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Some Results on c-Covers of A Pair of Groups

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Abstract Let G be a group and N be a normal subgroup of G. In this paper, we provide some results on c-covers of a pair of groups. Moreover, we prove that every c-perfect pair of groups (G, N) admits at least one c-cover and also we show that a c-cover of a pair of finite groups has a unique domain up to isomorphism under some assumptions.

Keywords Pair of groups · c-cover · c-nilpotent multiplier

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1 Introduction and Preliminaries

Let G be a group and $G \cong F/R$ for a free group F. The Schur multiplier G is isomorphic to $(R \cap F')/[R, F]$ (see [9]). Also, the abelian group

$$\mathcal{M}^{(c)}(G) = R \cap \gamma_{c+1}(F)/\gamma_{c+1}(R, F),$$

is said to the c-nilpotent multiplier of G, where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F, $\gamma_1(R, F) = R$, $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$ $(c \geq 1)$. In the case that $c = 1$, $\mathcal{M}^{(1)}(G) = \mathcal{M}(G)$ is the Schur multiplier of G (see [4]).

Let (N, G) be a pair of groups such that N is a normal subgroup of G. Then the Schur multiplier of (N, G) is defined to be the abelian group $\mathcal{M}(N, G)$ appears in the following natural exact sequence

$$\begin{aligned} &H_3(G) \to H_3(G/N) \to \mathcal{M}(N, G) \to \mathcal{M}(G) \\ &\to \mathcal{M}(G/N) \to N/[N, G] \to (G)^{ab} \to (G/N)^{ab} \to 1, \end{aligned}$$

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in which $H_3(-)$ is the third homology of a group with integer coefficients (see [5]). If $G \cong F/R$ is a free presentation of G and $N \cong S/R$, for some normal subgroup S of F, and N has a complement in G, one can see that $\mathcal{M}(N,G) \cong R \cap [S,F]/[R,F]$.

In the case $N = G$, the Schur multiplier of the pair (N,G) is the usual Schur multiplier of G.

If N has a complement in G, then we can define

$$\mathcal{M}^{(c)}(N,G) \cong \frac{R \cap [S,_c F]}{[R,_c F]}.$$

In particular, if $N = G$, then $\mathcal{M}^{(c)}(G,G) = \mathcal{M}^{(c)}(G)$ is the c-nilpotent multiplier of G (see [2,13,14] for more information). The notion of covering pairs was defined by Ellis [5], he proved that every pair of finite groups has a covering pair. In [8], the authors proved that every nilpotent pair of groups of class at most k with non-trivial c-nilpotent multiplier does not admit any c-covering pair, for all $c > k$. In this paper, we prove some new results on c-covering pairs of groups.

Let G and M be two groups with an action of G on M. Then the G-commutator subgroup and G-center subgroup of M are defined, respectively, as follows:

$$[M,G] = \langle [m,g] = m^g m^{-1} \mid m \in M, g \in G \rangle,$$
$$Z(M,G) = \{m \in M \mid m^g = m, \quad \forall g \in G\}.$$

We recall that the subgroups $[M,_c G]$ and $Z_c(M,G)$ for all $c \geq 1$, as follows:

$$[M,_c G] = \langle [m, g_1, \ldots, g_c] \mid m \in M, g_1, \ldots, g_c \in G \rangle,$$
$$Z_c(M,G) = \{m \in M \mid [m, g_1, \ldots, g_c] = 1, forall g_1, \ldots, g_c \in G\}.$$

Let (N,G) be a pair of groups. A relative c-central extension of the pair (N,G) is a homomorphism $\sigma : M \to G$ together with an action of G on M such that

(i) $\sigma(M) = N$
(ii) $\sigma(m^g) = g^{-1}\sigma(m)g$, for all $g \in G$, $m \in M$,
(iii) $m'^{\sigma(m)} = m^{-1}m'm$, for all $m, m' \in M$,
(iv) $\ker \sigma \subseteq Z_c(M,G)$.

In addition, the relative c-central extension $\sigma : M \to G$ is siad to be a c-cover of (N,G) if there exists a subgroup A of M such that

(i) $A \subseteq Z_c(M,G) \cap [M,_c G]$,
(ii) $A \cong \mathcal{M}^{(c)}(N,G)$,
(iii) $N \cong M/A$.

It is easy to see that 1-covering pair is the usual covering pair discussed in [5, 11].

Finally, a pair (N,G) of groups is called c-perfect, if $[N,_c G] = N$.

2 On c-covers of a pair of groups

Let (N, G) be a pair of groups with a free presentation $1 \to R \to F \xrightarrow{\pi} G \to 1$ such that $N \cong S/R$ for a normal subgroup S of F, in which $[R, S] \subseteq [R,_c F]$. Consider the group homomorphism

$$\begin{aligned} \delta :& S/[R,_c F] \to G \\ & s[R,_c F] \mapsto \pi(s). \end{aligned}$$

It is easy to see that δ is a relative c-central extension by an action of G on $S/[R,_c F]$, defined by $(s[R,_c F])^g = s^f[R,_c F]$, where $\pi(f) = g$.

By the above assumption, and a simple generalization of [12], we obtain the following Lemmas.

Lemma 1 *Let (N, G) be a pair of groups with $G \cong F/R$ and $N \cong S/R$. If $\sigma : M \to G$ is a relative c-central extension of the pair (N, G), then there exists a homomorphism $\beta : S/[R,_c F] \to M$ such that the following diagram is commutative*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R/[R,_c F] & \longrightarrow & S/[R,_c F] & \xrightarrow{\delta} & N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_N} & & \\ 1 & \longrightarrow & A & \longrightarrow & M & \xrightarrow{\sigma} & N & \longrightarrow & 1. \end{array}$$

where δ is the relative c-central extension defined above. In particular, if M is a perfect group with $\Phi(M) \neq M$, then β is an epimorphism.

Lemma 2 *Let (N, G) be a pair of groups, with $G \cong F/R$ and $N \cong S/R$. Then for every c-cover $\sigma : M \to G$ of (N, G) in which M is perfect and $\Phi(M) \neq M$, there is a normal subgroup T of F such that*

(i) $M \cong S/T$ *and* $\ker \sigma \cong R/T$,
(ii) $R/[R,_c F] = \mathcal{M}^c(N, G) \times T/[R,_c F]$.

Theorem 1 *Let (N, G) be a pair of groups and $\sigma_i : M_i \to G$ $(i = 1, 2)$ be two c-covers of the pair (N, G) such that $M_i' = M_i$ and $\Phi(M_i) \neq M_i$. If $\alpha : M_1 \to M_2$ is an epimorphism such that $\alpha(\ker \sigma_1) = \ker \sigma_2$, then α is an isomorphism.*

Proof Let $G \cong F/R$ be a free presentation of G and $N \cong S/R$, for a normal subgroup S of F. By using Lemma 2, there exist normal subgroups T_i $(i = 1, 2)$ of F such that

(i) $M_i \cong S/T_i$ and $\ker(\sigma_i) \cong R/T_i$,
(ii) $R/[R,_c F] = \mathcal{M}^{(c)}(N, G) \times T_i/[R,_c F]$.

So, we can consider the epimorphism $\alpha : S/T_1 \to S/T_2$ such that

$$\alpha(R/T_1) = R/T_2.$$

By using Lemma 1, there exists an epimorphism $\beta : S/[R,_c F] \to S/T_2$ such that

$$\ker \beta = T_2/[R,_c F].$$

Hence, the following diagram is commutative

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R/[R,_c F] & \longrightarrow & S/[R,_c F] & \longrightarrow & N & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \beta_1} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \beta_2} & & \\
1 & \longrightarrow & R/T_2 & \longrightarrow & S/T_2 & \longrightarrow & N & \longrightarrow & 1,
\end{array}$$

where β_1 and β_2 are the restricted and the induced homomorphisms of β, respectively. We can see that β_2 is an isomorphism. So, we obtain a homomorphism $\varphi : S \to S/T_1$ with $\alpha \circ \varphi = \beta \circ \gamma$, in which γ is the natural epimorphism from S onto $S/[R,_c F]$. So, φ induces a homomorphism $\overline{\varphi} : S/[R,_c F] \to S/T_1$. Thus, the following diagrams are commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R/[R,_c F] & \longrightarrow & S/[R,_c F] & \longrightarrow & N & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \varphi_1} & & \big\downarrow{\scriptstyle \overline{\varphi}} & & \big\downarrow{\scriptstyle \varphi_2} & & \\
1 & \longrightarrow & R/T_1 & \longrightarrow & S/T_1 & \longrightarrow & N & \longrightarrow & 1,
\end{array}$$

$$\begin{array}{ccc}
 & S/[R,_c F] & \\
{\scriptstyle \overline{\varphi}} \swarrow & & \searrow {\scriptstyle \beta} \\
S/T_1 & \xrightarrow{\ \ \alpha\ \ } & S/T_2
\end{array}$$

where φ_1 is restriction of $\overline{\varphi}$ and $\varphi_2 = (\alpha')^{-1} \circ \beta_2$ is an isomorphism, where $\alpha' : N \to N$ is the induced isomorphism by α. So, $\overline{\varphi}$ is onto. Put $\ker \overline{\varphi} = E/[R,_c F]$, for a normal subgroup E in S. We can see that $E \subseteq T_2$ and $E(R \cap [S,_c F]) = R$. Hence, $E = T_2$. Therefore, α is an isomorphism.

In the following result, we show that any c-perfect pair of groups has at least one c-covering pair. Indeed, in Theorem 2, we generalize [12, Theorem 2·4].

Theorem 2 *Any c-perfect pair of groups has at least one c-covering pair.*

Proof Let (N, G) be a pair of groups and $1 \to R \to F \overset{\pi}{\to} G \to 1$ be a free presentation of G in which $N \cong S/R$, for a normal subgroup S of F. We have the following relative c-central extension

$$1 \to R/[R,_c F] \to F/[R,_c F] \overset{\overline{\pi}}{\to} G \to 1.$$

Since $\overline{\pi}([S,_c F]/[R,_c F]) = N$, by restricting $\overline{\pi}$ to $[S,_c F]/[R,_c F]$, we obtain the following relative c-central extension of (N, G):

$$1 \to \mathcal{M}^{(c)}(N, G) \to [S,_c F]/[R,_c F] \to G \to 1.$$

Now, we can see that

$$[S,_c F]/[R,_c F] = [[S,_c F]/[R,_c F],_c G],$$

which completes the proof.

A relative c-central extension $\sigma : M \to G$ of a pair (N, G) is called universal, if for every relative c-central extension $\theta : K \to G$, there exists a unique homomorphism $\varphi : M \to K$ such that $\theta \circ \varphi = \sigma$.

In the following theorem, we show that if a pair (N, G) admits a universal relative c-central extension, then it is c-perfect.

Theorem 3 *Let (N, G) be a pair of groups and $\sigma : M \to G$ be a relative c-central extension of (N, G). If σ is universal, then (N, G) is c-perfect.*

Proof One can check that the exact sequence

$$1 \to \ker\sigma \times N/[N,_c G] \to M \times N/[N,_c G] \xrightarrow{\theta} G \to 1,$$

where $\theta(m, n[N,_c G]) = \sigma(m)$ for all $m \in M$ and $n \in N$, is a relative c-central extension of the pair (N, G). Define

$$\varphi_i : M \to M \times N/[N,_c G], for\, i = 1, 2$$

by $\varphi_1(m) = (m, 1)$ and $\varphi_2(m) = (m, \sigma(m)[N,_c G])$. Since $\theta \circ \varphi_i = \sigma$, we obtain $\varphi_1 = \varphi_2$. Thus, $[N,_c G] = N$.

Lemma 3 *([8], Lemma 3.1) Let (N, G) be a pair of groups and K be a normal subgroup of G such that $K \subseteq N$. Then the following sequence is exact:*

$$1 \to \mathcal{M}^{(c)}(K, G) \to \mathcal{M}^{(c)}(N, G) \to \mathcal{M}^{(c)}(\frac{N}{K}, \frac{G}{K}) \to \frac{(K \cap [N,_c G])}{[K,_c G]} \to 1.$$

Proposition 1 *Let (N, G) be a c-perfect pair of groups and $\mathcal{M}^{(c)}(N, G) = 1$. Also, let M be a normal subgroup of G succh that $M \subseteq N \cap Z_c(G)$. Then*

$$\mathcal{M}^{(c)}(N/M, G/M) \cong M,$$

and N is a c-covering pair of $(N/M, G/M)$.

Proof By Lemma 3, we can see that the following sequence is exact.

$$\mathcal{M}^{(c)}(N, G) \to \mathcal{M}^{(c)}(N/M, G/M) \to M \cap [N,_c G] \to 1.$$

Hence, we have

$$\mathcal{M}^{(c)}(N/M, G/M) \cong M \cap [N,_c G] = M,$$

which completes the proof.

Theorem 4 *Let (N,G) be a pair of groups with a free presentation $G \cong F/R$ and $N \cong S/R$. Also, let $\sigma : M \to G$ be a relative c-central extension of the pair (N,G) in which M is perfect and $\Phi(M) \neq M$. Then*

$$|\ker\sigma \cap [M,_c G]| \leq |\mathcal{M}^{(c)}(N,G)|.$$

Proof By Lemma 1, there exists an epimorphism $\beta : S/[R,_c F] \to M$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R/[R,_c F] & \longrightarrow & S/[R,_c F] & \longrightarrow & N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \beta_|} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_N} & & \\ 1 & \longrightarrow & \ker\sigma & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 1, \end{array}$$

Put $\ker\beta_| = E/[R,_c F]$, for a normal subgroup E of R. We have

$$M \cong \frac{S/[R,_c F]}{E/[R,_c F]} \qquad \text{and} \qquad \ker\sigma \cong \frac{R/[R,_c F]}{E/[R,_c F]}.$$

On the other hand,

$$\begin{aligned}(R \cap ([S,_c F]E))/E &\cong (R \cap [S,_c F])/(E \cap [S,_c F]) \\ &\cong \mathcal{M}^{(c)}(N,G)/((E \cap [S,_c F])/[R,_c F])\end{aligned}$$

So, we obtain

$$|\ker\sigma \cap [M,_c G]| \leq |\mathcal{M}^{(c)}(N,G)|.$$

In the following theorem, we give some conditions to show a relative c-central extension of a pair of groups is a homomorphic image of a c-covering pair.

Theorem 5 *Let $\sigma : M \to G$ be a relative c-central extension of a pair (N,G) of finite groups, in which M is perfect and $\Phi(M) \neq M$. Then M is a homomorphic image of the domain of a c-covering pair.*

Proof Let $G \cong F/R$ and $N \cong S/R$. By using Lemma 1, there is an epimorphism $\beta : S/[R,_c F] \to M$ such that the following diagram commute

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R/[R,_c F] & \longrightarrow & S/[R,_c F] & \xrightarrow{\delta} & N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \beta_|} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_N} & & \\ 1 & \longrightarrow & \ker\sigma & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 1, \end{array}$$

where δ is the relative c-central extension defined in Lemma 1.
Put $\ker\beta_| = \ker\beta = K/[R,_c F]$, for a normal subgroup K of R. We have

$$R/K \cong \frac{R/[R,_c F]}{K/[R,_c F]} \cong \ker\sigma \cong \frac{(R \cap [S,_c F])/[R,_c F]}{(K \cap [S,_c F])/[R,_c F]} \cong (R \cap [S,_c F])K/K,$$

On the other hand, $\ker\sigma$ is finite. So, we obtain $(R\cap[S,_c F])K=R$. Suppose that $T/[R,_c F]$ is a complement of $(R\cap[S,_c F])/[R,_c F]$ in $K/[R,_c F]$. So,

$$T\cap(R\cap[S,_c F])=[R,_c F] \quad \text{and} \quad (R\cap[S,_c F])T=R,$$

Thus,

$$R/[R,_c F]=\mathcal{M}^{(c)}(N,G)\times T/[R,_c F].$$

Now, we can see that $\theta: S/T\to F/R$ given by $\theta(sT)=sR$, together with the action

$$\begin{aligned}\overline{\theta}: S/T\times F/R&\to S/T\\ (s\cdot T, f\cdot R)&\mapsto [s,f]T,\end{aligned}$$

is a c-cover of the pair (N,G) such that $(S/T)/(K/T)\cong M$.

The following theorem guaranties the existence of c-cover for c-capable pair of groups. We recall from [7] that a pair (N,G) is said to be c-capable, if there exists a relative c-central extension $\varphi: M\to G$ with $\ker\varphi=Z_c(N,G)$.

Theorem 6 *Let (N,G) be a c-capable pair of groups. Then there exists a group K such that*

(i) $Z_c(K,G)\subseteq[K,_c G]$,
(ii) $K/Z_c(K,G)\cong N$.

Proof We can see that there exists a group T such that $N\cong T/Z_c(T,G)$. Let $F/M\cong T$ be a free presentation of T. Put $Z_c(T,G)\cong R/M$, for a normal subgroup R of F. Then $F/R\cong G$ is a free presentation of G. Let $E=H/[R,_c F]$ be a complement of $D=(R\cap[S,_c F])/[R,_c F]$ in $B=R/[R,_c F]$, where $S/R\cong N$. Since $E\subseteq B\subseteq Z(C)$, where $C=[S,_c F]/[R,_c F]$ and E is a normal subgroup of C, we obtain $B=D\times E$. Put $P=F/[R,_c F]$ and assume that $K=P/E$ and $W=B/E$. Now, we can see that $W\subseteq Z_c(K,G)$. Also, we have

$$Z_c(F/M,G/M)=R/M.$$

So, $[R,_c F]\subseteq M$. Let $(f[R,_c F])E\in Z_c(K,G)$, where $f\in F$. Then

$$[f,x_1,\ldots,x_c][R,_c F]\in E,$$

for all $x_i\in F$ $(1\le i\le c)$. Now

$$[f,x_1,\ldots,x_c][R,_c F]\in[S,_c F]/[R,_c F]\cap H/[R,_c F]=1.$$

Hence, $[f,x_1,\ldots,x_c]\in[R,_c F]\subseteq M$ and so,

$$fM\in Z_c(F/M,G/M)=R/M.$$

Hence, $f\in R$. Therefore,

$$Z_c(K,G)=B/E=D\times E/E\cong D=R\cap[S,_c F]/[R,_c F]\cong\mathcal{M}^{(c)}(N,G),$$

which completes the proof.

The following corollary is an immediate consequence of Theorem 6.

Corollary 1 *Any c-capable pair of groups has at least one c-covering pair.*

It is known that any two covering groups of a finite group G are isoclinic. Also, if G is a finite perfect pair of groups, then a covering group of G is also perfect. The notion of c-isoclinism of pairs of groups is introduced in [6] (see [1,15] for more information).

The pairs (N,G) and (N',G') of groups are said to be c-isoclinic if there exists isomorphisms

$$\alpha : G/Z_c(N,G) \to G'/Z_c(N',G'), \qquad \text{and} \qquad \beta : [N,_c G] \to [N',_c G'],$$

such that

$$\alpha(N/Z_c(N,G)) = N'/Z_c(N',G'), \quad \text{and} \quad \beta([n,g_1,\dots,g_c]) = [n',g'_1,\dots,g'_c],$$

whenever

$$\alpha(g_i Z_c(N,G)) = g'_i Z_c(N',G'),$$

for every $1 \le i \le c$ and

$$\alpha(nZ_c(N,G)) = n'Z_c(N',G'),$$

and we write $(N,G) \overset{c}{\sim} (N',G')$. Let $\sigma_i : M_i \to G$ $(i = 1,2)$ be two c-covers of a pair (N,G) of groups. In the following theorem, under some conditions we prove the pairs $(\ker \sigma_1, M_1)$ and $(\ker \sigma_2, M_2)$ are c-isoclinic. In Theorem 7, we extend [12, Theorem 2·8].

Theorem 7 *Let (N,G) be a pair of groups and $\sigma_i : M_i \to G$ $(i = 1,2)$ be two c-covers of (N,G) such that $M'_i = M_i$ and $\Phi(M_i) \neq M_i$. Then*

$$(\ker \sigma_1, M_1) \overset{c}{\sim} (\ker \sigma_2, M_2).$$

Proof Suppose that (N,G) and (N',G') are two pairs of groups and

$$\varphi : G \to G',$$

is an epimorphism such that $\varphi(N) = N'$ and $\ker \varphi \cap N = 1$.
Let $\alpha : G/Z_c(N,G) \to G'/Z_c(N',G')$ be given by

$$\alpha(gZ_c(N,G)) = \varphi(g)Z_c(N',G'),$$

and

$$\beta : [N,_c G] \to [N',_c G'],$$

by $\beta([n,g_1,\dots,g_c]) = [\varphi(n),\varphi(g_1),\dots,\varphi(g_c)]$. Then we have $(N,G) \overset{c}{\sim} (N',G')$. Now, if $\sigma : M \to G$ is a c-cover of the pair (N,G), then there exists an epimorphism $\varphi : S/[R,_c F] \to M$ such that

$$\varphi([S,_c F] \cap R/[R,_c F]) = \ker \sigma.$$

On the other hand,

$$\ker\varphi \cap [[S,_c F] \cap R/[R,_c F], S/[R,_c F]] = 1,$$

Hence,

$$(\ker\sigma, M) \overset{c}{\sim} ([S,_c F] \cap R/[R,_c F], S/[R,_c F]),$$

as clained.

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