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On the Order of the n-Center Factor Subgroup of An n-Abelian Group

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Abstract A group G is said to be n-abelian, if $(xy)^n = x^n y^n$, for any $x, y \in G$ and a positive integer n. In 1979, Fay and Waals introduced the n-potent and the n-center subgroups of a group G, denoted by G_n and $Z^n(G)$, respectively. In this paper, we show that the index of the n-center is bounded by an order power of the n-potent subgroup, for some classes of groups. In fact for all n-abelian groups G with finite n-potent subgroup, we prove that if $G/Z^n(G)$ is finitely generated, then $[G : Z^n(G)] \leq |G_n|^{d(G/Z^n(G))}$. Moreover, we conclude that $[G : Z^n(G)] \leq |G_n|^{2\log_2 |G_n|}$, for some n-capable group G.

Keywords Schur's theorem · n-abelian group · n-center subgroup · n-potent subgroup

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1 Introduction

A basic theorem of I. Schur [8] asserts that if the center of a group G has finite index, then the derived subgroup of G is finite. A question that naturally arises from Schur's theorem is whether the converse of theorem is valid? An extra special p-group of infinite order shows that the answer is negative (See [2]). One of the remarkable problems is finding conditions under which the converse of Schur's theorem holds. B.H. Neumann [5] provided a partial converse of Schur's theorem as follows
If G is finitely generated by k elements and $\gamma_2(G)$ is finite, then $G/Z(G)$ is finite and $|G/Z(G)| \leq |\gamma_2(G)|^k$.

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This result was generalized by P. Niroomand [6]. He proved that if G' is finite and $G/Z(G)$ is finitely generated, then $G/Z(G)$ is finite and

$$|G/Z(G)| \leq |G'|^{d(G/Z(G))},$$

in which $d(X)$ is the minimal number of generators of a group X. B. Sury [9] gave a completely elementary short proof of a further generalization of the Niroomand's result. M.K. Yadav [11] states another extension of the Neumann's result when $Z_2(G)/Z(G)$ is finitely generated. He [12] also provided other modifications of the converse of Schur's theorem as follows

Theorem 1 *For a group G the factor group $G/Z(G)$ is finite if any of the following holds true.*

(i) G' is finite and $Z_2(G)$ is abelian.
(ii) G' is finite and $Z_2(G) \leq G'$.
(iii) G' is finite and $Z_2(G)/Z(Z_2(G))$ is finitely generated.
(iv) $G/G'Z_2(G)$ is finite and $G/Z(Z_2(G))$ is finitely generated.

In 1979, T. H. Fay and G. L. Waals [1] introduced the notion of the n-potent and the n-center subgroups of a group G, for a positive integer n, respectively, as follows

$$G_n = \langle [x, y^n] | x, y \in G \rangle,$$
$$Z^n(G) = \{x \in G | xy^n = y^n x, \forall\ y \in G\},$$

where $[x, y] = x^{-1}y^{-1}xy$. It is easy to see that G_n is a fully invariant subgroup and $Z^n(G)$ is a characteristic subgroup of G.
It seems to be considerable to find the relationship between the n-center factor and n-potent subgroup.
In the present study we extend the Niroomand's result as a partial converse of Schur's theorem for the n-center factor and the n-potent subgroup. Moreover, we give some bounds for the order of the n-center factor in terms of the order of the n-potent subgroup. In particular, for an n-capable group $H = G/Z^n(G)$ such that G is an n-abelian group and $|H_n| = m$, we prove that

$$[H : Z^n(H)] \leq m^{2\log_2 m}.$$

2 Main Results

In this section, we first state a generalization of Niroomand's result as a partial converse of Schur's theorem, when G is an n-abelian group. A group G is said to be n-abelian, if $(xy)^n = x^n y^n$ for all $x, y \in G$, from which it follows that $[x^n, y] = [x, y]^n = [x^n, y^n]$. The origins of the subject may be traced back to 1944 and are associated with the name of Levi [3, 4].

Theorem 2 *Let G be an n-abelian group such that $d(G/Z^n(G))$ and G_n are finite. Then $G/Z^n(G)$ is finite and*

$$|G/Z^n(G)| \leq |G_n|^{d(G/Z^n(G))}.$$

Proof Let $G/Z^n(G) = \langle \bar{x}_1, \ldots, \bar{x}_t \rangle$, such that $\bar{x}_i = x_i Z^n(G)$, for all $1 \leq i \leq t$. Since G is n-abelian, for all $z \in Z^n(G)$ we have

$$[yz, x_i^n] = [y, x_i^n]^z [z, x_i^n] = [y, x_i^n][[y, x_i]^n, z] = [y, x_i^n].$$

Now, we can consider the well defined map $f : G/Z^n(G) \to G_n \times \cdots \times G_n$ (t-times) defined by $\bar{y} \mapsto ([y, x_1^n], \ldots, [y, x_t^n])$. It is enough to show that f is one to one. For this, let $\bar{x}, \bar{y} \in G/Z^n(G)$ and $f(\bar{x}) = f(\bar{y})$. So $[x, x_i^n] = [y, x_i^n]$ for all $1 \leq i \leq t$. Since $[x^{-1}, x_i^n] = [x_i^n, x]^{x^{-1}}$, we have

$$[yx^{-1}, x_i^n] = ([y, x_i^n][x_i^n, x])^{x^{-1}} = 1.$$

On the other hand, G is generated by x_i $(1 \leq i \leq t)$ module $Z^n(G)$ and G is n-abelian. Therefore $yx^{-1} \in Z^n(G)$, which completes the proof.

The following useful corollary is a consequence of the above theorem.

Corollary 1 *Let a group G be n-abelian and nilpotent such that $d(G/Z^n(G))$ and G_n are finite. Then $|G/Z^n(G)|$ divides $|G_n|^{d(G/Z^n(G))}$.*

Proof Since $G/Z^n(G)$ is a finite nilpotent group, so

$$G/Z^n(G) = P_1/Z^n(G) \times \cdots \times P_t/Z^n(G),$$

where $P_i/Z^n(G)$ $(1 \leq i \leq t)$ is a p_i-Sylow subgroup of $G/Z^n(G)$. It is immediate from Theorem 2, $|P_i/Z^n(P_i)| \leq |(P_i)_n|^{d(P_i/Z^n(P_i))}$ which means that $|P_i/Z^n(P_i)|$ divides $|(P_i)_n|^{d(P_i/Z^n(P_i))}$ $(1 \leq i \leq t)$. Notice that since G is n-abelian group, hence $G_n = (P_1)_n \ldots (P_t)_n$. On the other hand one can easily check that $d(P_i/Z^n(P_i)) \leq d(G/Z^n(G))$. Therefore we have

$$|G/Z^n(G)| \text{ divides } \prod_{i=1}^{t} |(P_i)_n|^{d(G/Z^n(G))} = |G_n|^{d(G/Z^n(G))}.$$

An important problem which goes back to Schur's theorem is finding the relationship between $G/Z(G)$ and G'. One of the best results is given by J. Wiegold in [10]. He showed that if $|G/Z(G)| = n$, then $|G'| \leq n^{\frac{1}{2} log_2^n}$. Also, some upper bounds for the order of $G/Z(G)$ can be fined in [6, 7, 12].

Here we extend some results in [7], for the n-center and the n-potent subgroups and give an upper bound for the order of $G/Z^n(G)$ in terms of the order of G_n, in a group G which is n-abelian and n-capable. Note that a group G is n-capable provided that $G \cong E/Z^n(E)$, for some group E.

Lemma 1 *Let G be a group and H be an n-abelian subgroup of G generated by k elements and $|G_n| = t$. Then $[G : C_G(H^n)] \leq t^k$.*

Proof Let $\{x_1, \dots, x_k\}$ be a set of generators of H. Denote by $Cl(x_i)$, the conjugacy class of x_i. Then we have

$$\begin{aligned}[G : C_G(H^n)] = [G : \bigcap_{i=1}^{k} C_G(x_i^n)] &\leq \prod_{i=1}^{k}[G : C_G(x_i^n)] \\ &= \prod_{i=1}^{k} |Cl(x_i^n)| \\ &\leq \prod_{i=1}^{k} |G_n|^k = t^k.\end{aligned}$$

Lemma 2 *Let G be a group and C be a proper subgroup of G. Then*

$$G_n = [G \setminus C, G^n].$$

Proof It is enough to show that $[c, g^n] \in [G\setminus C, G^n]$, for all $c, g \in G$. If $c \in G\setminus C$, then the assertion holds, immediately. Let $c \in C$. Then $[cx, g^n] \in [G \setminus C, G^n]$, for some element $x \in G \setminus C$. Therefore we have

$$[c, g^n] = ([x, g^n]^c)^{-1}[cx, g^n] \in [G \setminus C, G^n].$$

Lemma 3 *Let G be a group. Put $Z = G_n \cap Z^n(G)$ and let U, V be subgroups of G such that $Z \leq U, V \leq G_n$. Then there exist elements y, z of G with the following properties.*

(i) If $Z \neq U$, then $U \cap C_G(y) \neq U$.
(ii) If $V \neq G_n$, then $V \neq \langle V, [y, z]\rangle$.

Proof (i) Suppose that $Z \neq U$. Put $C = C_G(U)$. Now, C is a proper subgroup of G. Let $y \in G \setminus C$. Then $[y, x] \neq 1$, for some element x in U. Therefore $U \cap C_G(y) \neq U$.
(ii) Let $V \neq G_n$. If $Z = U$, then we can choose the element $[y, z^n]$ in G_n such that $[y, z^n] \notin V$ and the assertion holds. Suppose that $Z \neq U$. Thus by (i) we have $C_G(U) = C$ is a proper subgroup of G. Then Lemma 2 implies that $G_n = [G \setminus C, G^n]$, and so we have $V \neq [G \setminus C, G^n]$. Hence there exist elements $y \in G \setminus C, z \in G$ such that $[y, z^n] \notin V$. It follows that $V \neq \langle V, [y, z]\rangle$.

Lemma 4 *Let G be a group such that $G/Z^n(G)$ is n-abelian. Also, let*

$$Z = G_n \cap Z^n(G) \quad \textit{and} \quad [G_n : Z] = m.$$

Suppose that T is a subgroup of G such that $G_n \leq T$ and T/Z is n-abelian also the following properties hold.

(i) $G_n = T_n Z$.
(ii) $G_n \cap Z^n(T) = Z$.
(iii) $d(T/Z) = k$.

Then there exists a subgroup M of G such that

$$[G:M] \leq m^k \quad \text{and} \quad [M, G^n, G^n] = 1.$$

Proof Put $M/Z = C_{G/Z}((T/Z)^n)$. Then we have $|(G/Z)_n| = [G_n : Z] = m$. Therefore, Lemma 1 implies that

$$[G:M] = [G/Z : C_{G/Z}((T/Z)^n] = m^k.$$

On the other hand $[M, T^n] \leq Z \leq Z^n(G)$. Thus $[M, T^n, G^n] = 1$ and so $[M, T^n, T^n] = 1$. Then we have $[T^n, T^n, M^n] \leq [T^n, T^n, M] = 1$, by the Three Subgroup Lemma. Since $G/Z^n(G)$ is n-abelian, $[T, T^n, M] = 1$ so we have $[T_n, M^n] = 1$. This implies that $[G_n, M^n] = [T_n Z, M^n] = 1$ and hence $[T^n, G^n, M^n] = 1$. On the other hand, $[M^n, T^n, G^n] = 1$. Then by the Three Subgroup Lemma we have $[M^n, G^n, T^n] = [G^n, M^n, T^n] = 1$. Since $G/Z^n(G)$ is n-abelian we have $[M, G^n, T^n] = 1$. Therefore $[M, G^n] \leq Z^n(T) \cap G_n = Z$ and so $[M, G^n, G^n] = 1$.

Lemma 5 *Let G be an n-abelian group and $|G_n : Z^n(G) \cap G_n| = m$. Then there exists a subgroup T as in Lemma 4 with $k \leq 2\log_2 m$.*

Proof For $1 \leq i \leq l-1$, we define the elements y_{i+1}, z_{i+1} recursively by applying Lemma 3 for $V_i = \langle Z, [y_1, z_1]^n, \ldots, [y_i, z_i]^n \rangle$ and $U_i = C_{G_n}(V_i^n)$. Now we have

$$Z = V_0 < V_1 < V_2 < \cdots < V_l = G_n$$

and

$$G_n = U_0 > U_1 > U_1 > \cdots > U_l = Z,$$

where l is the smallest integer such that $V_l = G_n$ and $U_l = Z$.
Since $m = [G_n : Z] \geq 2^l$, we have $l \leq \log_2 m$. Put $T = \langle Z, y_1, z_1, \ldots, y_l, z_l \rangle$. Then it is easy to check that T has the required properties.

We have now accumulated all the information necessary to prove the following result.

Theorem 3 *Let G be an n-abelian group such that $[G_n : Z^n(G) \cap G_n] = m$. Then*

$$[G : Z_2^n(G)] \leq m^{2\log_2 m}.$$

Proof By Lemma 4 and Lemma 5 we can conclude that there exists a subgroup M of G such that $[M, G^n, G^n] = 1$ and $[G : M] \leq m^k$ with $k \leq 2\log_2 m$. Hence the assertion follows.

As a corollary, we easily obtain the following result.

Corollary 2 *Let $H = G/Z^n(G)$ in which G is an n-abelian group and*

$$|H_n| = m.$$

Then $[H : Z^n(H)] \leq m^{2\log_2 m}$.

Proof Owing to this $H = G/Z^n(G)$, we have

$$|H_n| = \Big|\frac{G_n Z^n(G)}{Z^n(G)}\Big| = \Big|\frac{G_n}{G_n \cap Z^n(G)}\Big| = m.$$

Notice that since G is n-abelian group, hence by Theorem 3 we have

$$[H : Z^n(H)] = [G : Z_2^n(G)] \leq m^{2log_2 m}.$$

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