \frac{2j+1}{l} \int_0^l y(t) L_{l,j}(t) dt, \quad j = 0, 1, 2, \ldots . \tag{4}$$

In practice, only the first $(m+1)$-terms of shifted Legendre polynomials are considered.

Hence we can write

$$y_m(t) \simeq \sum_{j=0}^{m} c_j L_{l,j}(t) = C^T \Phi_{l,m}(t) = C^T V X_t,$$

where $C^T = [c_0, c_1, \ldots, c_m]$ and V is a non-singular matrix given by

$$\Phi_{l,m}(t) = V X_t,$$

with a standard basic vector, $X_t = \left[1, t, t^2, \ldots, t^m\right]^T$.
Similarly a function of two independent variables $k(t,\tau)$ may be expressed in terms of the double shifted Legendre polynomials as

$$k(t,\tau) \simeq \sum_{i=0}^{m} \sum_{j=0}^{m} k_{i,j} L_{l,i}(t) L_{l,j}(\tau) = \Phi_{l,m}^T(t) K \Phi_{l,m}(\tau), \tag{5}$$

where K is a $(m+1) \times (m+1)$ matrix as

$$K = \begin{bmatrix} k_{00} & k_{01} & \ldots & k_{0m} \\ k_{10} & k_{11} & \ldots & k_{1m} \\ \vdots & \vdots & \ldots & \vdots \\ k_{m0} & k_{m1} & \ldots & k_{mm} \end{bmatrix},$$

where

$$k_{i,j} = \left(\frac{2i+1}{l}\right)\left(\frac{2j+1}{l}\right) \int_0^l \int_0^l k(t,\tau) L_{l,i}(t) L_{l,j}(\tau) dt d\tau. \tag{6}$$

Also, $k(t,\tau)$ can be expressed as

$$k(t,\tau) \simeq \Phi_{l,m}^T(t) K \Phi_{l,m}(\tau) = X_t^T V^T K V X_\tau,$$

where $V = [v_{i,j}]_{i,j=0,1,\ldots,m}$ is a non-singular matrix given by $\Phi_{l,m}(t) = V X_t$ with a standard basic vector, $X_t = \left[1, t, t^2, \ldots, t^m\right]^T$. If we take $\overline{K} = V^T K V$ then we can write $k(t,\tau) \simeq X_t^T \overline{K} X_\tau$.

3 Fractional calculus

Definition 1 [24] The Riemann-Liouville fractional integral operator J^α of order α is given by

$$J^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau, \quad \alpha > 0,$$
$$J^0 y(t) = y(t).$$

Definition 2 [24] The Caputo definition of fractional operator is given by

$$D^\alpha y(t) = \begin{cases} \dfrac{d^r y(t)}{dt^r}, & \alpha = r \in \mathbb{N}, \\ \dfrac{1}{\Gamma(r-\alpha)} \int_0^t \dfrac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}} d\tau, & 0 \leq r-1 < \alpha < r. \end{cases}$$

The Caputo fractional derivatives of order α is also defined as

$$D^\alpha y(t) = J^{r-\alpha} D^r y(t).$$

The relation between the Caputo operator and the Riemann-Liouville is given by the

$$D^\alpha J^\alpha y(t) = y(t),$$
$$D^\alpha J^\alpha y(t) = y(t) - \sum_{k=0}^{r-1} y^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0.$$

4 Operational matrices of shifted Legendre polynomials

In this section, we derive the operator matrix representation for the differential and integral parts seeming in the equation (1) using the operational Tau method based on shifted Legendre polynomials.

4.1 Matrix representation of (1)

As a consequence of the previous section, and aid of following lemma and theorems we derive formulas for numerical solvability of fractional integro-differential equation with weakly singular kernel (1) based on shifted Legendre polynomial of the operational Tau method.

Lemma 1 *Let $y_m(t) \simeq C^T V X_t$ be a polynomial where*

$$C^T = [c_0, c_1, \ldots, c_m, 0 \ldots] \quad \textit{and} \quad X_t = [1, t, \ldots]^T,$$

then we have

$$\frac{d^k}{dt^k} y_m(t) = C^T V \eta^k X_t,$$

$$t^k y_m(t) = C^T V \mu^k X_t,$$

$$k = 0, 1, 2, \ldots,$$

where

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ & 0 & 1 & 0 & \\ & & 0 & 1 & \\ & & & & \ddots \end{bmatrix},$$

and

$$\eta = \begin{bmatrix} 0 & & & \ldots \\ 1 & 0 & & \\ 0 & 2 & 0 & \\ 0 & 0 & 3 & \\ & & & \ddots \end{bmatrix}.$$

Proof see [18].

4.2 Matrix representation of integral parts

Lemma 2 *If Γ is the Gamma function, then we have*

$$\int_0^t \frac{\tau^m}{(t-\tau)^{\alpha-r+1}} \, \mathrm{d}\tau = \frac{\Gamma(r-\alpha)\Gamma(m+1)}{\Gamma(m-\alpha+r+1)} t^{m-\alpha+r}, \qquad m = 0, 1, 2, \ldots.$$

Proof With integration by parts and using $\Gamma(\alpha) = (\alpha-1)!$ for $\alpha > 0$ it can easily be obtained.

Theorem 1 *Let $\Phi_{l,m}(t) = V X_t$ be the shifted Legendre vector then*

$$\int_0^t \frac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}} d\tau \simeq C^T V \eta^r \Gamma A V X_t, \tag{7}$$

where Γ is a diagonal matrix with elements

$$\Gamma_{i,i} = \frac{\Gamma(r-\alpha)\Gamma(i+1)}{\Gamma(i-\alpha+r+1)}, \qquad i = 0, 1, 2, \ldots, m,$$

and

$$A = \Big[B_0, B_1, \ldots, B_m\Big]^T, \qquad B_j = \Big[t_{j,0}, t_{j,1}, \ldots, t_{j,m}],$$

which $t_{j,i}$, $i, j = 0, 1, \ldots, m$ are the coefficients of $L_{l,i}$, $i = 0, 1, \ldots, m$ in expansion of $t^{j-\alpha+r}$.

Proof

$$\begin{aligned}
\int_0^t \frac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau \simeq \int_0^t \frac{C^T V \eta^r X_\tau}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau \\
&= C^T V \eta^r \int_0^t \frac{X_\tau}{(t-\tau)^{\alpha-r+1}}\mathrm{d}\tau \\
&= C^T V \eta^r \int_0^t \frac{[1,\tau,\ldots,\tau^m]^T}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau \\
&= C^T V \eta^r \Bigg[\int_0^t \frac{1}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau, \int_0^t \frac{\tau}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau,\ldots, \\
&\qquad \int_0^t \frac{\tau^m}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau\Bigg]^T,
\end{aligned}$$

by using lemma (2) we can write

$$\begin{aligned}
\int_0^t \frac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}}\,\mathrm{d}\tau \simeq C^T V \eta^r \Bigg[\frac{\Gamma(r-\alpha)\Gamma(1)}{\Gamma(r-\alpha+1)}t^{r-\alpha}, \frac{\Gamma(r-\alpha)\Gamma(2)}{\Gamma(r-\alpha+2)}t^{r-\alpha+1},\ldots, \\
&\qquad \frac{\Gamma(r-\alpha)\Gamma(m+1)}{\Gamma(m-\alpha+r+1)}t^{m-\alpha+r}\Bigg]^T \\
&= C^T V \eta^r \Gamma \Pi, \qquad (8)
\end{aligned}$$

where

$$\Pi = \left[t^{r-\alpha}, t^{r-\alpha+1}, \ldots, t^{m-\alpha+r}\right]^T.$$

By approximating $t^{j-\alpha+r}$, $j = 0, 1, \ldots, m$, we get

$$t^{j-\alpha+r} \simeq \sum_{i=0}^{m} t_{j,i} L_{l,i}(t) = B_j \Phi_{l,m}(t), \qquad B_j = [t_{j,0}, t_{j,1}, \ldots, t_{j,m}],$$

we obtain

$$\begin{aligned}
&\Pi = [B_0 V X_t, B_1 V X_t, \ldots, B_m V X_t]^T = A\Phi_{l,m}(t), \\
&A = [B_0, B_1, \ldots, B_m]^T.
\end{aligned} \qquad (9)$$

By substituting (9) into (8) we obtain

$$\int_0^t \frac{y^{(r)}(\tau)}{(t-\tau)^{\alpha-r+1}} d\tau \simeq C^T V \eta^r \Gamma A V X_t, \qquad (10)$$

Theorem 2 *Let $\Phi_{l,m}(t) = V X_t$ be the shifted Legendre vector then*

$$\int_0^t \frac{y(\tau)}{(t-\tau)^\beta} d\tau \simeq C^T V \Delta A V X_t, \qquad (11)$$

where Δ is a diagonal matrix with elements

$$\Delta_{i,i} = \frac{\Gamma(1-\beta)\Gamma(i+1)}{\Gamma(i-\beta+2)}, \qquad i = 0,1,2,\ldots,m,$$

and

$$A = \Big[B_0, B_1, \ldots, B_m\Big]^T, \qquad B_j = \Big[t_{j,0}, t_{j,1}, \ldots, t_{j,m}\Big],$$

which $t_{j,i}$, $i,j = 0,1,\ldots,m$ are the coefficients of $L_{l,i}$, $i = 0,1,\ldots,m$ in expansion of $t^{j-\beta+1}$.

Proof The proof is similar to proof in previous theorem.

Theorem 3 *Let the analytic function $y(t)$ and $k(t,\tau)$ be expressed as*

$$y(t) \simeq \sum_{j=0}^{m} c_j L_{l,j}(t) = C^T \Phi_{l,m}(t) = C^T V X_t,$$

$$k(t,\tau) \simeq \Phi_{l,m}^T(t) K \Phi_{l,m}(\tau) = X_t^T V^T K V X_\tau = X_t^T \overline{K} X_\tau = \sum_{i=0}^{m}\sum_{j=0}^{m} \overline{K}_{i,j}\tau^i t^j,$$

where $C = [c_0, c_1, \ldots, c_m]$ and $V = [v_{i,j}]_{i,j=0}^m$ is a non-singular matrix and $X_t = [1, t, t^2, \ldots, t^m]^T$, then we have

$$\int_0^1 k(t,\tau)y(\tau)d\tau \simeq C^T V M X_t, \tag{12}$$

where

$$M = \begin{bmatrix} \sum\limits_{j=0}^{m} \dfrac{\overline{K}_{0,j}}{j+1} & \cdots\ \cdots & \sum\limits_{j=0}^{m} \dfrac{\overline{K}_{m,j}}{j+1} \\ \vdots & \vdots\ \ \vdots & \vdots \\ \sum\limits_{j=0}^{m} \dfrac{\overline{K}_{0,j}}{j+m+1} & \cdots\ \cdots & \sum\limits_{j=0}^{m} \dfrac{\overline{K}_{m,j}}{j+m+1} \end{bmatrix}.$$

Proof Using Lemma (1) we have

$$\begin{aligned} &y(\tau) \simeq C^T V X_\tau, \\ &k(t,\tau)y(\tau) \simeq C^T V\Big[k(t,\tau), \tau k(t,\tau), \tau^2 k(t,\tau), \ldots, \tau^m k(t,\tau)\Big]^T, \\ &k(t,\tau)\tau^n = \sum_{i=0}^{m}\sum_{j=0}^{m} \overline{K}_{i,j} t^j \tau^{n+i}. \end{aligned}$$

So, the desired integration term can be written as

$$\int_0^1 k(t,\tau)y(\tau)d\tau \simeq C^T V\Big(\Big[\sum_{i=0}^{m}\sum_{j=0}^{m} \overline{K}_{i,j} t^j \frac{1}{n+i+1}\Big]_{n=0}^{m}\Big)^T.$$

On the other hand, we can show that

$$\Big(\Big[\sum_{i=0}^{m}\sum_{j=0}^{m}\overline{K}_{i,j}\frac{t^j}{n+i+1}\Big]_{n=0}^{m}\Big)^T = MX_t,$$

which M is the following matrix

$$M = \begin{bmatrix} \sum\limits_{j=0}^{m}\dfrac{\overline{K}_{0,j}}{j+1} & \cdots\ \cdots & \sum\limits_{j=0}^{m}\dfrac{\overline{K}_{m,j}}{j+1} \\ \vdots & \vdots\ \ \vdots & \vdots \\ \sum\limits_{j=0}^{m}\dfrac{\overline{K}_{0,j}}{j+m+1} & \cdots\ \cdots & \sum\limits_{j=0}^{m}\dfrac{\overline{K}_{m,j}}{j+m+1} \end{bmatrix}.$$

So, we have

$$\int_0^1 k(t,\tau)y(\tau)d\tau \simeq C^T VMX_t.$$

4.3 Matrix representation for the supplementary conditions

Let $y(t) \simeq \sum\limits_{j=0}^{m} c_j L_{l,j}(t) = C^T VX_t$ on the left hand side of (2), it can be written as

$$\begin{aligned} &y^{(s)}(0) = d_j, && s = 0,1,\ldots,r-1,\\ &C^T V\eta^j X_0 = d_j, && j = 0,1,\ldots,r-1. \end{aligned}$$

Let $H_j = \eta^j X_0$ where $X_0 = [1,0,0,\ldots,0]^T$ thus the j-th condition number of (2) is converted to

$$C^T VH_j = 0 \quad j = 0,1,\ldots,r-1.$$

Now, by setting H as the matrix with columns H_j, $j = 0,1,\ldots,r-1$ and by setting $d = [d_1,d_2,\ldots,d_j]$, as the vector that contains right-hand side of supplementary conditions, they take the form

$$C^T VH = d. \tag{13}$$

Now, Let us start our algorithm to solve (1) and (2).

We assume that the functions $f(t)$, generally are polynomial. Otherwise, we can approximate it by polynomials to any degree of accuracy (by interpolation or Taylor series or other suitable method). Now, we approximate $f(t)$ by the shifted Legendre polynomials as

$$f(t) \simeq \sum_{j=0}^{m} f_j L_{l,j}(t) = FVX_t, \tag{14}$$

where $F = [f_0, f_1, \dots, f_m]$ and f_j are given in (4).

Using (7), (11), (12), (14), and substituting in equation (1), it is easy to obtain that

$$\frac{1}{\Gamma(r-\alpha)} C^T V \eta^r \Gamma A V X_t = F V X_t + \lambda_1 C^T V \Delta A V X_t + \lambda_2 C^T V M X_t,$$

thus, the matrix vector multiplication representation for the (1) is as follows

$$C^T V_1 \Phi_{l,m}(t) = F \Phi_{l,m}(t) + C^T V_2 \Phi_{l,m}(t) + C^T V_3 \Phi_{l,m}(t),$$

where $V_1 = \dfrac{1}{\Gamma(r-\alpha)} V \eta^r \Gamma A$, $V_2 = \lambda_1 V \Delta A$ and $V_3 = \lambda_2 V M V^{-1}$. As we pointed out in Section 2, the orthogonality of $\left\{L_{l,i}(t)\right\}_{i=0}^{m-1}$, so we have

$$C^T V_1 = F + C^T V_2 + C^T V_3,$$

also from equation (13) we have following system

$$\begin{cases} C^T[V_1 - V_2 - V_3] = F, \\ C^T V H = d. \end{cases} \tag{15}$$

Now, setting

$$\begin{aligned} &\psi = V_1 - V_2 - V_3, \\ &\overline{H} = V H, \\ &G = [\overline{H_1}, \overline{H_2}, \dots, \overline{H_r}, \psi_1, \psi_2, \dots, \psi_{m+1-r}], \\ &g = [d_1, d_2, \dots, d_r, F_0, F_1, \dots, F_{m-r}], \end{aligned}$$

where $\overline{H_i}$ denotes the i-th column of $\overline{H}$, system of (15) can be written as $C^T G = g$, which must be solved for the unknown coefficients $c_0, c_1, \dots, c_m$.

4.4 Algorithm of shifted Legendre Tau approximation

Step 1. Choose m, form the set of shifted Legendre polynomials $\left\{L_{l,i}(t)\right\}_{i=0}^{m}$, and let the approximate solution be $y_m(t) \simeq \sum_{i=0}^{m} c_i L_{l,i}(t)$.

Step 2. Compute the non singular coefficient matrix V with respect to $X_t = \left[1, t, t^2, \dots, t^m\right]^T$, such that $\Phi_{l,m}(t) = V X_t$.

Step 3. By using orthogonality condition of $\{L_{l,i}(t)\}_{i=0}^{m}$ as

$$f(t) \simeq \sum_{j=0}^{m} f_j L_{l,j}(t)$$

where

$$f_j = \frac{2j+1}{l} \int_0^l f(t) L_{l,j}(t) dt, \quad j = 0, 1, 2, \ldots, m,$$

compute $F = [f_0, f_1, \ldots, f_m]$.

Step 4. Compute the matrices M, η, A, Γ, and Δ from Lemmas (1), (2), and Theorems (1), (2), and (3) then set $V_1 = \frac{1}{\Gamma(r-\alpha)} V \eta^r \Gamma A$, $V_2 = \lambda_1 V \Delta A$, and $V_3 = \lambda_2 V M V^{-1}$.

Step 5. Let $C^T = [c_0, c_1, \ldots, c_m]$ and obtain the entries of the vector solution C^T from the $C^T G = g$ where $G = [\overline{H_1}, \overline{H_2}, \ldots, \overline{H_r}, \psi_1, \psi_2, \ldots, \psi_{m+1-r}]$ and $g = [d_1, d_2, \ldots, d_r, F_0, F_1, \ldots, F_{m-r}]$, $\overline{H_i}$ denotes the i-th column of matrix VH and ψ_i denotes the i-th column of matrix $V_1 - V_2 - V_3$.

5 Convergence analysis

In this section we present the shifted Legendre expansion of a function $y(t)$ with bounded second derivative, converges uniformly to $y(t)$. Also we state the estimate error for the proposed method.

Theorem 4 *A continuous function $y(t) \in [0, l]$, with bounded second derivative, say $\left|\frac{d^2 y(t)}{dt^2}\right| \leq \alpha$, can be expanded as an infinite sum of shifted Legendre polynomials and the series $\sum_{i=0}^{\infty} c_i L_{l,i}(t)$ converges uniformly to the $y(t)$. Furthermore, we have*

$$\int_0^l \Big(y(t) - \sum_{i=0}^{m} c_i L_{l,i}(t) \Big)^2 dt \leq \alpha l^2 \sqrt{\frac{3l}{8}} \sqrt{\sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4}}$$

Proof From (4), it follows that

$$c_i = \left(\frac{2i+1}{l} \right) \int_0^l y(t) L_{l,i}(t) dt, \quad i = 0, 1, \ldots, m.$$

By partial integration and using following equation

$$L'_{l,i+1} - L'_{l,i-1} = \frac{2}{l}(2i+1) L_{l,i}(t)$$

we have

$$
\begin{aligned}
c_i &= \frac{2i+1}{l} \times \frac{l}{2(2i+1)} \int_0^l y(t)\Big(L'_{l,i+1}(t) - L'_{l,i-1}(t)\Big)dt \\
&= \frac{1}{2}\Big(y(t)\Big(L_{l,i+1}(t) - L_{l,i-1}(t)\Big)\Big|_0^l - \int_0^l \Big(L_{l,i+1}(t) - L_{l,i-1}(t)\Big)\frac{dy}{dt}dt \\
&= -\frac{1}{2}\int_0^l \frac{l}{2(2i+3)}\Big(L'_{l,i+2}(t) - L'_{l,i}(t)\Big)\frac{dy}{dt}dt \\
&+ \frac{l}{2}\int_0^l \frac{l}{2(2i-1)}\Big(L'_{l,i}(t) - L'_{l,i-2}(t)\Big)\frac{dy}{dt}dt \\
&= \frac{l}{4}\int_0^l \frac{d^2y(t)}{dt^2}\Big(\frac{L_{l,i+2}(t) - L_{l,i}(t)}{2i+3}\Big)dt \\
&- \frac{l}{4}\int_0^l \frac{d^2y(t)}{dt^2}\Big(\frac{L_{l,i}(t) - L_{l,i-1}(t)}{2i-1}\Big)dt.
\end{aligned}
$$

Now, let $Q_{l,i}(t) = (2i-1)L_{l,i+2}(t) - 2(2i+1)L_{l,i}(t) + (2i+3)L_{l,i-2}(t)$ then we have

$$
c_i = \frac{l}{4(2i+3)(2i-1)} \int_0^l \frac{d^2y(t)}{dt^2} Q_{l,i}(t)dt,
$$

thus

$$
\begin{aligned}
|c_i| &\le \frac{l}{4(2i+3)(2i-1)} \int_0^l \Big|\frac{d^2y(t)}{dt^2}\Big| |Q_{l,i}(t)|dt \\
&\le \frac{l\alpha}{4(2i+3)(2i-1)} \int_0^l |Q_{l,i}(t)|dt.
\end{aligned}
$$

Also we have

$$
\begin{aligned}
\Big(\int_0^l |Q_i(t)|dt\Big)^2 &= \Big(\int_0^l |(2i-1)L_{l,i+2}(t) - 2(2i+1)L_{l,i}(t) + (2i+3)L_{l,i-2}(t)|dt\Big)^2 \\
&\le \Big(\int_0^l (1)^2 dt\Big)\Big(\int_0^l (2i-1)^2 L^2_{l,i+2}(t) + (4i+2)^2 L^2_{l,i}(t) + (2i+3)^2 L^2_{l,i-2}(t)\Big)dt \\
&\le l\Big(\frac{(2i-1)^2 l}{2i+5} + \frac{(4i+2)^2 l}{2i+1} + \frac{(2i+3)^2 l}{2i-3}\Big) \\
&\le \frac{6l^2(2i+3)^2}{2i-3}.
\end{aligned}
$$

Then we get

$$
\int_0^l |Q_i(t)|dt \le \frac{\sqrt{6}\, l(2i+3)}{\sqrt{2i-3}}.
$$

Thus we obtain

$$
|c_i| \le \frac{l\alpha}{4(2i+3)(2i-1)} \times \frac{\sqrt{6}\, l(2i+3)}{\sqrt{2i-3}} = \frac{l^2\alpha\sqrt{6}}{4\sqrt{(2i-3)^3}}.
$$

Consequently, $\sum\limits_{i=0}^{\infty} c_i$ is absolute convergent and thus the expansion of the function converges uniformly.
Also, we let

$$\varepsilon_n = \Big(\int_0^l (y(t) - \sum_{i=0}^{m} c_i L_{l,i}(t))^2 dt\Big)^{1/2},$$

where

$$\begin{aligned}
\varepsilon_n^2 &= \int_0^l \Big(y(t) - \sum_{i=0}^{m} c_i L_{l,i}(t)\Big)^2 dt \\
&= \int_0^l \Big(\sum_{i=0}^{\infty} c_i L_{l,i}(t) - \sum_{i=0}^{m} c_i L_{l,i}(t)\Big)^2 dt \\
&= \int_0^l \Big(\sum_{i=m+1}^{\infty} c_i L_{l,i}(t)\Big)^2 dt \\
&= \int_0^l \sum_{i=m+1}^{\infty} c_i^2 L_{l,i}^2(t) dt \\
&= \sum_{i=m+1}^{\infty} c_i^2 \int_0^l L_{l,i}^2(t) dt \\
&= \sum_{i=m+1}^{\infty} c_i^2 \frac{l}{(2i+1)} \\
&\leq \sum_{i=m+1}^{\infty} \frac{6\alpha^2 l^5}{16(2i-3)^3(2i+1)} \\
&\leq \frac{6\alpha^2 l^5}{16} \sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4}
\end{aligned}$$

Then we have

$$\varepsilon_n \leq \alpha l^2 \sqrt{\frac{3l}{8}} \sqrt{\sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4}}.$$

Now, we give the estimate error of shifted Legendre Tau method for the weakly singular Volterra integro-differential equation (1).
Firstly, we define

$$e_m(t) = y(t) - y_m(t), \tag{16}$$

the error function of the Legendre Tau approximation $y_m(t)$ to $y(t)$, where $y(t)$ is the exact solution of (1). Therefore by using equations (16), (1), and

(2) we have,

$$D^{\alpha}(e_m(t)+y_m(t)) = f(t)+\lambda_1\int_0^t \frac{e_m(\tau)+y_m(\tau)}{(t-s)^{\beta}}d\tau+\lambda_2\int_0^1 k(t,\tau)(e_m(\tau)+y_m(\tau))d\tau,$$

$$(e_m+y_m)^{(s)}(0) = d_j, \quad s=0,1,\dots,r-1, \quad j=0,1,\dots,r-1.$$

Also, we have

$$D^{\alpha}(e_m(t)) = H_m(t)+\lambda_1\int_0^t \frac{e_m(\tau)}{(t-s)^{\beta}}d\tau+\lambda_2\int_0^1 k(t,\tau)e_m(\tau)d\tau,$$

$$(e_m)^{(s)}(0) = 0, \quad s=0,1,\dots,r-1,$$

$H_m(t)$ is a perturbation term associated with $y_m(t)$ and can be obtained with following formulae

$$H_m(t) = f(t)-D^{\alpha}(y_m(t))+\lambda_1\int_0^t \frac{y_m(\tau)}{(t-s)^{\beta}}ds+\lambda_2\int_0^1 k(t,\tau)y_m(\tau)d\tau.$$

We proceed to find an approximation $(e_{m,N})(t)$ to the $e_m(t)$ in the same as we did for the solutions of equations (1) and (2) (N denotes the Tau degree of $e_m(t)$).

6 Numerical results and comparisons

In this section, we present five numerical examples to demonstrate the accuracy of the proposed method. The results show that this method, by selecting a few number of shifted Legendre polynomials is accurate.

Example 1 As a first application, we offer the following fractional order integro-differential equation with weakly singular kernel [25,24]

$$\begin{cases} D^{\,0.25}y(t) = f(t)+\frac{1}{2}\int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau+\frac{1}{3}\int_0^1 (t-\tau)y(\tau)\mathrm{d}\tau, \\ y(0)=0, \end{cases}$$

with the exact solution $y(t)=t^3+t^2$. In this example we have

$$f(t) = \frac{\Gamma(3)t^{1.75}}{\Gamma(2.75)}+\frac{\Gamma(4)t^{2.75}}{\Gamma(3.75)}-\frac{\Gamma(3)t^{\frac{5}{2}}\sqrt{\pi}}{2\Gamma(\frac{7}{2})}-\frac{\Gamma(4)t^{\frac{7}{2}}\sqrt{\pi}}{2\Gamma(\frac{9}{2})}-\frac{7t}{36}+\frac{3}{20},$$

$\lambda_1=\frac{1}{2}$, $\lambda_2=\frac{1}{3}$, and $k(t,\tau)=t-\tau$. Table 1 shows the comparison between the absolute errors obtained by second Chebyshev wavelets (SCW), Cosine and Sine (CAS) wavelet and our method (SLT). From Table 1 we can see clearly that the shifted Legendre Tau method can reach a higher degree of accuracy than the SCW and CAS wavelet methods. From Figures 1 and 2 we infer that the approximate solutions converge to the exact solution.

Table 1 The absolute errors obtained at different points with different methods.

t	SLT ($m = 6$)	SCW ($m' = 6$)	CAS ($m' = 6$)
0	3.85×10^{-17}	8.35×10^{-3}	3.05×10^{-2}
1/6	3.16×10^{-15}	1.25×10^{-3}	4.40×10^{-2}
2/6	1.10×10^{-14}	9.36×10^{-3}	3.87×10^{-2}
3/6	7.21×10^{-15}	2.24×10^{-2}	1.57×10^{-2}
4/6	6.77×10^{-15}	1.95×10^{-2}	2.85×10^{-2}
5/6	6.21×10^{-15}	3.25×10^{-2}	9.88×10^{-2}

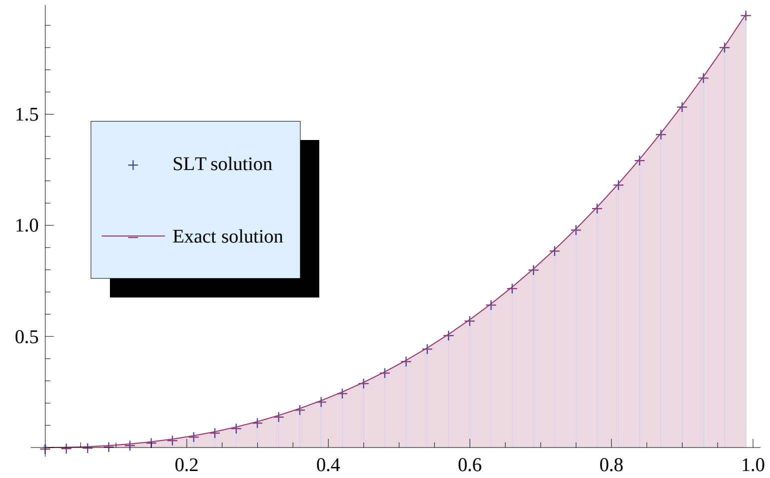

Fig. 1 Comparison of numerical and exact solutions of Example 1 for $m = 6$

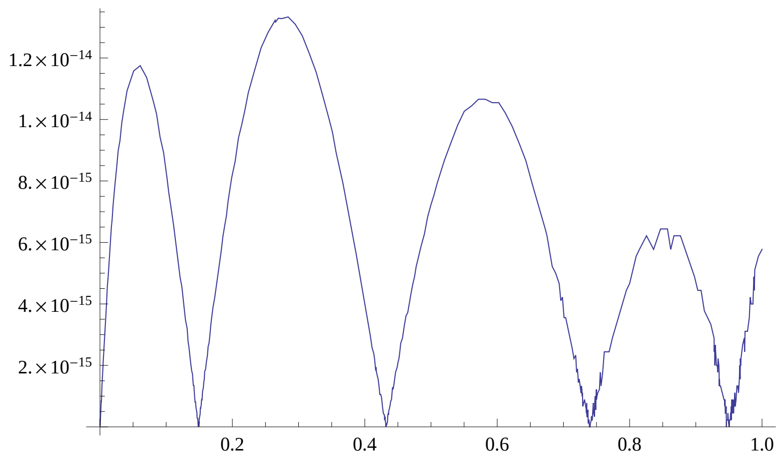

Fig. 2 The absolute error of Example 1 for $m = 6$

Example 2 Consider the following equation [25, 24]

$$\begin{cases} D^{\,0.15}y(t) = f(t) + \frac{1}{4}\int_0^t \dfrac{y(\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau + \frac{1}{7}\int_0^1 e^{t+\tau}y(\tau)\mathrm{d}\tau, \\ y(0) = 0, \end{cases}$$

with the exact solution $y(t) = t^2 - t$ and

$$f(t) = \frac{\Gamma(3)t^{1.85}}{\Gamma(2.85)} - \frac{\Gamma(2)t^{0.85}}{\Gamma(1.85)} - \frac{\Gamma(3)t^{\frac{5}{2}}\sqrt{\pi}}{4\Gamma(\frac{7}{2})} + \frac{\Gamma(2)t^{\frac{3}{2}}\sqrt{\pi}}{4\Gamma(\frac{5}{2})} - \frac{e^{t+1} - 3e^t}{7}.$$

We apply SLT method to solve this problem. The comparisons of exact solution and numerical solutions in different points with $m = 6$ are shown in Table 2. Figure 3 shows the error function of this example with $m = 6$.

Table 2 The numerical solutions at different points with different methods for Example 2.

t	SLT ($m = 6$)	SCW ($m' = 6$)	CAS ($m' = 6$)	Exact solution
0	0	-1.5509×10^{-2}	6.1480×10^{-3}	0
1/6	-1.3889×10^{-1}	-1.4412×10^{-1}	-1.1115×10^{-1}	-1.3889×10^{-1}
2/6	-2.2222×10^{-1}	-2.2290×10^{-1}	-1.8713×10^{-1}	-2.2222×10^{-1}
3/6	-2.5000×10^{-1}	-2.4994×10^{-1}	-2.1160×10^{-1}	-2.5000×10^{-1}
4/6	-2.2222×10^{-1}	-2.2128×10^{-1}	-1.8346×10^{-1}	-2.2222×10^{-1}
5/6	-1.3889×10^{-1}	-1.3718×10^{-1}	-1.0222×10^{-1}	-1.3889×10^{-1}

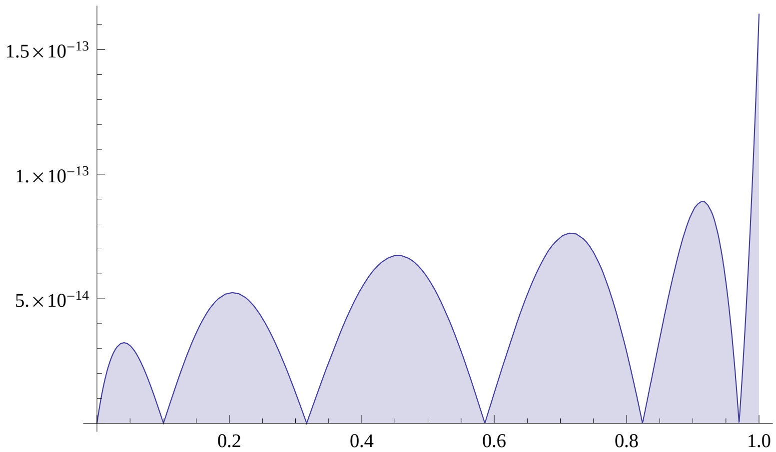

Fig. 3 The absolute error of Example 2 for $m = 6$

Example 3 Consider the following fractional order integro-differential equation with a weakly singular kernel

$$\begin{cases} D^{0.5}y(t) = f(t) - \int_0^t \dfrac{y(\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau - \int_0^1 (t\tau)y(\tau)\mathrm{d}\tau, \\ y(0) = 1, \end{cases}$$

with

$$f(t) = 2\sqrt{t} - \frac{t}{3} - 2.06\,t^{3.5} - \frac{256\,t^{9/2}}{315}.$$

In this example we have $\lambda_1 = \lambda_2 = 1$, and $k(t,\tau) = t\tau$. The exact solution is $u(t) = 1 - t^4$. We apply SLT method to solve this equation and the results are given in Table 3 for different choices of t. The numerical solution and exact solution have been compared in Figure 4.

Table 3 Absolute errors for Example 3.

t	Error by SLT for $m = 4$
0	0
0.1	6.66×10^{-16}
0.2	5.55×10^{-16}
0.3	1.11×10^{-16}
0.4	5.55×10^{-16}
0.5	8.88×10^{-16}
0.6	1.11×10^{-15}
0.7	8.88×10^{-16}
0.8	3.33×10^{-16}
0.9	5.55×10^{-17}
1	3.71×10^{-16}

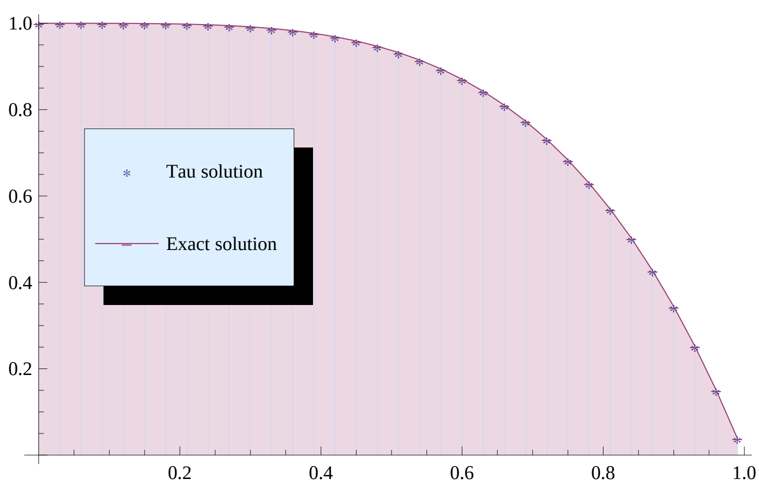

Fig. 4 Comparison of numerical and exact solutions of Example 3 for $m = 8$

Example 4 Consider the following fractional order integro-differential equation with a weakly singular kernel

$$\begin{cases} D^{1.25}y(t) = f(t) + \frac{1}{2}\int_0^t \dfrac{y(\tau)}{\sqrt[3]{t-\tau}}\mathrm{d}\tau - \int_0^1 (t+\tau^2)y(\tau)\mathrm{d}\tau, \\ y(0) = 1, \quad y'(0) = 0, \end{cases}$$

with

$$f(t) = \frac{2}{3} - \frac{3t}{2} + 7.46\,t^{1.75} - \frac{3\,t^{2/3}}{440}(110 + 81t^3).$$

In this example we have $\lambda_1 = \lambda_2 = 1$, and $k(t,\tau) = t + \tau^2$. The exact solution is $u(t) = 2t^3 + 1$. In this example, we implement the SLT method to solve this kind of fractional integro-differential equation for $m = 4$. The numerical results and absolute errors for different choices of t have been provided in Table 4. Also, the numerical results for $m = 3$ are shown in Figure 5. We can find easily that the numerical solutions are more and more close to the exact solution.

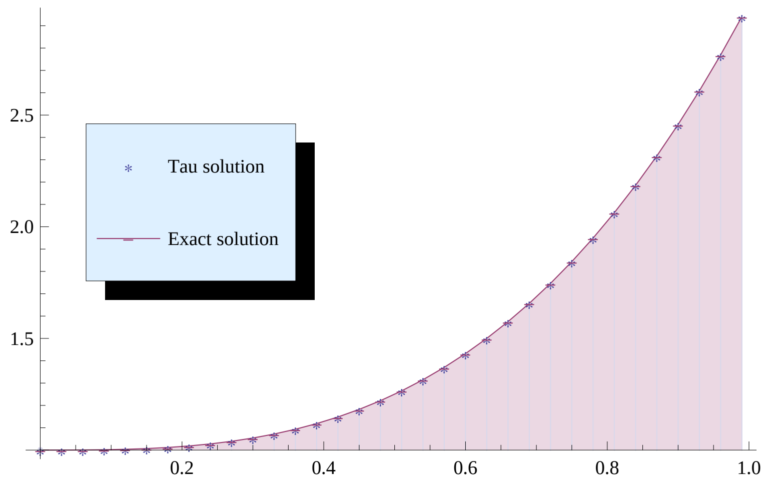

Fig. 5 Comparison of numerical and exact solutions of Example 4 for $m = 3$

Table 4 Absolute errors for Example 4.

t	Error by SLT for $m = 4$
0	2.22×10^{-16}
0.1	2.22×10^{-16}
0.2	2.22×10^{-16}
0.3	0
0.4	0
0.5	0
0.6	2.22×10^{-16}
0.7	2.22×10^{-16}
0.8	4.44×10^{-16}
0.9	8.88×10^{-16}
1	4.44×10^{-16}

Example 5 Consider the following fractional order integro-differential equation with a weakly singular kernel

$$\begin{cases} D^{\alpha} y(t) = f(t) + \int_0^t \dfrac{y(\tau)}{\sqrt{t-\tau}} \mathrm{d}\tau - \int_0^1 \cos(t+\tau) y(\tau) \mathrm{d}\tau, \\ y(0) = 0, \quad y'(0) = 0, \end{cases}$$

where

$$f(t) = \frac{-16}{15} t^{5/2} + 2\cos(1-t) + \frac{2(2+\sqrt{3})t^{2-\sqrt{3}}}{\Gamma(2-\sqrt{3})} - \sin(1-t) - 2\sin t.$$

The exact solution of this example for $\alpha = \sqrt{3}$ is $y(t) = t^2$. In this example we have $\lambda_1 = 1$, $\lambda_2 = -1$, and $k(t,\tau) = \cos(t-\tau)$. The comparison of numerical results for $\alpha = 1.25$, $\alpha = 1.6$, $\alpha = 1.7$, $\alpha = \sqrt{3}$, and the exact solution for $\alpha = \sqrt{3}$ are shown in Figure 6. The absolute errors for different choices of t are shown in Table 5. As, we expected, Tau method has produced an accurate approximation of the exact solution.

Table 5 Absolute errors for Example 5.

t	Error by SLT for $m = 4$
0	2.77×10^{-17}
0.1	4.68×10^{-17}
0.2	6.93×10^{-18}
0.3	5.55×10^{-17}
0.4	2.77×10^{-17}
0.5	0
0.6	5.55×10^{-17}
0.7	1.11×10^{-16}
0.8	1.11×10^{-16}
0.9	2.22×10^{-16}
1	2.22×10^{-16}

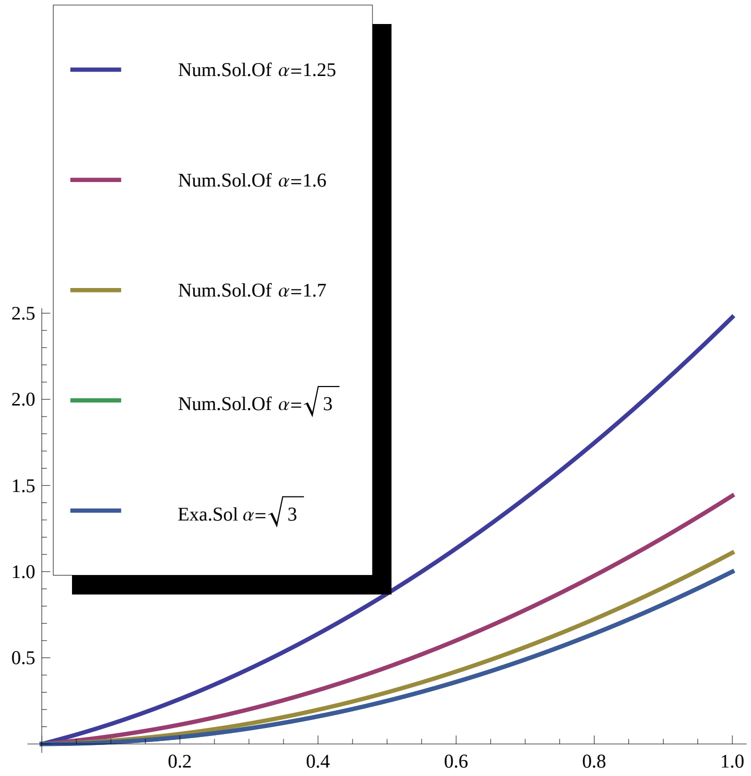

Fig. 6 Numerical solutions (different α) and exact solution ($\alpha = \sqrt{3}$) for Example 5.

7 Conclusion

In this work, shifted Legendre Tau method have been applied to solve a class of fractional integro-differential equations with weakly singular kernel. The most important contribution of our work is that we transform the initial problem into a linear algebraic system equations, and we compare our numerical results with the results obtained in [24], [25]. We can see that the numerical results obtained are better than in [24], [25]. The illustrative examples show that the approximations are in very good coincidence with the exact solution.

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