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Shifted Legendre Tau Method for Solving the Stochastic Weakly Singular Integro-Differential Equations

Ruhangiz Azimi · Mostafa Mohagheghy Nezhad · Saedeh Foadian

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Abstract In this paper, the stochastic weakly singular integro-differential equation is discussed. The shifted Legendre Tau method is introduced for finding the unknown function. For this purpose, shifted Legendre polynomials and their properties are introduced. The proposed method is based on expanding the approximate solution as the elements of shifted Legendre polynomials. We reduce the problem to set of algebraic equations by using operational matrices. Also, the convergence analysis of shifted Legendre polynomials and error estimation for this method have been discussed. Finally, several numerical examples are given to demonstrate the high accuracy of the method.

Keywords Stochastic integro-differential equation · Shifted Legendre Tau method · Weakly singular kernel · Integro-differential equation.

Mathematics Subject Classification (2010) 65R20 · 45D05

1 Introduction

Stochastic functional equations have been an interesting research area in different fields, e.g. geophysics, biology, chemistry, epidemiology, microelectronic, finance, and medical. Modeling such phenomena requires the use of various

R. Azimi (Corresponding Author)
Adib Mazandaran Institute of Higher Education, Sari, Iran.
Tel.: +981133110809
Fax: +981133033960
E-mail: ruhangiz.azimi@gmail.com

M. Mohagheghy Nezhad
Adib Mazandaran Institute of Higher Education, Sari, Iran.
E-mail: mohaqeqi@gmail.com

S. Foadian
School of Mathematics and Computer Science, Damghan University, Damghan, Iran.
E-mail: s.foadian@std.du.ac.ir

stochastic differential equations [13,14,4,5,21,9], stochastic integral equations or stochastic integro-differential equations [20,26,10,27,24].

Since in many cases it is difficult to derive an explicit form of the solution to these class of equations, numerical approximation becomes a practical way to face this difficulty. Many published papers have been devoted to describe numerical solution of stochastic differential and integral equations [8,1,18,11, 12,6].

The Tau method that is a way to solve linear and nonlinear functional equations is one of the important types of the spectral method that express the solution of the problem as a linear combination of orthogonal or non-orthogonal basis functions. The main advantage of using orthogonal basis is that the problem under consideration is reduced into solving a system of linear or nonlinear algebraic equation [23]. Recently, different orthogonal basis functions such as block pulse functions, Fourier series, Walsh functions, orthogonal polynomials, and wavelets, were utilized to approximate solution of functional equations [3,16,17,2]. Shifted Legendre polynomials have been widely applied for solving functional equations [15,25].
In this paper the shifted Legendre polynomials will be used for solving the stochastic weakly singular integro-differential equation as follows

$$u_t(x,t) + a\,u_x(x,t) - (b+\beta\frac{dB}{dt})u_{xx}(x,t) = \int_0^t K(t-s)u(x,s)\mathrm{d}s + f(x,t), \tag{1}$$

where, a, b, and β are considered to be real constants. The integral term is called memory term, the kernel is a weakly singular kernel

$$K(t-s) = (t-s)^{-\alpha}, \quad 0<\alpha<1,$$

subject to the initial condition

$$u(x,0) = g_0(x), \quad 0 \le x \le l, \tag{2}$$

and the boundary conditions

$$u(0,t) = f_0(t), \quad u(l,t) = f_1(t), \quad t \ge 0, \tag{3}$$

where, $g_0(x)$, $f_0(t)$, $f_1(t)$, and $f(x,t)$ are the stochastic processes defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $u(x,t)$ is an unknown stochastic function to be determined and $B(t)$ is a one-dimensional Brownian motion process.
A real-valued stochastic process $B(t)$, $t \in [0,T]$ is called Brownian motion, if it satisfies the following properties [8,19]

(i) $B(0) = 0$ (with the probability 1).
(ii) For $0 \le s < t \le T$ the random variable given by the increment $B(t) - B(s)$ is normally distributed with mean zero and variance $t-s$; equivalently, $B(t) - B(s) \sim \sqrt{t-s}N(0,1)$, where $N(0,1)$ denotes a normally distributed random variable with zero mean and unit variance.

(iii) For $0 \leq s < t \leq u < v \leq T$ the increments $B(t) - B(s)$ and $B(v) - B(u)$ are independent.
(iv) The function $t \to B(t)$ is continuous functions of t.

The paper is organized as follows. In the next section, the shifted Legendre polynomials and their properties are described. In section 3, we construct the operational matrices of Legendre polynomials. In section 4, by using shifted Legendre Tau method we construct and develop an algorithm for the solution of the stochastic weakly singular integro-differential equation with boundary conditions. We obtain the error estimatoin for this method in section 5. Some numerical examples are solved using the method of this article in section 6. Finally, a conclusion is given in section 7.

2 Properties of shifted Legendre polynomials

It is well-known that the classical Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formulae

$$L_0(z) = 1, \; L_1(z) = z,$$

$$L_{i+1}(z) = \frac{2i+1}{i+1} z \; L_i(z) - \frac{i}{i+1} L_{i-1}(z), \quad i = 1, 2, \ldots .$$

Assume $z \in [z_a, z_b]$ and let $z^{\sim} = \frac{2z - z_a - z_b}{z_b - z_a}$. Then $\{L_i(z^{\sim})\}$ are called the shifted Legendre polynomials on $[z_a, z_b]$. In this paper, we mainly consider the shifted Legendre polynomials defined on $[0, l]$.

For $x \in [0, l]$, let $L_{l,i}(x) = L_i(\frac{2x-l}{l})$, $i = 0, 1, 2, \ldots$. Then the shifted Legendre polynomials $\{L_{l,i}(x)\}$ are defined by

$$\begin{aligned} &L_{l,0}(x) = 1, \\ &L_{l,1}(x) = \frac{2x - l}{l}, \\ &L_{l,i+1}(x) = \frac{(2i+1)(2x-l)}{(i+1)l} L_{l,i}(x) - \frac{i}{i+1} L_{l,i-1}(x), \quad i = 1, 2, \ldots . \end{aligned}$$

The set of $L_{l,i}(x)$ is a complete $L^2(0, l)$-orthogonal system, namely

$$\int_0^l L_{l,i}(x) L_{l,j}(x) dx = \begin{cases} \dfrac{l}{2i+1}, & i = j, \\ 0, & i \neq j. \end{cases}$$

So, we define $\Pi_m = \text{span}\{L_{l,0}, L_{l,1}, \ldots, L_{l,m}\}$. Thus, for any $y(x) \in L^2(0, l)$, we write

$$y(x) = \sum_{j=0}^{\infty} c_j L_{l,j}(x),$$

where the coefficients c_j are given by

$$c_j = \frac{2j+1}{l} \int_0^l y(x) L_{l,j}(x) dx, \quad j = 0, 1, 2, \dots . \tag{4}$$

In practice, only the first $(m+1)$-terms of shifted Legendre polynomials are considered. Hence we can write

$$y_m(x) \simeq \sum_{j=0}^{m} c_j L_{l,j}(x),$$

which alternatively may be written in the matrix form

$$y_m(x) \simeq C^T \Phi_{l,m}(x), \qquad C^T = [c_0, c_1, \dots, c_m],$$

with

$$\Phi_{l,m}(x) = [L_{l,0}, L_{l,1}, \dots, L_{l,m}]^T = V X_x, \tag{5}$$

where V is the coefficient matrix of shifted Legendre polynomials as follows

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 2 & 0 & 0 & \dots \\ 1 & -6 & 6 & 0 & \dots \\ -1 & 12 & -30 & 20 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and $X_x = \left[1, x, x^2, \dots, x^m\right]^T$, $(.)^T$ stands for the transpose.

Similarly a function of two independent variables $u(x,t)$ which is infinitely differentiable for $0 \le x \le l$ and $0 \le t \le \tau$ may be expressed in terms of the double shifted Legendre polynomials as

$$u_{n,m}(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} L_{l,i}(x) L_{\tau,j}(t). \tag{6}$$

If the infinite series in (6) is truncated, then it can be written as

$$u_{n,m}(x,t) \simeq \sum_{i=0}^{n} \sum_{j=0}^{m} a_{i,j} L_{l,i}(x) L_{\tau,j}(t) = \Phi_{l,n}^T(x) A \Phi_{\tau,m}(t), \tag{7}$$

where the shifted Legendre vectors $\Phi_{\tau,n}(x)$ and $\Phi_{l,m}(x)$ are defined similarly to (5). Also the shifted Legendre coefficient matrix A is given by

$$A = \begin{bmatrix} a_{00} & a_{01} & \dots & a_{0m} \\ a_{10} & a_{11} & \dots & a_{1m} \\ \vdots & \vdots & \dots & \vdots \\ a_{n0} & a_{n1} & \dots & a_{nm} \end{bmatrix},$$

where

$$a_{i,j} = \left(\frac{2i+1}{\tau}\right)\left(\frac{2j+1}{l}\right)\int_0^\tau \int_0^l u(x,t)L_{\tau,i}(t)L_{l,j}(x)dxdt. \tag{8}$$

Now, we present the shifted Legendre expansion of a function $u(x,t)$ with bounded mixed fourth partial derivative, converges uniformly to $u(x,t)$.

Theorem 1 *(convergence theorem) If a continuous function $u(x,t)$, defined on $[0,l]\times[0,\tau]$, has bounded mixed fourth partial derivative $\frac{\partial^4 u(x,t)}{\partial x^2\ \partial t^2}$, then the shifted Legendre expansion of the function as*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}L_{l,i}(x)L_{\tau,j}(t),$$

converges uniformly to the $u(x,t)$.

Proof Let $u(x,t)$ be a function defined on $[0,l]\times[0,\tau]$ such that $\left|\frac{\partial^4 u(x,t)}{\partial x^2\ \partial t^2}\right| \leq \alpha$, where α is a positive constant and

$$a_{i,j} = \left(\frac{2i+1}{\tau}\right)\left(\frac{2j+1}{l}\right)\int_0^\tau \int_0^l u(x,t)L_{\tau,i}(t)L_{l,j}(x)dxdt,$$

for $i = 0,1,\ldots,n$ and $j = 0,1,\ldots,m$. By partial integration and using following equation

$$L'_{l,i+1} - L'_{l,i-1} = \frac{2}{l}(2i+1)L_{l,i}(x),$$

we have

$$\begin{aligned}
a_{i,j} &= \frac{2j+1}{2\tau}\int_0^\tau u(x,t)\Big(L_{l,i+1}(x) - L_{l,i-1}(x)\Big)\Big|_0^l L_{\tau,j}(t)dt \\
&- \frac{2j+1}{2\tau}\int_0^\tau\int_0^l \frac{\partial u(x,t)}{\partial x}\Big(L_{l,i+1}(x) - L_{l,i-1}(x)\Big)L_{\tau,j}(t)dxdt \\
&= -\frac{2j+1}{2\tau}\int_0^\tau\int_0^l \frac{\partial u(x,t)}{\partial x}\Big(L_{l,i+1}(x) - L_{l,i-1}(x)\Big)L_{\tau,j}(t)dxdt \\
&= -\frac{(2j+1)l}{4\tau}\int_0^\tau \frac{\partial u(x,t)}{\partial x}\Big(\frac{L_{l,i+2}(x) - L_{l,i}(x)}{2i+3} - \frac{L_{l,i}(x) - L_{l,i-2}(x)}{2i-1}\Big)\Big|_0^l L_{\tau,j}(t)dt \\
&+ \frac{(2j+1)l}{4\tau}\int_0^\tau\int_0^l \frac{\partial^2 u(x,t)}{\partial x^2}\Big(\frac{L_{l,i+2}(x) - L_{l,i}(x)}{2i+3} - \frac{L_{l,i}(x) - L_{l,i-2}(x)}{2i-1}\Big)L_{\tau,j}(t)dxdt \\
&= \frac{(2j+1)l}{4\tau}\int_0^\tau\int_0^l \frac{\partial^2 u(x,t)}{\partial x^2}\Big(\frac{L_{l,i+2}(x) - L_{l,i}(x)}{2i+3} - \frac{L_{l,i}(x) - L_{l,i-2}(x)}{2i-1}\Big)L_{\tau,j}(t)dxdt.
\end{aligned}$$

Now, let $Q_{l,i}(x) = (2i-1)L_{l,i+2} - 2(2i+1)L_{l,i}(x) + (2i+3)L_{l,i-2}(x)$ then we have

$$\begin{aligned}
a_{i,j} &= \frac{(2j+1)l}{4\tau(2i+3)(2i-1)}\int_0^\tau\int_0^l \frac{\partial^2 u(x,t)}{\partial x^2}Q_{l,i}(x)L_{\tau,j}(t)dxdt \\
&= \frac{l\tau}{16(2i+3)(2i-1)(2j+3)(2j-1)}\int_0^l\int_0^\tau \frac{\partial^4 u(x,t)}{\partial t^2 \partial x^2}Q_{l,i}(x)Q_{\tau,j}(t)dtdx.
\end{aligned}$$

Thus

$$|a_{i,j}| \leq \frac{l\tau}{16(2i+3)(2i-1)(2j+3)(2j-1)} \int_0^l \int_0^\tau \Big|\frac{\partial^4 u(x,t)}{\partial t^2 \partial x^2}\Big| |Q_{l,i}(x)||Q_{\tau,j}(t)| dt dx$$
$$\leq \frac{l\tau\alpha}{16(2i+3)(2i-1)(2j+3)(2j-1)} \int_0^l |Q_{l,i}(x)| dx \int_0^\tau |Q_{\tau,j}(t)| dt.$$

Also we have

$$\Big(\int_0^l |Q_i(x)|dx\Big)^2 = \Big(\int_0^l |(2i-1)L_{l,i+2}(x) - 2(2i+1)L_{l,i}(x) + (2i+3)L_{l,i-2}(x)|dx\Big)^2$$
$$\leq \Big(\int_0^l (1)^2 dx\Big)\Big(\int_0^l (2i-1)^2 L_{l,i+2}(x)^2 + (4i+2)^2 L_{l,i}(x)^2 + (2i+3)^2 L_{l,i-2}(x)^2\Big)dx$$
$$\leq l\Big(\frac{(2i-1)^2 l}{2i+5} + \frac{(4i+2)^2 l}{2i+1} + \frac{(2i+3)^2 l}{2i-3}\Big)$$
$$\leq \frac{6l^2(2i+3)^2}{2i-3}.$$

Then we get

$$\int_0^l |Q_i(x)|dx \leq \frac{\sqrt{6}l(2i+3)}{\sqrt{2i-3}}.$$

Thus we obtain

$$|a_{i,j}| \leq \frac{l\tau\alpha}{16(2i+3)(2i-1)(2j+3)(2j-1)} \times \frac{\sqrt{6}l(2i+3)}{\sqrt{2i-3}} \times \frac{\sqrt{6}\tau(2j+3)}{\sqrt{2j-3}}$$
$$= \frac{3l^2\tau^2\alpha}{8\sqrt{(2i-3)^3}\sqrt{(2j-3)^3}}.$$

Consequently, $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}$ is absolute convergent and thus the expansion of the function converges uniformly.

Theorem 2 *Let $u(x,t)$ be a continuous function defined on $[0,l] \times [0,\tau]$ with bounded mixed fourth partial derivative, say $\left|\frac{\partial^4 u(x,t)}{\partial x^2\ \partial t^2}\right| \leq \alpha$, then we have the following accuracy estimation*

$$\varepsilon_n \leq \frac{3\alpha l^2\tau^2}{8}\sqrt{\sum_{i=n+1}^{\infty}\frac{1}{(2i-3)^4}\sum_{j=m+1}^{\infty}\frac{1}{(2j-3)^4}},$$

where

$$\varepsilon_n = \Big(\int_0^\tau \int_0^l (u(x,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i,j}L_{l,i}(x)L_{\tau,j}(t))^2 dx dt\Big)^{1/2}. \qquad (9)$$

Also in the case of $n = m$ the error bound is $\varepsilon_n \leq \frac{3\alpha l^2\tau^2}{8}\sum_{i=n+1}^{\infty}\frac{1}{(2i-3)^4}$.

Proof

$$
\begin{aligned}
\varepsilon_n^2 &= \int_0^\tau \int_0^l \Big(u(x,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i,j} L_{l,i}(x) L_{\tau,j}(t)\Big)^2 dxdt \\
&= \int_0^\tau \int_0^l \Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j} L_{l,i}(x) L_{\tau,j}(t) - \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i,j} L_{l,i}(x) L_{\tau,j}(t)\Big)^2 dxdt \\
&= \int_0^\tau \int_0^l \Big(\sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} a_{i,j} L_{l,i}(x) L_{\tau,j}(t)\Big)^2 dxdt \\
&= \int_0^\tau \int_0^l \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} a_{i,j}^2 L_{l,i}^2(x) L_{\tau,j}^2(t) dxdt \\
&= \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} a_{i,j}^2 \int_0^l L_{l,i}^2(x)dx \int_0^\tau L_{\tau,j}^2(t)dt \\
&= \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} a_{i,j}^2 \frac{l\tau}{(2i+1)(2j+1)} \\
&\leq \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} \frac{9\alpha^2 l^4 \tau^4}{64(2i-3)^3(2j-3)^3(2i+1)(2j+1)} \\
&\leq \frac{9\alpha^2 l^4 \tau^4}{64} \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} \frac{1}{(2i-3)^4(2j-3)^4} \\
&= \frac{9\alpha^2 l^4 \tau^4}{64} \sum_{i=n+1}^{\infty} \frac{1}{(2i-3)^4} \sum_{j=m+1}^{\infty} \frac{1}{(2j-3)^4}.
\end{aligned}
$$

Then we have

$$
\varepsilon_n \leq \frac{3\alpha l^2 \tau^2}{8} \sqrt{\sum_{i=n+1}^{\infty} \frac{1}{(2i-3)^4} \sum_{j=m+1}^{\infty} \frac{1}{(2j-3)^4}}, \tag{10}
$$

which in the case of $n = m$, $\varepsilon_n \leq \frac{3\alpha l^2 \tau^2}{8} \sum_{i=n+1}^{\infty} \frac{1}{(2i-3)^4}$.

3 Operational matrices of shifted Legendre polynomials

In this section, we make the operational matrix of stochastic weakly singular integro-differential equation of the shifted Legendre vector.

3.1 Matrix representation of partial differential part

The derivative of the vector $\Phi_{l,m}(x)$ can be expressed by

$$\frac{d}{dx}\Phi_{l,m}(x) = D\Phi_{l,m}(x), \tag{11}$$

where D is the $(m+1)\times(m+1)$ operational matrix of derivative given by

$$D = (d_{i,j}) = \begin{cases} \dfrac{2(2j+1)}{l}, & \text{for } j = i-k, \quad k = \begin{cases} 1,3,\dots,m, & \text{if m odd,} \\ 1,3,\dots,m-1, & \text{if m even.} \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

For example, for odd m, we have

$$D = \frac{2}{l}\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 3 & 0 & 7 & \dots & 2m-3 & 0 & 0 \\ 1 & 0 & 5 & 0 & \dots & 0 & 2m-1 & 0 \end{bmatrix}.$$

Theorem 3 *Let $\Phi_{l,m}(x)$ be the shifted Legendre vector and*

$$u_{n,m}(x,t) = \Phi^T_{l,n}(x)A\Phi_{\tau,m}(t),$$

then

$$\frac{\partial^r}{\partial x^r}u_{n,m}(x,t) = \Phi^T_{l,n}(x)(D^T)^r A\Phi_{\tau,m}(t). \tag{12}$$

Proof From equations (7) and (11) we have

$$\begin{aligned} \frac{\partial^r}{\partial x^r}u_{n,m}(x,t) &= \frac{\partial^{r-1}}{\partial x^{r-1}}\Big(\Phi^T_{l,n}(x)D^T A\Phi_{\tau,m}(t)\Big) \\ &= \frac{\partial^{r-2}}{\partial x^{r-2}}\Big(\frac{\partial}{\partial x}\Phi^T_{l,n}(x)D^T A\Phi_{\tau,m}(t)\Big) \\ &= \frac{\partial^{r-2}}{\partial x^{r-2}}\Big(\Phi^T_{l,n}(x)(D^T)^2 A\Phi_{\tau,m}(t)\Big) \\ &\;\;\vdots \\ &= \frac{\partial}{\partial x}\Big(\Phi^T_{l,n}(x)(D^T)^{r-1} A\Phi_{\tau,m}(t)\Big) \\ &= \Phi^T_{l,n}(x)(D^T)^r A\Phi_{\tau,m}(t). \end{aligned}$$

Corollary 1 *Let $\Phi_{l,m}(x)$ be the shifted Legendre vector and*

$$u_{n,m}(x,t) = \Phi^T_{l,n}(x) A \Phi_{\tau,m}(t),$$

then

$$\frac{\partial^r}{\partial t^r} u_{n,m}(x,t) = \Phi^T_{l,n}(x) A D^r \Phi_{\tau,m}(t). \tag{13}$$

Proof From equations (7) and (11) we have

$$\begin{aligned}
\frac{\partial^r}{\partial t^r} u_{n,m}(x,t) &= \frac{\partial^{r-1}}{\partial t^{r-1}} \Big(\Phi^T_{l,n}(x) A D \Phi_{\tau,m}(t) \Big) \\
&= \frac{\partial^{r-2}}{\partial t^{r-2}} \Big(\Phi^T_{l,n}(x) A D^2 \Phi_{\tau,m}(t) \Big) \\
&\ \ \vdots \\
&= \frac{\partial}{\partial t} \Big(\Phi^T_{l,n}(x) A D^{r-1} \Phi_{\tau,m}(t) \Big) \\
&= \Phi^T_{l,n}(x) A D^r \Phi_{\tau,m}(t).
\end{aligned}$$

Lemma 1 *Let $y_m(t) = C^T V X_t$ be a polynomial where*

$$C^T = [c_0, c_1, \dots, c_m, 0, \dots], \qquad X_t = [1, t, t^2, \dots]^T,$$

then we have

$$\begin{aligned}
\frac{d^k}{dt^k} y_m(t) &= C^T V \eta^k X_t, \\
t^k y_m(t) &= C^T V \mu^k X_t, \\
k &= 0, 1, 2, \dots,
\end{aligned}$$

where

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ & 0 & 1 & 0 & \\ & & 0 & 1 & \\ & & & & \ddots \end{bmatrix} \quad \text{and} \quad \eta = \begin{bmatrix} 0 & & & & \dots \\ 1 & 0 & & & \\ 0 & 2 & 0 & & \\ 0 & 0 & 3 & & \\ & & & & \ddots \end{bmatrix}.$$

Proof see [22].

Lemma 2 *Let $\Phi_{l,m}(x)$ be the shifted Legendre vector and*

$$u_{n,m}(x,t) \simeq \Phi^T_{l,n}(x) A \Phi_{\tau,m}(t),$$

then

$$t^s \frac{\partial^r}{\partial x^r} u_{n,m}(x,t) \simeq \Phi^T_{l,n}(x) (D^T)^r A \mu_s \Phi_{\tau,m}(t), \tag{14}$$

where $\mu_s = V \mu^s V^{-1}$ and μ is given in Lemma 1.

Proof From Theorem (3), Lemma (1), and equations (7) and (11) we have

$$\begin{aligned}
t^s \frac{\partial^r}{\partial x^r} u_{n,m}(x,t) &\simeq t^s \left(\Phi_{l,n}^T(x)(D^T)^r A\Phi_{\tau,m}(t)\right) \\
&\simeq \Phi_{l,n}^T(x)(D^T)^r A t^s \Phi_{\tau,m}(t) \\
&\simeq \Phi_{l,n}^T(x)(D^T)^r A t^s V X_t \\
&\simeq \Phi_{l,n}^T(x)(D^T)^r A V t^s X_t \\
&\simeq \Phi_{l,n}^T(x)(D^T)^r A V \mu^s X_t \\
&\simeq \Phi_{l,n}^T(x)(D^T)^r A V \mu^s V^{-1} V X_t \\
&\simeq \Phi_{l,n}^T(x)(D^T)^r A \mu_s \Phi_{\tau,m}(t).
\end{aligned}$$

3.2 Matrix representation of integral part

Lemma 3 *If Γ is the Gamma function, then we have*

$$\int_0^t \frac{s^m}{(t-s)^\alpha}\,\mathrm{d}s = \frac{\Gamma(1-\alpha)\Gamma(m+1)}{\Gamma(m-\alpha+2)} t^{m-\alpha+1}, \qquad m = 0,1,2,\ldots.$$

Proof With integration by parts and using $\Gamma(\alpha) = (\alpha-1)!$ it can easily be obtained.

Theorem 4 *Let $\Phi_{l,m}(x) = VX_x$ be the shifted Legendre vector then*

$$\int_0^t \frac{u(x,s)}{(t-s)^\alpha} ds \simeq \Phi_{l,n}^T(x) A V U K \Phi_{\tau,m}(t), \tag{15}$$

where U is a diagonal matrix with elements

$$U_{i,i} = \frac{\Gamma(1-\alpha)\Gamma(i+1)}{\Gamma(i-\alpha+2)}, \qquad i = 0,1,2,\ldots,m,$$

and

$$K = \Big[B_0, B_1, \ldots, B_m\Big]^T, \qquad B_j = \Big[t_{j,0}, t_{j,1}, \ldots, t_{j,m}],$$

which $t_{j,i}$, $i,j = 0,1,\ldots,m$ are the coefficients of $L_{\tau,i}$, $i = 0,1,\ldots,m$ in expansion of $t^{j-\alpha+1}$.

Proof

$$\begin{aligned}
\int_0^t \frac{u(x,s)}{(t-s)^\alpha}\,\mathrm{d}s &\simeq \int_0^t \frac{\Phi_{l,n}^T(x) A \Phi_{\tau,m}(s)}{(t-s)^\alpha}\,\mathrm{d}s \\
&= \Phi_{l,n}^T(x) A \int_0^t \frac{[L_{\tau,0}(s), L_{\tau,1}(s), \ldots, L_{\tau,m}(s)]^T}{(t-s)^\alpha} \mathrm{d}s \\
&= \Phi_{l,n}^T(x) A \int_0^t \frac{V\,[1, s, \ldots, s^m]^T}{(t-s)^\alpha}\,\mathrm{d}s \\
&= \Phi_{l,n}^T(x) A V \left[\int_0^t \frac{1}{(t-s)^\alpha}\,\mathrm{d}s, \int_0^t \frac{s}{(t-s)^\alpha}\,\mathrm{d}s, \ldots, \int_0^t \frac{s^m}{(t-s)^\alpha}\,\mathrm{d}s\right]^T,
\end{aligned}$$

by using lemma (3) we can write

$$\int_0^t \frac{u(x,s)}{(t-s)^\alpha}\,\mathrm{d}s \simeq \Phi_{l,n}^T(x)AV\left[\frac{\Gamma(1-\alpha)\Gamma(1)}{\Gamma(-\alpha+2)}t^{-\alpha+1}, \frac{\Gamma(1-\alpha)\Gamma(2)}{\Gamma(-\alpha+3)}t^{-\alpha+2}, \dots, \frac{\Gamma(1-\alpha)\Gamma(m+1)}{\Gamma(m-\alpha+2)}t^{m-\alpha+1}\right]^T$$
$$= \Phi_{l,n}^T(x)AVU\Pi, \tag{16}$$

where

$$\Pi = \left[t^{-\alpha+1}, t^{-\alpha+2}, \dots, t^{m-\alpha+1}\right]^T.$$

By approximating $t^{j-\alpha+1}$, $j = 0, 1, \dots, m$, we get

$$t^{j-\alpha+1} \simeq \sum_{i=0}^{m} t_{j,i}L_{\tau,i}(t) = B_j\Phi_{\tau,m}(t),$$
$$B_j = [t_{j,0}, t_{j,1}, \dots, t_{j,m}],$$

we obtain

$$\Pi = [B_0\Phi_{\tau,m}(t), B_1\Phi_{\tau,m}(t), \dots, B_m\Phi_{\tau,m}(t)]^T = K\Phi_{\tau,m}(t),$$
$$K = [B_0, B_1, \dots, B_m]^T. \tag{17}$$

By substituting (17) into (16) we obtain

$$\int_0^t \frac{u(x,s)}{(t-s)^\alpha}\,\mathrm{d}s \simeq \Phi_{l,n}^T(x)AVUK\Phi_{\tau,m}(t). \tag{18}$$

4 Description of the proposed method

In this section, a new algorithm for solving stochastic weakly singular equations is proposed based on shifted Legendre polynomials.
Consider the stochastic weakly singular integro-differential equation with a weakly singular kernel (1). Let us start our algorithm to solve (1)-(3).

Now, we approximate the functions $f(x,t)$, $g_0(x)$, $f_0(t)$, and $f_1(t)$ by the shifted Legendre polynomials as

$$f_{n,m}(x,t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{m} f_{i,j}L_{l,i}(x)L_{\tau,j}(t) = \Phi_{l,n}^T(x)F\Phi_{\tau,m}(t),$$
$$g_0(x) \simeq \sum_{i=0}^{n} g_iL_{l,i}(x) = \Phi_{l,n}^T(x)G,$$
$$f_0(t) \simeq \sum_{i=0}^{m} p_iL_{l,i}(t) = P\Phi_{\tau,m}(t),$$
$$f_1(t) \simeq \sum_{i=0}^{m} r_iL_{l,i}(t) = R\Phi_{\tau,m}(t), \tag{19}$$

where F, G, P, and R are known matrices which can be written as

$$P = [p_0, p_1, \dots, p_m], \qquad R = [r_0, r_1, \dots, r_m], \qquad G = [g_0, g_1, \dots, g_n]^T,$$

$$F = \begin{bmatrix} f_{00} & f_{01} & \cdots & f_{0m} \\ f_{10} & f_{11} & \cdots & f_{1m} \\ \vdots & \vdots & \dots & \vdots \\ f_{n0} & f_{n1} & \cdots & f_{nm} \end{bmatrix},$$

where p_j, r_j, and g_j are given as in (4) but f_{ij} are given as in (8).

Now, we consider discretized Brownian motion, where $B(t)$ is determined at $N_1 + 1$ distinct values and utilized an interpolation to construct $B(t)$. Let $t_i = ih$, $i = 0, 1, 2, \dots, N_1$, $h = \frac{T}{N_1}$ and B_i denote $B(t_i)$. Condition (i) in introduction says that $B_0 = 0$ with probability 1, and condition (ii) and (iii) tell us that

$$B_i = B_{i-1} + dB_i, \quad i = 1, 2, \dots, N_1,$$

where each dB_i is an independent random variable of the form $\sqrt{h}N(0,1)$. For approximation $\frac{dB}{dt}$ we perform the following steps

1) Let $B_0 = 0$.
2) Let $B_i = B_{i-1} + \text{Random[Normal Distribution}[0, \sqrt{\Delta t}]]$. That's mean each B_i will be obtained by the sum of the previous value with a random amount in the interval $[0,1]$ which distributed with mean 0 and variance $\sqrt{\Delta t}$.
3) Let data $= \{(0,0), (\Delta t, B_1), \dots, (N_1 \Delta t, B_{N_1})\}$.
4) Now, we obtain a polynomial interpolating from these points which is an approximation for the $B(t)$ function. $B(t)$ is not differentiable but we approximate it as a polynomial and show it by $\tilde{B}(t)$.
5) Let $\frac{d\tilde{B}}{dt} = \sum_{i=0}^{J} p_i t^i$ then we have

$$\begin{aligned} \frac{d\tilde{B}}{dt} u_{xx}(x,t) &= \sum_{i=0}^{J} p_i t^i \Big(\Phi_{l,n}^T(x) (D^T)^2 A \Phi_{\tau,m}(t) \Big) \\ &= \Phi_{l,n}^T(x) \Big(\sum_{i=0}^{J} p_i (D^T)^2 A \mu_i \Big) \Phi_{\tau,m}(t). \end{aligned} \tag{20}$$

The Mathematica program for constructing Brownian motion is as follows

$\Delta t = \dfrac{1}{N_1};$

$B[0] = 0;$

$B[i\_] := B[i-1] + \text{Random[Normal Distribution}[0, \sqrt{\Delta t}]];$

$\text{data} = \text{Table}[\{i\Delta t, B[i]\}, \{i, 0, N_1\}];$

$B[t\_] := \text{Interpolating Polynomial}[data, t];$

Now, using (12), (13), (15), (19), (20), and substituting in equation (1), it is easy to obtain that

$$\begin{aligned}\Phi_{l,n}^T(x)AVUk\Phi_{\tau,m}(t) + \Phi_{l,n}^T(x)F\Phi_{\tau,m}(t) =& \Phi_{l,n}^T(x)AD\Phi_{\tau,m}(t)\\ &+ a\Phi_{l,n}^T(x)D^TA\Phi_{\tau,m}(t)\\ &- b\Phi_{l,n}^T(x)(D^T)^2A\Phi_{\tau,m}(t)\\ &- \beta\Phi_{l,n}^T(x)\Delta\Phi_{\tau,m}(t),\end{aligned}$$

where $\Delta = \sum_{i=0}^{J} p_i(D^T)^2A\mu_i$. Hence the residual $R_{n,m}(x,t)$ for (1) can be written as

$$\begin{aligned}R_{n,m}(x,t) &= \Phi_{l,n}^T(x)\left[AD + aD^TA - b(D^T)^2A - \beta\Delta - AVUk - F\right]\Phi_{\tau,m}(t)\\ &= \Phi_{l,n}^T(x)H\Phi_{\tau,m}(t),\end{aligned}$$

where

$$H = A\,(D - VUk) + \left(aD^T - b(D^T)^2\right)A - \beta\Delta - F.$$

For finding a typical matrix formulation, similar to the typical tau method, we eliminate one last column and two last rows of the matrix H, then we generate $(n-1)\times m$ algebraic equations by using the following algebraic equations

$$H_{ij} = 0, \quad i = 0,1,\ldots,n-2, \quad j = 0,1,\ldots,m-1,$$

namely

$$\int_0^l\int_0^\tau R_{n,m}(x,t)L_{\tau,i}(t)L_{l,j}(x)\mathrm{d}t\mathrm{d}x = 0. \tag{21}$$

Also, by substituting equations (7) and (19) in equations (2) and (3) we have

$$\begin{aligned}\Phi_{l,n}^T(x)A\Phi_{\tau,m}(0) &= \Phi_{l,n}^T(x)G,\\ \Phi_{l,n}^T(0)A\Phi_{\tau,m}(t) &= P\Phi_{\tau,m}^T(t),\\ \Phi_{l,n}^T(1)A\Phi_{\tau,m}(t) &= R\Phi_{\tau,m}^T(t),\end{aligned}$$

which implies that

$$A\Phi_{\tau,m}(0) = G, \tag{22}$$

$$\Phi_{l,n}^T(0)A = P, \tag{23}$$

$$\Phi_{l,n}^T(1)A = R. \tag{24}$$

We can find $n+1$ linear algebraic equations from (22), m linear algebraic equations by choosing m equations from (23), similarly m equations from (24) and finally $(n-1)\times m$ equations from (21). Since the number of the unknown coefficients a_{ij} is equal to $(n+1)\times(m+1)$ we generate a system of $(n+1)\times(m+1)$ equations. Consequently $u_{n,m}(x,t)$ given in (7) can be calculated. In

our implementation, we have solved this system using the Mathematica Solve function.
In all the considered examples in section 6, this function has succeeded to obtain an accurate approximate solution of the system. We summarize the algorithm of the method as follows.

Algorithm of the method

Step 1. Choose the set of shifted Legendre polynomials $\{L_{l,i}(x)\}_{i=0}^{n}$, $\{L_{\tau,j}(t)\}_{j=0}^{m}$, and let the approximate solution $u_{n,m}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_{i,j}L_{l,i}(x)L_{\tau,j}(t)$.

Step 2. Find the coefficient matrix V respect to $X_x = \left[1, x, x^2, \ldots, x^m\right]^T$, such that $\Phi_{l,m}(x) = VX_x$.

Step 3. Using equations (13), (12), (14) (15), and (19) convert problem (1) and boundary conditions (2) and (3) to an algebraic system.

Step 4. Linearize the supplementary conditions in the same way as mentioned in Step 3.

Step 5. We can find $n+1$ linear algebraic equations from (22), $2m$ equations from (23) and (24), and $m(n-1)$ equations from (21) in the obtained system.

Step 6. Solve the system obtained from Steps 4 and 5 to find the unknown coefficients $a_{i,j}$, $i = 0, 1, \ldots, n$, and $j = 0, 1, \ldots, m$.

5 Error analysis for Tau method

In this section, we state the error analysis for the solution of stochastic weakly singular integro-differential equations (2) and (3). Let

$$e_{n,m}(x,t) = u(x,t) - u_{n,m}(x,t). \tag{25}$$

If $u_{m,n}(x,t)$ is a good approximation for $u(x,t)$ then for a given $\varepsilon > 0$, Max $|e_{n,m}(x,t)| < \varepsilon$. To this end, we are looking for an approximation for $e_{n,m}(x,t)$ by using the same method we used for approximation of $u(x,t)$. Firstly, we obtain from equation (25) that

$$u(x,t) = e_{n,m}(x,t) + u_{n,m}(x,t). \tag{26}$$

Therefore by using equations (26) and (1) we have,

$$\begin{aligned}&(e_{n,m})_t(x,t) + a(e_{n,m})_x(x,t) - (b + \beta\frac{dB}{dt})(e_{n,m})_{xx}(x,t)\\&= \int_0^t k(t-s)e_{n,m}(x,s)\mathrm{d}s + H_{n,m}(x,t),\end{aligned}$$

where, $H_{n,m}(x,t)$ is a perturbation term associated with $u_{n,m}(x,t)$ and can be obtained with following formulae

$$H_{n,m}(x,t) = \int_0^t k(t-s)u_{n,m}(x,s)\mathrm{d}s + f(x,t) - (u_{n,m})_t(x,t) - a(u_{n,m})_x(x,t) + (b+\beta\frac{dB}{dt})(u_{n,m})_{xx}(x,t),$$

which $\frac{dB}{dt}$ is the same approximation polynomial that is mentioned in previous section, and the boundary conditions

$$\begin{aligned} e_{n,m}(x,0) &= u(x,0) - u_{n,m}(x,0) \\ &= g_0(x) - u_{n,m}(x,0), \\ e_{n,m}(0,t) &= u(0,t) - u_{n,m}(0,t) \\ &= f_0(t) - u_{n,m}(0,t), \\ e_{n,m}(l,t) &= u(l,t) - u_{n,m}(l,t) \\ &= f_1(t) - u_{n,m}(l,t). \end{aligned}$$

We proceed to find an approximation $(e_{n,m})_{n_1,m_1}(x,t)$ to the $e_{n,m}(x,t)$ in the same as we did for the solutions of equations (1)-(3) ((n_1, m_1) denotes the Tau degree of $e_{n,m}(x,t)$).

6 Numerical results and comparisons

In this section, we present three numerical examples to demonstrate the accuracy of the proposed method. The results show that this method, by selecting afew number of shifted Legendre polynomials is accurate. Let $t_n = nk$, $n = 0,1,2,\ldots,M$, $k = \frac{T}{M}$, $x_i = ih$, $i = 0,1,2,\ldots,N$, and $h = \frac{l}{N}$ where M, N respectively denotes the final time level t_M and the final space level x_N, $N+1$ is the number of nodes. In order to check the accuracy of the proposed method, the maximum absolute errors and Mean squared errors between the exact solution $u(x,t)$ and the approximate solution $u_{n,m}(x,t)$ are given by the following definitions.

Maximum norm error: $\|e_M\|_\infty = \max_{0\le i\le N} |u(x_i,t_M) - u_{n,m}(x_i,t_M)|$.

Mean squared error: $\|e_M\|_2 = \frac{1}{N}\left(\sum_{i=0}^{N} \mid u(x_i,t_M) - u_{n,m}(x_i,t_M)\mid^2\right)^{1/2}$.

Example 1 As a first application, we offer the following stochastic weakly singular integro-differential equation

$$u_t(x,t)+au_x(x,t)-(b+\beta\frac{dB}{dt})u_{xx}(x,t) = \int_0^t \frac{u(x,s)}{\sqrt[3]{t-s}}\mathrm{d}s+f(x,t),\ x\in[0,1],\ t>0,$$

with $a = 0.005$, $b = 0.5$, and the following initial condition

$$u(x, 0) = 1 - \cos 2\pi x + 2\pi^2 x(1 - x), \quad 0 \le x \le 1,$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad 0 \le t \le 1,$$

$$\begin{aligned} f(x,t) =& 2(1+t)(1+2\pi^2(1-x)x - \cos 2\pi x) \\ &+ \frac{3}{40} t^{2/3}(20 + 3t(8+3t))(-1 + 2\pi^2(-1+x)x + \cos 2\pi x) \\ &- 0.5(1+t)^2(-4\pi^2 + 4\pi^2 \cos 2\pi x) \\ &+ 0.005(1+t)^2(2\pi^2(1-x) - 2\pi^2 x + 2\pi \sin 2\pi x). \end{aligned}$$

The deterministic solution in kind of $\beta = 0$ (in the absence of the noise term) is

$$u(x, t) = (t + 1)^2(1 - \cos 2\pi x + 2\pi^2 x(1 - x)).$$

The maximum absolute errors and Mean squared errors between the deterministic solution $u(x, t)$ and the approximate solution $u_{n,m}(x, t)$ for $\beta = 0$ with various choices of $(n = m)$ and two different grid sizes $N = 100$, $M = 50$, and $N = 50$, $M = 100$, are presented in Table 1. Also this problem is solved by proposed method for $N_1 = 3$, $m = n = 7$. The behaviour of the approximation solutions together with contour plots for different values of β are shown in Figures 1-3. The absolute errors of the approximate solution for $\beta = 0$ at some different points $(x_i, t_j) \in [0, 1] \times [0, 1]$ are shown in Table 4.

Table 1 $\|e_M\|_\infty$ is the Maximum norm error and $\|e_M\|_2$ is Mean squared error and $\beta = 0$

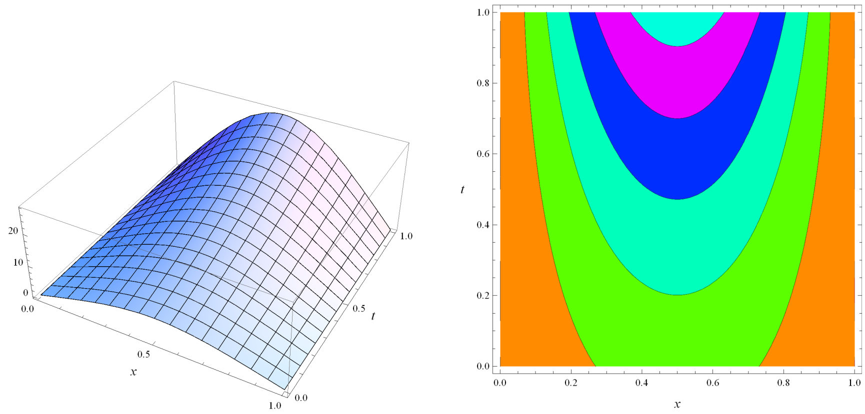
	$M = 50$	$N = 100$	$M = 100$	$N = 50$
$n = m$	$\|e_M\|_\infty$	$\|e_M\|_2$	$\|e_M\|_\infty$	$\|e_M\|_2$
7	2.32×10^{-2}	1.31×10^{-3}	2.33×10^{-2}	1.85×10^{-3}
9	6.34×10^{-4}	3.56×10^{-5}	6.34×10^{-4}	5.04×10^{-5}
11	1.58×10^{-6}	8.65×10^{-8}	1.19×10^{-5}	9.41×10^{-7}
13	3.18×10^{-7}	1.32×10^{-8}	3.18×10^{-7}	1.87×10^{-8}
15	2.12×10^{-7}	1.27×10^{-8}	2.07×10^{-7}	1.79×10^{-8}

Example 2 Consider the following stochastic weakly singular integro-differential equation

$$u_t(x,t) + au_x(x,t) - (b + \beta\frac{dB}{dt})u_{xx}(x,t) = \int_0^t \frac{u(x,s)}{\sqrt[4]{t-s}} \mathrm{d}s + f(x,t), \ x \in [0,1], \ t > 0,$$

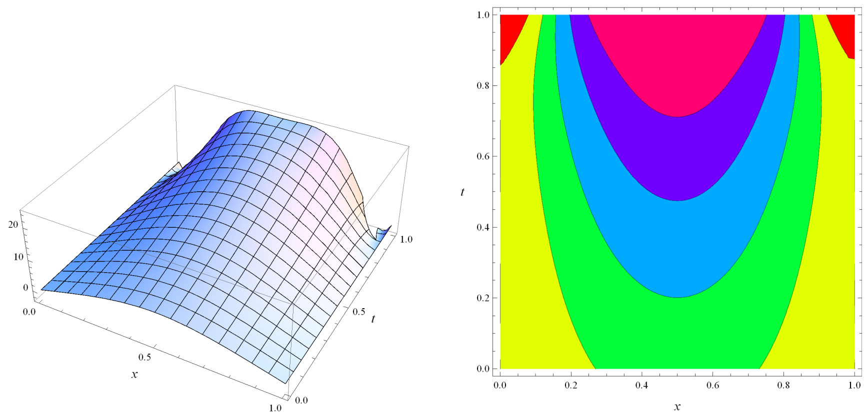
with $a = 0.5$ and $b = 0.001$, the following initial condition

$$u(x, 0) = 2\sin^2 \pi x, \quad 0 \le x \le 1,$$



(a) approximation solution (b) contour plot

Fig. 1 The graphs of the approximate solution (left side) and contour plot (right side) of Example 1 for $\beta = 0.0001$



(a) approximation solution (b) contour plot

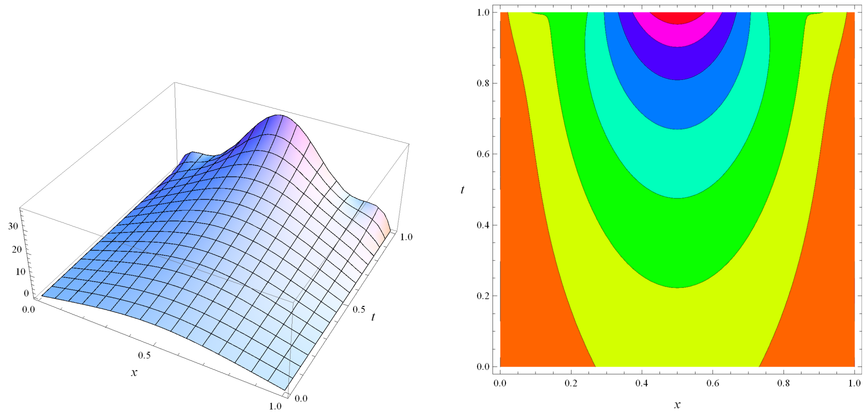
Fig. 2 The graphs of the approximate solution (left side) and contour plot (right side) of Example 1 for $\beta = 0.001$.

boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \leq t \leq 1,$$

and

$$\begin{aligned} f(x,t) =& 6.28319(1+t+t^2)\cos \pi x \sin \pi x + 2(1+2t)\sin^2 \pi x \\ &- \frac{8}{231} t^{3/4}(77+4t(11+8t))\sin \pi x^2 \\ &- 0.002(1+t+t^2)(2\pi)^2 \cos \pi x^2 - 2(\pi)^2 \sin^2 \pi x. \end{aligned}$$



(a) approximation solution (b) contour plot

Fig. 3 The graphs of the approximate solution (left side) and contour plot (right side) of Example 1 for $\beta = 0.01$

The deterministic solution in kind of $\beta = 0$ (in the absence of the noise term) is

$$u(x, t) = 2(t^2 + t + 1)\sin^2 \pi x.$$

In this example, we implement our method to solve stochastic weakly singular integro-differential equation. The results of this example for $\beta = 0$ with various choices of $(n = m)$ are shown in Table 2. Also, the graph of the maximum absolute error function is shown in Fig 4. This problem is solved by proposed method for $N_1 = 3$ and $m = n = 7$. The behaviour of the approximation solutions together with contour plots for different values of β are shown in Figures 5 and 6. The absolute errors of the approximate solution for $\beta = 0$ at some different points $(x_i, t_j) \in [0, 1] \times [0, 1]$ are shown in Table 4.

Table 2 Absolute error($|u(x, 0) - u_{n,m}(x, 0)|$) for different choices of n, m and $\beta = 0$.

x	$m = n = 9$	$m = n = 11$	$m = n = 13$
0	1.12×10^{-4}	2.39×10^{-6}	-1.87×10^{-8}
0.1	2.11×10^{-5}	6.90×10^{-7}	4.66×10^{-10}
0.2	5.20×10^{-6}	5.68×10^{-7}	-1.31×10^{-9}
0.3	2.12×10^{-5}	5.88×10^{-7}	2.96×10^{-9}
0.4	2.77×10^{-5}	4.17×10^{-7}	4.14×10^{-9}
0.5	1.18×10^{-14}	4.25×10^{-14}	-9.43×10^{-13}
0.6	2.77×10^{-5}	4.17×10^{-7}	-1.07×10^{-9}
0.7	2.12×10^{-5}	5.88×10^{-7}	-7.89×10^{-9}
0.8	5.20×10^{-6}	5.68×10^{-7}	-1.79×10^{-9}
0.9	2.11×10^{-5}	6.90×10^{-7}	-1.5×10^{-10}
1	1.12×10^{-4}	2.39×10^{-6}	-5.45×10^{-8}

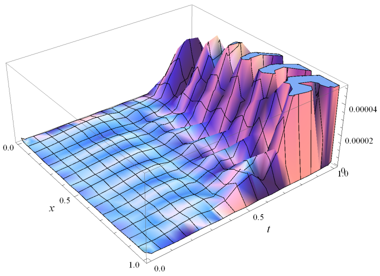
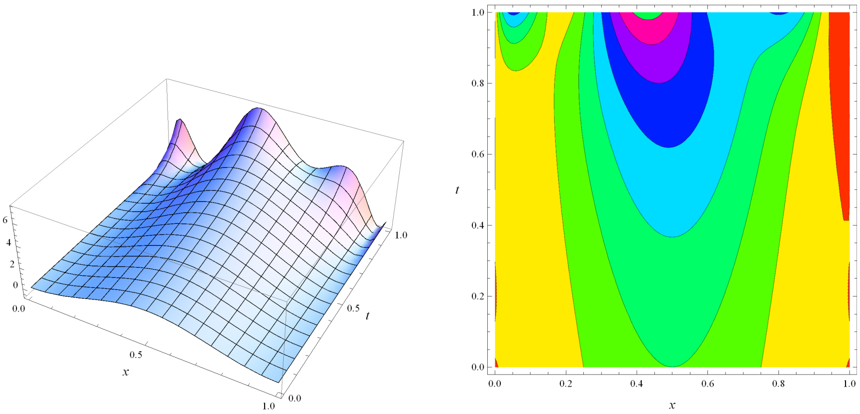
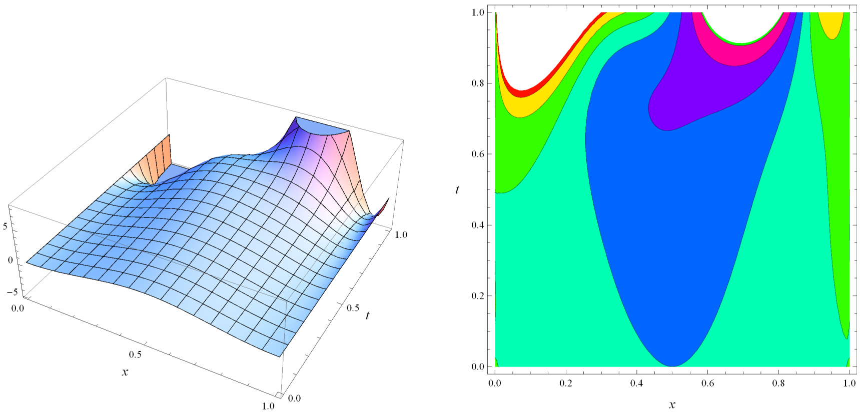


Fig. 4 Error function ($|u(x,t) - u_{n,m}(x,t)|$) for the Example 2, when $m = n = 11$ and $\beta = 0$.



(a) approximation solution (b) contour plot

Fig. 5 The graphs of the approximate solution (left side) and contour plot (right side) of Example 2 for $\beta = 0.0001$



(a) approximation solution (b) contour plot

Fig. 6 The graphs of the approximate solution (left side) and contour plot (right side) of Example 2 for $\beta = 0.001$

Example 3 Consider the following stochastic weakly singular integro-differential equation

$$u_t(x,t)+au_x(x,t)-(b+\beta\frac{dB}{dt})u_{xx}(x,t) = \int_0^t \frac{u(x,s)}{\sqrt{t-s}}\mathrm{d}s+f(x,t),\ x\in[0,1]\,,\ t>0,$$

with $a = 1$ and $b = 1$ and initial condition

$$u(x,0) = \sin\pi x, \quad 0\le x\le 1,$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0\le t\le 1,$$

$$f(x,t) = e^t\pi\cos\pi x + e^t\sin\pi x + e^t\pi^2\sin\pi x - e^t\sqrt{\pi}\mathrm{erf}(\sqrt{t})\sin\pi x.$$

The deterministic solution in kind of $\beta = 0$ (in the absence of the noise term) is

$$u(x,t) = \sin\pi x\ e^t.$$

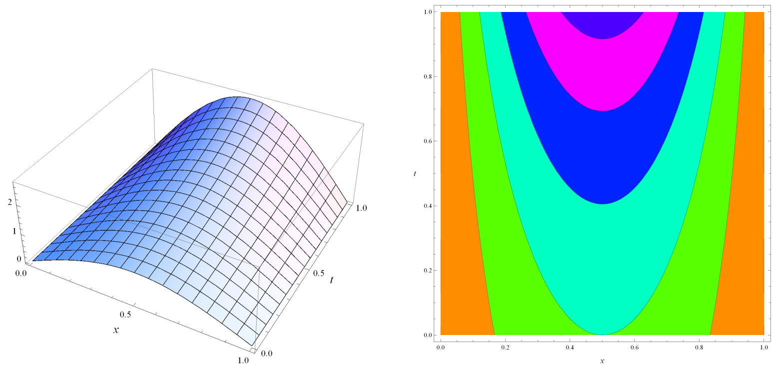
We have solved this problem for $m = n = 9$ and compute $u(x,t) - u_{9,9}(x,t)$ for different t and x (see Table 3). This problem is solved by proposed method for $N_1 = 3$ and $m = n = 7$. The behaviour of the approximation solutions together with contour plots for different values of β are shown in Figures 7-9. The absolute errors of the approximate solution for $\beta = 0$ at some different points $(x_i,t_j)\in[0,1]\times[0,1]$ are shown in Table 4.

Table 3 Error function $u(x,t) - u_{9,9}(x,t)$ for different t and x of Example 3 and $\beta = 0$.

$m = n = 9$					
t	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
0	3.96×10^{-8}	-3.22×10^{-8}	1.27×10^{-8}	1.72×10^{-8}	-3.27×10^{-8}
0.25	-5.18×10^{-8}	-1.34×10^{-7}	1.68×10^{-7}	-2.04×10^{-8}	-2.96×10^{-7}
0.5	-3.68×10^{-8}	-1.30×10^{-7}	2.16×10^{-7}	1.78×10^{-8}	-2.91×10^{-7}
0.75	-5.73×10^{-9}	-9.54×10^{-8}	3.32×10^{-7}	1.19×10^{-7}	-2.41×10^{-7}
1	-8.89×10^{-8}	-2.47×10^{-7}	4.79×10^{-7}	6.64×10^{-8}	-5.48×10^{-7}

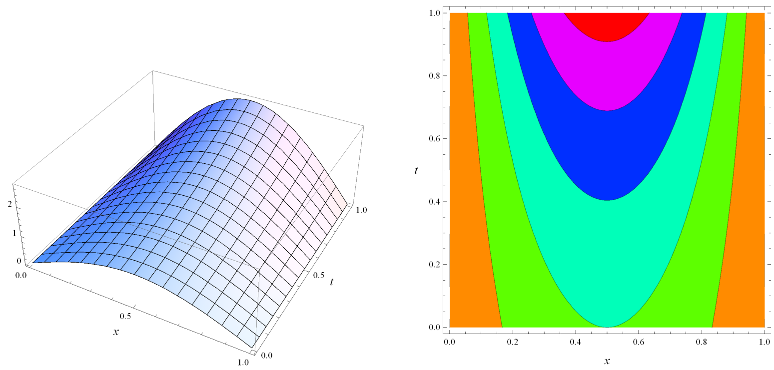
7 Conclusion

In this research, a new computational method based on the shifted Legendre polynomials together with the Tau method was proposed for solving a class of stochastic weakly singular integro-differential equation. To this end, operational matrices of partial derivatives and integral parts was derived. The main advantage of the proposed method was that it transformed the problem under study into solving a linear system of algebraic equations to achieve an



(a) approximation solution (b) contour plot

Fig. 7 The graphs of the approximate solution (left side) and contour plot (right side) of Example 3 for $\beta = 0.0001$

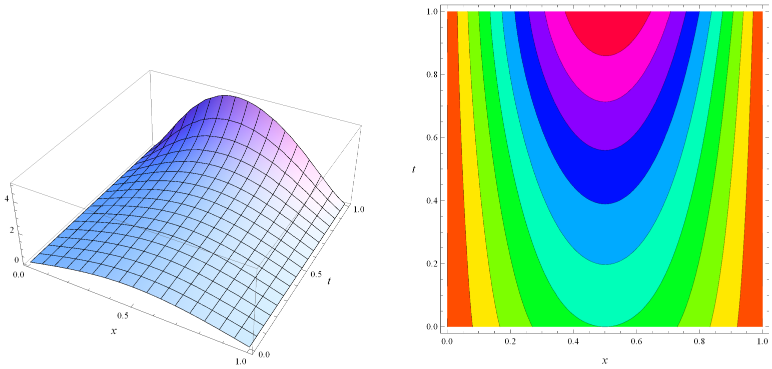


(a) approximation solution (b) contour plot

Fig. 8 The graphs of the approximate solution (left side) and contour plot (right side) of Example 3 for $\beta = 0.001$

Table 4 The absolute errors of the approximate solution for $\beta = 0.0001$ at some different points $(x_i, t_i) \in [0, 1] \times [0, 1]$.

$\beta = 0.0001, m = n = 7$			
(x_i, t_i)	$e(x_i, t_i)$ for Ex.1	$e(x_i, t_i)$ for Ex.2	$e(x_i, t_i)$ for Ex.3
$(0, 0)$	3.48×10^{-3}	3.48×10^{-3}	1.70×10^{-5}
$(0.1, 0.1)$	4.51×10^{-3}	1.08×10^{-3}	1.23×10^{-5}
$(0.2, 0.2)$	1.28×10^{-4}	8.95×10^{-4}	8.13×10^{-5}
$(0.3, 0.3)$	5.28×10^{-3}	1.88×10^{-3}	2.30×10^{-4}
$(0.4, .4)$	9.90×10^{-4}	1.91×10^{-3}	3.35×10^{-4}
$(0.5, 0.5)$	1.23×10^{-3}	5.92×10^{-4}	3.84×10^{-4}
$(0.6, 0.6)$	1.54×10^{-2}	6.39×10^{-3}	3.68×10^{-4}
$(0.7, 0.7)$	4.80×10^{-3}	2.08×10^{-2}	2.07×10^{-4}
$(0.8, 0.8)$	6.04×10^{-3}	1.89×10^{-1}	7.60×10^{-5}
$(0.9, 0.9)$	4.25×10^{-1}	3.28×10^{-1}	9.67×10^{-5}
$(1, 1)$	3.48×10^{-3}	3.48×10^{-3}	1.70×10^{-5}



(a) approximation solution (b) contour plot

Fig. 9 The graphs of the approximate solution (left side) and contour plot (right side) of Example 3 for $\beta = 0.01$

approximate solution of the problem. Illustrative examples were included to demonstrate the efficiency and accuracy of the proposed method. The performance of the proposed method for the considered problems was measured by calculating the maximum norm error and Mean squared error. Moreover, in cases that exact solutions were existed, the results of the proposed method were in a good agreement with the exact solutions.

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