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An Upper Bound for the Index of the Second n-Center Subgroup of An n-Abelian Group

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Abstract Let n be a positive integer. A group G is said to be n-abelian, if $(xy)^n = x^n y^n$, for any $x, y \in G$. In 1979, Fay and Waals introduced the n-potent and the n-center subgroups of a group G, as $G_n = \langle [x, y^n] | x, y \in G \rangle$, $Z^n(G) = \{x \in G | xy^n = y^n x, \forall y \in G\}$, respectively. Also, the second n-center subgroup, $Z_2^n(G)$, is defined by $Z_2^n(G)/Z^n(G) = Z^n(G/Z^n(G))$. In this paper, we give an upper bound for the index of the second n-center subgroup of any n-abelian group G in terms of the order of n-potent subgroup G_n.

Keywords n-abelian group · n-center subgroup · n-potent subgroup.

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1 Introduction

Abelian groups are important in the theory of group. For that reason many generalizations have been considered and exploited. One, in particular, is the idea of an n-abelian group. This concept has first been discussed by F. Levi [6,7] and it will play an important role in our discussion. If n is an integer and $n \geq 1$, then a group G is said to be n-abelian if $(xy)^n = x^n y^n$, for all elements x and y in G, from which it follows that $[x^n, y] = [x, y]^n = [x^n, y^n]$. Thus a group is 2-abelian if and only if it is abelian, while non abelian n-abelian groups do exist for every $n > 2$. Other self-evident fact about n-abelian groups are that every n-abelian group is $(1-n)$-abelian, and conversely. Indeed, n-abelian groups have been classified by Alperin [1]. A detailed introduction to n-abelian groups can be found in Baer's paper [2].

In this article we use two other concepts, the n-potent and the n-center subgroups of a group G, that have been introduced by Fay and Waals [3]. For

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a positive integer n, the n-potent and the n-center subgroups of a group G are defined respectively, as follows

$$G_n = \langle [x, y^n] | x, y \in G \rangle,$$
$$Z^n(G) = \{x \in G | xy^n = y^n x, \forall y \in G\},$$

where $[x, y] = x^{-1}y^{-1}xy$. It is easy to see that G_n is a fully invariant subgroup and $Z^n(G)$ is a characteristic subgroup of group G. The second n-center subgroup, $Z_2^n(G)$, is defined by $Z_2^n(G)/Z^n(G) = Z^n(G/Z^n(G))$. The concepts of n-potent and n-center subgroups are extensions of the important subgroups G' and $Z(G)$, respectively. One of the considerable problems in the group theory is the study on the relationship between the derived subgroup and the center factor group. In 1951, B.H Neumann [9] used an implicit idea of I. Schur [14] and proved that G' is finite when $[G : Z(G)]$ is finite. This important result has been known as the Schur's theorem. We know the converse is not true for infinite extra special groups (see [4]). However P. Hall in [4] showed that if G' is finite then $[G : Z_2(G)]$ is bounded in terms of $|G'|$. The first explicit bound was given by I.D. Macdonald [8]. Improving this bound, K. Podoski and B. Szegedy [11] proved that $[G : Z_2(G)] \leq |G'|^{2\log_2 |G'|}$. They [10] also proved that if G is a (not necessarily finite) group with $[G' : G' \cap Z(G)] = n$, then $[G : Z_2(G)] \leq n^{2\log_2 n}$.

Our motivation of writing this paper is to study the relation between the orders of the second n-center factor and n-potent subgroup of an n-abelian group. We first prove that if $G = H/Z^n(H)$ in which H is a finite n-abelian group, then the index of the n-center subgroup is bounded by some function of the order of n-potent subgroup. Then we find an upper bound for the index of the second n-center subgroup of any n-abelian group G in terms of the order of n-potent subgroup G_n.

2 Main Results

This section is devoted to obtain our main results. We begin with a key lemma.

Lemma 1 *Let G be an n-abelian group such that $|G_n| = t$ is finite and A be an n-central subgroup of G such that $[G : A] = m$ is finite. Then*

$$[G : Z^n(G)] \leq m^{1+log_2 t}.$$

Proof First, we find a subset $X \subseteq G$ such that

$$G =< X, A > \quad \text{and} \quad |X| \leq \log_2 m.$$

For this, put $A_0 = A$ and recursively construct subgroups A_i such that

$$A_i = \langle A_{i-1}, x_i \rangle,$$

where x_i is chosen arbitrarily in $G \setminus A_{i-1}$. So we have

$$A = A_0 < A_1 < \cdots < A_r = G,$$

for some integer r. It is easy to see that $G =< X, A >$, in which

$$X = \{x_1, x_2, \dots, x_r\}.$$

On the other hand, since $[A_i : A_{i-1}] \geq 2$, we have

$$m = [G : A] = [A_r : A_{r-1}] \cdots [A_1 : A_0] \geq 2^r.$$

Hence,

$$r \leq \log_2 m. \tag{1}$$

Put $\bar{X} = \{x_1^n, x_2^n, \dots, x_r^n\}$. Since $(x_i^n)^g = x_i^n[x_i^n, g]$, each conjugacy class of x_i^n in G is contained in some coset of G_n in G, and thus each of the classes has cardinality no longer than $|G_n|$. This implies that $[G : C_G(x_i^n)] \leq |G_n|$ for all $x \in X$ and so we have

$$[G : C_G(\bar{X})] = [G : \bigcap_{1\leq i\leq r} C_G(x_i^n)] \leq \prod_{1\leq i\leq r} [G : C_G(x_i^n)] \leq |G_n|^r. \tag{2}$$

Now we claim that $A \cap C_G(\bar{X})$ is a subgroup of $Z^n(G)$.
Suppose that $a \in A \cap C_G(\bar{X})$. Then $[a, x^n] = [a, b^n] = 1$, for all $x \in X$ and $b \in A$. On the other hand, G is an n-abelian group which is generated by A and X, and A is n-central. So we have $[a, g^n] = 1$ for all $g \in G$. Therefore, $a \in Z^n(G)$ and the claim is proved. Then by using (1) and (2), we have

$$\begin{aligned} [G : Z^n(G)] &\leq [G : A \cap C_G(\bar{X})] \\ &= [G : A][A : A \cap C_G(\bar{X})] \\ &\leq [G : A]|G_n|^r \\ &\leq \mathrm{mt}^{log_2 m} = \mathrm{mm}^{log_2 t}. \end{aligned}$$

Therefore the desired assertion follows.

We come now to the Main Result of this paper. In order to prove the Main Result, we first consider n-abelian groups with trivial n-center.

Theorem 1 *Let G be an n-abelian group. There exits a function $f(t)$ defined on natural numbers such that if $Z^n(G) = 1$ and $|G_n| = t$ is finite, then $|G| \leq f(t)$.*

Proof Put $C = C_G(G_n)$ and $m = [G : C]$. Consider the map

$$\varphi : G \to Aut(G_n),$$

defined by $g \mapsto \varphi(g)$, in which $\varphi(g)(x) = x^g$, for all $x \in G_n$ and $g \in G$. It is easy to check that φ is a homomorphism with $\ker \varphi = C$. Hence, G/C is isomorphic to a subgroup of $Aut(G_n)$. This implies that $m \leq t!$ and so m is bounded by a function of t. On the other hand, $[G, G^n, C] = [G_n, C] = 1$. It follows that $[C, G^n, C] = 1$. Then by the Three Subgroups Lemma, we have $[C, C, G^n] = 1$ and so $[C, C^n, G^n] = 1$. Thus $[C, C^n] \leq Z^n(G) = 1$ and we conclude that C is an n-central subgroup of G. Now by applying Lemma 1 for the n-central subgroup $C \subseteq G$ of index m, we conclude that $|G| = [G : Z^n(G)] \leq m^{1+\log_2 t}$. Then the result follows, because m is bounded by $t!$.

A group G is said to be n-capable if there exists a group E such that $G \cong E/Z^n(E)$ (See more details in [12]). Before we state the Main Result of the paper, we define a function $b(t)$, recursively. Assume that $f(t)$ is the function defined in Theorem 1. Let $b(1) = 1$, and for $t > 1$, we define

$$b(t) = \max\{f(t), (tM)^{1+\log_2 t}\},$$

in which M is the maximum value of $b(t/q)$, where q runs over all prime divisors of t. It is easy to see that $f(t) \leq b(t)$ and $b(k) \leq b(t)$, if k divides t.

Theorem 2 *Let $G = H/Z^n(H)$ in which H is a finite n-abelian group. Then there exists a function $b(t)$ defined on the natural numbers such that*

$$[G : Z^n(G)] \leq b(|G_n|).$$

Proof Let $|G_n| = t$. We have

$$\begin{aligned} G_n = [G, G^n] &= [H/Z^n(H), H^n Z^n(H)/Z^n(H)] \\ &= [H, H^n]Z^n(H)/Z^n(H). \end{aligned}$$

Putting $U = H_n Z^n(H)$, we have

$$|U/Z^n(H)| = t. \tag{3}$$

We use induction on the order of H to show that $[G : Z^n(G)] \leq b(t)$. If $H = 1$, then $G = 1$ and the inequality is trivially true. Now suppose that $|H| \geq 1$ and the assertion holds for all groups with order less than $|H|$. We prove the assertion for H. We can consider $Z(H) > 1$. Because, if $Z(H) = 1$, then one can see easily $Z(G) = 1$. Hence G is capable and by [5, Theorem A], the result holds. Now, since $Z(H) > 1$, there exists a normal subgroup T of G such that $T \leq Z(H) \leq Z^n(H)$ and $|T| = p$ where p is a prime number.
Put $Y/T = Z^n(H/T)$. Then Y is a normal subgroup of H and

$$H/Y = (H/T)/Z^n(H/T),$$

is n-capable. Also, since $Z^n(H)T/T = Z^n(H/T) = Y/T$, we have

$$Z^n(H) \leq Y \cap U.$$

Now, we consider two cases.
Case 1. Suppose $Z^n(H) = Y \cap U$. Then we have

$$\begin{aligned} (H/Y)_n = H_n Y/Y &= H_n Z^n(H)Y/Y \\ &= UY/Y \cong U/(Y \cap U) \\ &= U/Z^n(H). \end{aligned}$$

Hence, by using (3), we have $|(H/Y)_n| = t$.
On the other hand, $|H/Y| \leq |H/T| < |H|$ and H/Y is an n-abelian group. Therefore, by induction hypothesis, $[H/Y : Z^n(H/Y)] \leq b(t)$. Assume that $A/Y = Z^n(H/Y)$. Then we have $[H : A] \leq b(t)$.

Also, since $[A, H^n] \leq Y \cap U = Z^n(H)$, we have $A/Z^n(H) \leq Z^n(H/Z^n(H))$. This implies that

$$[G : Z^n(G)] \leq [H/Z^n(H) : A/Z^n(H)] = [H : A] \leq b(t),$$

and the assertion holds.
Case 2. Suppose $Z^n(H) < Y \cap U$.
Let y be an element of $(Y \cap U)\ Z^n(H)$. Set $C = C_H(y^n) \leq H$. Since

$$[Y, H^n] \leq T \leq Z^n(H),$$

we have $[h_1, y^n, h_2] = 1$, for all $h_1, h_2 \in H$. So the map $h \mapsto [h, y^n]$ defines a homomorphism φ from H into T with $\ker \varphi = C$. Thus H/C is isomorphic to a subgroup of T. On the other hand, since y does not belong to $Z^n(H)$, we have $C < H$. Therefore

$$[H : C] = p. \tag{4}$$

It is easy to see that

$$U = [H, H^n]Z^n(H) \subseteq C_H(y^n) = C.$$

Also, since $[H^n, Y] \subseteq T$, we have $1 = [h^n, y]^p = [h^n, y^p]$ for all $h \in H$. This implies that $y^p \in Z^n(H)$ and so y has order p modulo $Z^n(H)$. Let $X = Z^n(C)$. Then $y \in X \cap U$ and so $yZ^n(H) \in (X \cap U)/Z^n(H)$. It follows that $p = |yZ^n(H)|$ divides $[X \cap U : Z^n(H)]$. This implies that $[U : X \cap U]$ is a divisor of $[U : Z^n(H)]/p = t/p$. On the other hand, X is a normal subgroup of H and

$$(H/X)_n = H_n X/X = UX/X \cong U/(X \cap U).$$

Hence $|(H/X)_n| = |U/(X \cap U)|$ divides t/p. It follows that $|(C/X)_n|$ divides t/p. Also C/X is an n-capable group such that $|C/X| < |H|$. Therefore by induction hypothesis we have

$$[C/X : Z^n(C/X)] \leq b(|(C/X)_n|) \leq b(t/p).$$

Put $V/X = Z^n(C/X)$. Then we have

$$[C : V] \leq b(t/p). \tag{5}$$

By using (4), H/C is cyclic. So there exists an element $h \in H \setminus C$ such that $H = \langle h, C\rangle = \langle h\rangle C$. Set $S/X = C_{V/X}(h^n X)$. Then we have

$$[V : S] = [V/X : C_{V/X}(h^n X)] \leq |(V/X)_n| \leq |(H/X)_n| \leq t/p.$$

Hence applying (4) and (5), we have

$$[H : S] = [H : C][C : V][V : S] \leq tb(t/p) \leq tM, \tag{6}$$

where M is the maximum value of $b(t/q)$, as q runs over all prime divisors of t. Now we claim that

$$[S, H^n] \leq X = Z^n(C).$$

Let $a \in S$ and $b \in H$ be arbitrary elements. Then $b = h^r c$ for an integer r and $c \in C$. Since $S \leq V$ and $[V, C^n] \leq X$, we have $[a, c^n] \in X$. Also, by the definition of S, $[a, h^n] \in X$. Then by commutator calculus we can conclude that $[a, b^n] \in X$ and the claim is proved.
On the other hand, $S/X \leq V/X \leq C/X$ and so $S \leq C$. Therefor we have $[S, H^n, S^n] = 1$. Hence, by the Three Subgroup Lemma, we have

$$[S, S^n] \leq Z^n(H).$$

It follows that $S/Z^n(H)$ is n-central. Then Lemma 1, implies that

$$[G : Z^n(G)] \leq [H/Z^n(H) : S/Z^n(H)]^{1+log_2 t} = [H : S]^{1+log_2 t}.$$

Then by using (6) we have $[G : Z^n(G)] \leq (tM)^{1+log_2 t}$ and the required assertion follows.

As an immediate consequence, we have the following interesting result.

Corollary 1 *Let G be an n-abeian group. Then the index of the second n-center of G is bounded above by some function of $|G_n/(G_n \cap Z^n(G))|$.*

Proof By consider the factor group $G/Z^n(G)$ and applying Theorem 2, we have

$$[G : Z_2^n(G)] = [G/Z^n(G) : Z^n(G/Z^n(G))] \leq b(|(G/Z^n(G))_n|).$$

Now, since

$$(G/Z^n(G))_n = G_n Z^n(G)/Z^n(G) \cong G_n/(G_n \cap Z^n(G)),$$

the result follows.

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