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Numerical Solution of Newell–Whitehead–Segel Equation

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Abstract The Newell–Whitehead–Segel (NWS) equation is an important model arising in fluid mechanics. Various researchers worked on approximate solutions to this model by using different methods. In this paper, the Sine-Cosine wavelets method is applied for solving numerically the NWS equation. The Sine-Cosine wavelet operational matrix of integration is obtained and used to transform the equations into a system of algebraic equations. To demonstrate the effectiveness and applicability of this method, two numerical examples are included.

Keywords Newell-Whitehead-Segel equation · Numerical method · Sine-Cosine wavelets · Operational matrix · Function approximation

Mathematics Subject Classification (2010) 65Nxx · 65T60

1 Introduction

In natural phenomena, nonequilibrium systems are usually shown in many extended states; uniform, oscillatory, chaotic, and pattern states. Many stripe patterns, e.g., ripples in the sand, stripes of seashells, occurs in a variety of spatially extended systems which can be described by a set of equation called amplitude equations. One of the most important amplitude equations is the

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NWS equation which describes the appearance of the stripe pattern in two-dimensional systems. Moreover, this equation was applied to several problems in various systems, e.g., Rayleigh-Benard convection, Faraday instability, nonlinear optics, chemical reactions, and biological systems.
The general type of NWS equation is given by

$$u_t = \varepsilon u_{xx} + au + bu^q, \qquad (x,t) \in (0,1) \times (0,t_F), \tag{1}$$

with the initial and Dirichlet boundary conditions

$$u(x,0) = g(x), \qquad x \in [0,1], \tag{2}$$

$$u(0,t) = f_1(t), \qquad u(1,t) = f_2(t), \qquad t \in [0,t_F], \tag{3}$$

where a, b, and ε are real numbers and q is a positive integer. Also, t_F represents the final time, $g(x)$, $f_1(t)$, and $f_2(t)$, are differentiable known functions.

In recent years various methods and techniques are developed to solve this nonlinear, parabolic partial differential equation. For $a = -b = \varepsilon = 1$, and $q = 3$, equation (1) becomes the Allen-Cahn equation. This equation arises in many scientific applications such as mathematical biology, quantum mechanics, and plasma physics. It is well known that phenomena of plasma media and fluid dynamics are modeled by kink-shaped and Tanh solution or bell-shaped sech solutions. The Allen-Cahn equation serves as a model for the study of phase separation in isothermal, isotropic, and binary mixtures such as molten alloys [24]. Several methods have been suggested to solve the NWS equation [11,18,13].

One way to solve equations numerically is to use wavelets. The basic idea of wavelets goes back to the early 1960s [5,4]. The wavelet analysis is the decomposition of a function into shifted and scaled versions of the basic wavelet. Also, the wavelet basis is an orthogonal basis for $\mathscr{L}^2(\mathbb{R})$ and is generated by the translation and dilatation of the basic wavelet. There are developments concerning the multiresolution analysis algorithm based on wavelets [6] and the construction of compactly supported orthonormal wavelet basis [14]. So far, several problems have been solved numerically using different wavelets, for example, we can refer to references [3,1,16,7,17,9,22,23].

In the present paper, we apply the Sine-Cosine wavelets method for solving equation (1) with the initial and boundary conditions (2) and (3). Sine-Cosine wavelets has been used and showed efficiency to solve various problems. Azizi and Pourgholi have applied Sine-Cosine wavelets method for solving the Drinfel'd–Sokolov–Wilson System [2]. Razzaghi and Yousefi in [19] have employed a Sine-Cosine wavelet to solve variational problems. Tavassoli Kajani et al. [12] have proposed a method based on Sine-Cosine wavelet for solving integro-differential equations. A numerical evaluation of Hankel transform for seismology has been given in [10] using the Sine-Cosine wavelets approach. Amir and Umer Saeed in [21] have used Sine-Cosine wavelets to solve the fractional nonlinear oscillator equations, and so on.

The paper is organized as follows: in Section 2, we describe the Sine-Cosine wavelets and function approximation. The convergence analysis of the Sine-Cosine wavelets is given in Section 3. In Section 4, the application and procedure of implementation of the method, are presented. The numerical results are reported in Section 5 and finally, the conclusions are summarized in Section 6.

2 Sine-Cosine wavelets

In this section first, we give some necessary mathematical preliminaries of Sine-Cosine wavelets which are used further in this paper, then function approximation and the operational matrix via this conception are introduced.

Wavelets are mathematical functions that are constructed using dilation and translation of a single function called the mother wavelet denoted by $\varphi(x)$. If the dilation parameter is a and the translation parameter is b, we have the following family of continuous wavelets [8]:

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\varphi\Big(\frac{x-b}{a}\Big), \qquad a, b \in \mathbb{R}, \quad a \neq 0.$$

If the parameters a and b are restricted to take values $a = a_0^{-k}$ and $b = nb_0a_0^{-k}$, a family of discrete wavelets which forms a wavelet basis for $\mathscr{L}^2(\mathbb{R})$ is obtained as:

$$\psi_{k,n}(x) = |a_0|^{\frac{k}{2}}\varphi(a_0^k x - nb_0),$$

where $a_0 > 1$, $b_0 > 0$, and n and k are positive integers. Especially, if the dilation parameter is 2 and the translation parameter is 1, the set $\{\psi_{k,n}(x)\}$ forms an orthonormal basis.

Sine-Cosine wavelets are defined on interval $x \in [0, 1)$ as ([10])

$$\psi_{n,m}(x) = \begin{cases} 2^{k+1/2} f_m(2^k x - n), & x \in [\frac{n}{2^k}, \frac{n+1}{2^k}), \\ 0, & \text{elsewhere}, \end{cases} \tag{4}$$

where

$$f_m(x) = \begin{cases} \frac{1}{\sqrt{2}}, & m = 0, \\ \cos(2m\pi x), & m = 1, 2, \ldots, L, \\ \sin(2(m-L)\pi x), & m = L+1, L+2, \ldots, 2L, \end{cases} \tag{5}$$

and L is any positive integer, $k = 0, 1, 2, \ldots,$ is the level of resolution, $n = 0, 1, 2, \ldots, 2^k - 1$, is the translation parameter, $m = 0, 1, 2, \ldots, 2L$ and x is the normalized time. Sine-Cosine wavelets have compact support and forms a orthonormal basis of $\mathscr{L}^2([0, 1))$.

2.1 Function approximation

Since the set of Sine-Cosine wavelets forms an orthonormal basis, this implies that any function $u(x) \in \mathscr{L}^2([0,1))$ can be expanded as

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{2L} c_{n,m} \psi_{n,m}(x), \tag{6}$$

where $c_{n,m} =< u, \psi_{n,m} >= \int_0^1 u(x)\psi_{n,m}(x)\, dx$. By truncating the infinite series (6) at levels $n = 2^k - 1$, we obtain an approximate representation for $u(x)$ as

$$u(x) \simeq \sum_{n=0}^{2^k-1} \sum_{m=0}^{2L} c_{n,m} \psi_{n,m}(x) = \mathbf{C}^T \mathbf{\Psi}(x). \tag{7}$$

where $\mathbf{C}$ and $\mathbf{\Psi}$ are $(2^k(2L+1) \times 1)$-vectors. given by

$$\begin{aligned} \mathbf{C} &= \Big[c_{0,0}, \dots, c_{0,2L}, c_{1,0}, \dots, c_{1,2L}, \dots\dots, c_{2^k-1,0}, \dots, c_{2^k-1,2L}\Big]^T, \\ \mathbf{\Psi} &= \Big[\psi_{0,0}, \dots, \psi_{0,2L}, \psi_{1,0}, \dots, \psi_{1,2L}, \dots\dots, \psi_{2^k-1,0}, \dots, \psi_{2^k-1,2L}\Big]^T. \end{aligned} \tag{8}$$

2.2 Sine–Cosine wavelets operational matrix of integration

The integration of the vector $\mathbf{\Psi}(x)$, can be obtained as

$$\int_0^x \mathbf{\Psi}(s)\, ds = P\mathbf{\Psi}(x),$$

where P is $2^k(2L+1) \times 2^k(2L+1)$ operational matrix given by

$$P = \frac{1}{2^{k+1/2}} \begin{pmatrix} \mathbf{F} & \mathbf{S} & \cdots & \mathbf{S} \\ 0 & \mathbf{F} & \cdots & \mathbf{S} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{F} \end{pmatrix},$$

where $\mathbf{S}$ and $\mathbf{F}$ are $(2L+1) \times (2L+1)$ matrices (see [12]). For example, when $k = 1$ and $L = 2$, we have

$$P_{10\times 10} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{\pi} & -\frac{1}{2\pi} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\pi} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4\pi} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2\pi} & -\frac{1}{2\pi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4\pi} & 0 & -\frac{1}{4\pi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{\pi} & -\frac{1}{2\pi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4\pi} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\pi} & -\frac{1}{2\pi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4\pi} & 0 & -\frac{1}{4\pi} & 0 & 0 \end{bmatrix}.$$

3 Convergence analysis of the Sine-Cosine wavelets

In this section, we get the convergence of the Sine-Cosine wavelets approximation of a function. For this purpose, we state and prove the following convergence theorem.

Theorem 1 *Let $k \to \infty$, then the series solution* (7) *converges to $u(x)$.*

Proof Let $\mathcal{S}_{k,L}(x)$ be a sequence of partial sums of $c_{n,m}\psi_{n,m}(x)$ as

$$\mathcal{S}_{k,L}(x) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{2L} c_{n,m}\psi_{n,m}(x). \tag{9}$$

We prove that $\mathcal{S}_{k,L}$ is a Cauchy sequence in Hilbert space $\mathscr{L}^2([0,1))$ and then, we show that $\mathcal{S}_{k,L}$ converges to $u(x)$, when $k \to \infty$. In order to reach the first aim, let $\mathcal{S}_{k',L}$ be arbitrary sums of $c_{n,m}\psi_{n,m}(x)$ with $k > k'$, then

$$\begin{aligned}
\left\| \mathcal{S}_{k,L} - \mathcal{S}_{k',L} \right\|^2 &= \left\| \sum_{n=2^{k'}}^{2^k-1} \sum_{m=0}^{2L} c_{n,m}\psi_{n,m}(x) \right\|^2 \\
&= \left\langle \sum_{n=2^{k'}}^{2^k-1} \sum_{m=0}^{2L} c_{n,m}\psi_{n,m}(x), \sum_{i=2^{k'}}^{2^k-1} \sum_{j=0}^{2L} c_{i,j}\psi_{i,j}(x) \right\rangle \\
&= \sum_{n=2^{k'}}^{2^k-1} \sum_{m=0}^{2L} \sum_{i=2^{k'}}^{2^k-1} \sum_{j=0}^{2L} c_{n,m}\bar{c}_{i,j} \left\langle \psi_{n,m}, \psi_{i,j} \right\rangle \\
&= \sum_{n=2^{k'}}^{2^k-1} \sum_{m=0}^{2L} |c_{n,m}|^2.
\end{aligned}$$

From the Bessel's inequality, we have $\sum\limits_{n=0}^{\infty} \sum\limits_{m=0}^{2L} |c_{n,m}|^2$ is convergent, and hence

$$\left\| \mathcal{S}_{k,L} - \mathcal{S}_{k',L} \right\|^2 \to 0 \quad \text{as} \quad k \to \infty.$$

This implies that $\mathcal{S}_{k,L}$ is a Cauchy sequence and hence, it converges to a function in $\mathscr{L}^2([0,1))$, say, $\mathcal{U}(x)$. We need to show that $\mathcal{U}(x) = u(x)$. For this end,

$$\begin{aligned}
\left\langle \mathcal{U} - u, \psi_{n,m} \right\rangle &= \left\langle \mathcal{U}, \psi_{n,m} \right\rangle - \left\langle u, \psi_{n,m} \right\rangle \\
&= \lim_{k \to \infty} \left\langle \mathcal{S}_{k,L}, \psi_{n,m} \right\rangle - c_{n,m} \\
&= c_{n,m} - c_{n,m} = 0.
\end{aligned}$$

Therefore, $\sum\limits_{n=0}^{2^k-1} \sum\limits_{m=0}^{2L} c_{n,m}\psi_{n,m}(x)$ converges to $u(x)$ as $k \to \infty$.

4 Application of the method

In this section, we present our method for solving equation (1). To show the applications of Sine-Cosine wavelets in solving this equation, we divide the interval $[0, t_F]$ into N equal parts of length $\Delta t = \frac{t_F}{N}$ and denote $t_j = (j-1)\Delta t$, $j = 1, 2, \dots, N+1$. Now, we assume that $\dot{u}''(x,t)$ can be expanded in terms of Sine-Cosine wavelets as [2]

$$\dot{u}''(x,t) \cong \sum_{n=0}^{2^k-1} \sum_{m=0}^{2L} c_{n,m}\psi_{n,m}(x) = \mathbf{C}^T\mathbf{\Psi}(x), \tag{10}$$

where . and $'$ mean differentiation with respect to t and x, respectively.
By integrating equation (10) once with respect to t from t_j to t and twice with respect to x from 0 to x, and using the boundary conditions (3), we obtain

$$u''(x,t) = (t-t_j)\mathbf{C}^T\mathbf{\Psi}(x) + u''(x,t_j), \tag{11}$$

$$\dot{u}(x,t) = \mathbf{C}^T P^2\mathbf{\Psi}(x) + xf_2'(t) + (1-x)f_1'(t), \tag{12}$$

$$\begin{aligned} u(x,t) = {} & (t-t_j)\mathbf{C}^T P^2\mathbf{\Psi}(x) + x[f_2(t) - f_2(t_j)] + (1-x)[f_1(t) - f_1(t_j)] \\ & + u(x,t_j). \end{aligned} \tag{13}$$

Discretizing the results (11)-(13) at the collocation points

$$x_i = \frac{2i-1}{2\mathcal{M}}, \qquad i = 1, 2, \dots, \mathcal{M} = 2^k(2L+1),$$

and $t \to t_{j+1}$ we have

$$u''(x_i,t_{j+1}) = \Delta t\mathbf{C}^T\mathbf{\Psi}(x_i) + u''(x_i,t_j), \tag{14}$$

$$\dot{u}(x_i,t_{j+1}) = \mathbf{C}^T P^2\mathbf{\Psi}(x_i) + x_i f_2'(t_{j+1}) + (1-x_i)f_1'(t_{j+1}), \tag{15}$$

$$\begin{aligned} u(x_i,t_{j+1}) = {} & \Delta t\mathbf{C}^T P^2\mathbf{\Psi}(x_i) + x_i[f_2(t_{j+1}) - f_2(t_j)] \\ & + (1-x_i)[f_1(t_{j+1}) - f_1(t_j)] + u(x_i,t_j). \end{aligned} \tag{16}$$

To linearized the nonlinear term u^q in equation (1), we use the linearization form given by Rubin and Graves [20] as follows

$$u^q = (1-q)u^q(x,t_j) + qu^{q-1}(x,t_j)u(x,t_{j+1}). \tag{17}$$

Using linear expression (17), the discrete form of equation (1) considering x_i and t_{j+1} as follows

$$\dot{u}(x_i,t_{j+1}) - \varepsilon u''(x_i,t_{j+1}) - \Big(a + bqu^{q-1}(x_i,t_j)\Big)u(x_i,t_{j+1}) = b(1-q)u^q(x_i,t_j) \tag{18}$$

Now, by using equations (14)-(16), equation (18) leads to

$$\mathcal{A}\mathbf{C} = \mathcal{B}, \tag{19}$$

where

$$\mathcal{A} = \Big[\Big(1 - \varDelta t(a + bqu^{q-1}(x_i, t_j))\Big)P^2\mathbf{\Psi}(x_i) - \varepsilon\varDelta t\mathbf{\Psi}(x_i)\Big],$$

$$\begin{aligned}\mathcal{B} = {} & b(1-q)u^q(x_i,t_j) - \Big[x_i f_2'(t_{j+1}) + (1-x_i)f_1'(t_{j+1})\Big] + \varepsilon u''(x_i,t_j) \\ & + (a + bqu^{q-1}(x_i,t_j))\Big[u(x_i,t_j) + x_i\Big[f_2(t_{j+1}) - f_2(t_j)\Big] \\ & \qquad + (1-x_i)\Big[f_1(t_{j+1}) - f_1(t_j)\Big]\Big].\end{aligned}$$

From the equation (19), the coefficients $\mathbf{C}$ can be calculated. Finally, putting the calculated coefficients into the equation (16), we can successively calculate the approximate solution.

5 Numerical results

In this section, the numerical examples are discussed to demonstrate the capability, consistency, and efficiency of the presented method which described in Section 4.

Example 1 In this example, equation (1) is considered for the parameters $a = 2$, $b = -3$, $\varepsilon = 1$, and $q = 2$. In this case, NWS equation is written as

$$u_t = u_{xx} + 2u - 3u^2, \qquad (x,t) \in [0,1] \times [0,1],$$

with the exact solution ([15])

$$u(x,t) = \frac{2\,\lambda\,\mathrm{e}^{2\,t}}{3\,\left(\lambda\,\mathrm{e}^{2\,t} - \lambda + \frac{2}{3}\right)},$$

where λ is an arbitrary constant.

Example 2 If the parameters $a = -b = \varepsilon = 1$, and $q = 3$ are written into equation (1), the equation is reduced to the Allen-Cahn equation which is written as

$$u_t = u_{xx} + u - u^3, \qquad (x,t) \in [0,1] \times [0,1],$$

with the exact solution ([15])

$$u(x,t) = \frac{1}{1 + \mathrm{e}^{-\frac{\sqrt{2}}{2}\left(x + \frac{3\sqrt{2}}{2}t\right) + c_0}},$$

where c_0 is integration constant.

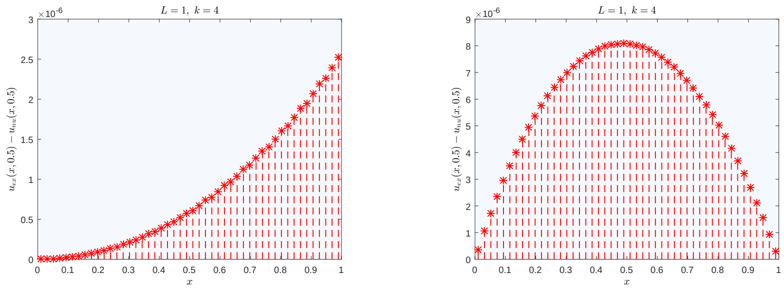
Tables 1 and 2 show the comparison among the exact and numerical solutions $u(x,t)$ at different points. Also, the difference between the exact and numerical results for $u(x,t)$, are shown graphically in Fig. 1.

Table 1: Comparison between the exact and numerical solutions at $t = 0.1$ when $L = 1$ and $k = 2$.

	Example 1			Example 2		
x	$u_{ex}(x,t)$	$u_{nu}(x,t)$	$\|u_{ex}(x,t) - u_{nu}(x,t)\|$	$u_{ex}(x,t)$	$u_{nu}(x,t)$	$\|u_{ex}(x,t) - u_{nu}(x,t)\|$
0.125	0.916897	0.916897	$3.468194e-08$	0.999736	0.999737	$9.704408e-07$
0.375	0.916897	0.916898	$3.108614e-07$	0.999779	0.999780	$1.928517e-06$
0.625	0.916897	0.916898	$8.939672e-07$	0.999814	0.999816	$1.911994e-06$
0.875	0.916897	0.916899	$1.848915e-06$	0.999845	0.999845	$9.841383e-07$

Table 2: Comparison between the exact and numerical solutions at $x = 0.458$ when $L = 1$ and $k = 2$.

	Example 1			Example 2		
t	$u_{ex}(x,t)$	$u_{nu}(x,t)$	$\|u_{ex}(x,t) - u_{nu}(x,t)\|$	$u_{ex}(x,t)$	$u_{nu}(x,t)$	$\|u_{ex}(x,t) - u_{nu}(x,t)\|$
0.1	0.916897	0.916898	$5.081713e-07$	0.999791	0.999793	$2.031222e-06$
0.2	0.858487	0.858488	$7.938125e-07$	0.999820	0.999824	$3.845009e-06$
0.3	0.815931	0.815932	$9.751446e-07$	0.999845	0.999851	$5.347303e-06$
0.4	0.784107	0.784108	$1.104137e-06$	0.999867	0.999873	$6.578716e-06$
0.5	0.759844	0.759845	$1.205749e-06$	0.999885	0.999893	$7.574116e-06$
0.6	0.741069	0.741070	$1.292842e-06$	0.999901	0.999910	$8.363406e-06$
0.7	0.726374	0.726375	$1.372439e-06$	0.999915	0.999924	$8.972209e-06$
0.8	0.714770	0.714771	$1.448567e-06$	0.999927	0.999936	$9.422454e-06$
0.9	0.705542	0.705543	$1.523644e-06$	0.999937	0.999947	$9.732881e-06$
1	0.698162	0.698164	$1.599182e-06$	0.999946	0.999956	$9.919475e-06$



(a) Example 1 (b) Example 2

Fig. 1: Difference between the exact and numerical solutions, at time $t = 0.5$, when $L = 1$ and $k = 4$.

6 Conclusion

In this paper, we used the Sine-Cosine wavelets method to solve the NWS equation (1). The obtained results demonstrate the accuracy of this method and its stability compared to the exact solutions. The results show that this method can be a powerful mathematical tool for finding the numerical solutions of nonlinear equations.

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