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Jacobi Wavelets Method for Numerical Solution of Weakly Singular Volterra Integral Equation

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Abstract In this paper, the first and second kind weakly singular Volterra integral equations are approximated by using the Jacobi wavelets method. First, the operational matrixes for fractional integration and product for Jacobi wavelets are computed with a new matrix approach, and then, it applied to solve numerically the first and second kind Volterra integral equations involving singularity. Illustrative numerical experiments with comparison are included to indicate the validity and practicability of the method.

Keywords hybrid functions · Jacobi polynomials · operational matrix · singular Volterra integral equation · wavelets.

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1 Introduction

In this paper the following weakly singular Volterra integral equation of the first or second kind is studied

$$wu(x) = v(x) + \int_0^x (x-t)^{\alpha-1} k(x,t)u(t)dt, \quad 0 < \alpha \leq 1, \quad 0 \leq x \leq L, \quad (1)$$

where $w = 0$ or 1 is constant, $k(x,t)$ is the smooth kernel and the non homogeneous part of the equation $v(x)$ is a smooth function, and $u(x)$ is the unknown function.

For the first time, an integral equation associated with the tautochrone problem was modeled by Abel (1825). Latter, the theory of integral equations was developed by Volterra, Fredholm, Hilbert, and others. But Volterra was the first one who studied the integral equations systematically and proposed the theorems for the existence and uniqueness of the solution [1].

In (1), in case of $w = 0$ and $w = 1$ the first kind and the second kind Volterra integral equations are obtained respectively. The case $w = 0$ is associated with the Abel integral equation as a special case of the first kind of Volterra integral equations. This equation frequently appears in applications such as electro- chemistry, crystal growth, and heat conduction [2].

So far, several methods have been proposed for analytical and numerical solution of integral equations. Some of these methods are Adomian decomposition methods, Taylor expansion methods, Spectral methods, Spline approximation methods, Fixed point methods, and Variational iteration methods (see [3–10] and references there in).

Recently, Usta [11] has applied Bernstein approximation method for Numerical solution of fractional Volterra integral equations. Dehbozorgi et al. [12] used an hp-version collocation approach for the numerical solution of nonlinear weakly singular Volterra integral equations.

In this paper, we use Jacobi wavelets to solve the fractional Volterra integral equation (1). We first introduce a new approach for the operational matrix of product and integration for Jacobi wavelets in matrix format, which reduces the computational coast, and then we apply it to the numerical approximation of fractional Volterra integral equation (1). The remaining of this manuscript is as follows. In Section 2, we stated some basic mathematical preliminaries that we need to establish our method which is followed by recalling the essential definitions from Jacobi polynomials and wavelets. In Section 3, we applied the suggested method to the fractional Volterra integral equation. In Section 4 the convergence and error analysis were presented. The numerical illustrations were presented in Section 5. Finally, in Section 6, the conclusion was derived.

2 Prelimineries

In this part, the definition of fractional calculus, Jacobi polynomials, wavelets, and their attributes were explained. As well as, the essential descriptions from the operational matrix of fractional integral and derivative were defined.

2.1 Fractional calculus

The Riemann-Liouville fractional integral I_x^α of order $0 \leq \alpha < 1$ is presented with [13]

$$I_x^\alpha u(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s) ds, & \alpha > 0, \\ u(x), & \alpha = 0. \end{cases}$$

One of the fundamental attributes of the operator I_x^α is

$$I_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} x^{\beta+\alpha}.$$

The Caputo fractional derivative of order $\alpha > 0$ is determined as [13]

$$D_x^\alpha u(x) = I_x^{n-\alpha} \frac{d^n}{dt^n} u(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where n is an integer $(n-1 < \alpha \leq n)$ and $u^{(n)} \in L^1[0,b]$.
The main relationship between the Riemann-Liouville integral operator and Caputo fractional derivative is as follows

$$I_x^\alpha D_x^\alpha u(x) = u(x) - \sum_{r=0}^{n-1} u^{(r)}(0^+) \frac{x^r}{r!}, \qquad (n-1 < \alpha \leq n),$$

where $u^{(r)} \in L^1[0,b]$, $r = 0, 1, \ldots, n-1$.

2.2 Shifted Jacobi polynomials (SJPs) and their attributes

The shifted Jacobi polynomials on $[0,1]$, can be specified with the following analytic formula [14]

$$P_j^{(\mu,\nu)}(x) = \sum_{k=0}^{j} \gamma_k^j x^k,$$

where

$$\gamma_k^j = \frac{(-1)^{j-k} \Gamma(j+\nu+1) \Gamma(j+k+\mu+\nu+1)}{\Gamma(k+\nu+1) \Gamma(j+\mu+\nu+1)(j-k)! k!}, \quad j = 0, 1, \ldots$$

The SJPs are orthogonal with respect to the weight function $w(x) = (1-x)^\mu x^\nu$, $0 \leq x \leq 1$, that is

$$\int_0^1 P_j^{(\mu,\nu)}(x) P_k^{(\mu,\nu)}(x) w(x) dx = \theta_j \delta_{jk},$$

where δ_{jk} is the well-known Kronecker delta function and

$$\theta_j = \frac{\Gamma(j+\mu+1) \Gamma(j+\nu+1)}{\Gamma(2j+\mu+\nu+1) j! \Gamma(j+\mu+\nu+1)}, \quad j = 0, 1, \ldots$$

The SJPs on $[0,1]$ form an orthogonal base for $L^2[0,1]$, however are not normal. To simplify the next calculations, they are normalized as the following formula

$$\overline{P}_j^{(\mu,\nu)}(x) = \frac{1}{\sqrt{\theta}_j} P_j^{(\mu,\nu)}(x) = \frac{1}{\sqrt{\theta}_j} \sum_{k=0}^{j} \gamma_k^j x^k.$$

The normalized SJPs (NSJPs) on $[0,1]$ form an orthonormal base for $L^2[0,1]$. A function $u(x)$, square integrable on $[0,1]$ can be explained in phrases of NSJPs as follows

$$u(x) = \sum_{j=0}^{\infty} c_j \overline{P}_j^{(\mu,\nu)}(x),$$

where

$$c_j = \int_0^1 u(x)\overline{P}_j^{(\mu,\nu)}(x) w(x) dx, \qquad j = 0, 1, \ldots.$$

Generally in perfect, only the first $N+1$ terms of NSJPs are used, thus $u(x)$ is approximated by $u_N(x)$ as follows

$$u(x) \simeq u_N(x) = \sum_{j=0}^{N} c_j \overline{P}_j^{(\mu,\nu)}(x) = \Psi^T(x) C = C^T \Psi(x), \tag{2}$$

where the vectors C and $\Psi(x)$ are defined by

$$C = [c_0, c_1, \ldots, c_N]^\top, \qquad \Psi(x) = [\overline{P}_0^{(\mu,\nu)}(x), \overline{P}_1^{(\mu,\nu)}(x), \ldots, \overline{P}_N^{(\mu,\nu)}(x)]^\top.$$

Several efficient lemmas and theorems which were offered by the novelist in [14], are described as follows.

Lemma 1 *([15]) If $p \geq 0$, then*

$$\int_0^1 x^p \overline{P}_n^{(\mu,\nu)}(x) w(x) dx =$$
$$\frac{1}{\sqrt{\theta}_n} \sum_{l=0}^{n} \frac{(-1)^{n-1} \Gamma(n+\nu+1)\Gamma(n+l+\mu+\nu+1)\Gamma(p+l+\nu+1)\Gamma(\mu+1)}{\Gamma(l+\nu+1)\Gamma(n+\mu+\nu+1)\Gamma(p+l+\mu+\nu+2)(n-l)!l!}.$$

Lemma 2 *([15]) If $\overline{P}_i^{(\mu,\nu)}(x)$ and $\overline{P}_s^{(\mu,\nu)}(x)$ are severally ith and sth normalized SJPs, then the product of $\overline{P}_i^{(\mu,\nu)}(x)$ and $\overline{P}_s^{(\mu,\nu)}(x)$ can be defined as*

$$\overline{P}_i^{(\mu,\nu)}(x) \overline{P}_s^{(\mu,\nu)}(x) = \frac{1}{\sqrt{\theta_i \theta_s}} \sum_{r=0}^{i+s} \rho_r^{(i,s)} x^r,$$

where factors $\rho_r^{(i,s)}$ are expressed as follows

Algorithm 1 product of $\overline{P}_i^{(\mu,\nu)}(x)$ and $\overline{P}_s^{(\mu,\nu)}(x)$

if $i \geq s$ **then**
  for $r = 0, 1, \ldots, i+s$ **do**
    if $r > i$ **then**
      $\rho_r^{(i,s)} = \sum_{l=r-i}^{s} \gamma_{r-l}^{(i)} \gamma_l^{(s)}$,
    else
      $r_1 = \min\{r, s\}$,
      $\rho_r^{(i,s)} = \sum_{l=0}^{r_1} \gamma_{r-l}^{(i)} \gamma_l^{(s)}$,
    end if
    if $i < s$: **then**
      for $r = 0, 1, 2, \ldots, i+s$ **do**
        if $r \leq i$ **then**
          $r_1 = \min\{r, s\}$,
          $\rho_r^{(i,s)} = \sum_{l=0}^{r_1} \gamma_{r-l}^{(i)} \gamma_l^{(s)}$,
        else
          $r_2 = \min\{r, s\}$,
          $\rho_r^{(i,s)} = \sum_{l=r-i}^{r_2} \gamma_{r-l}^{(i)} \gamma_l^{(s)}$,
        end if
      end for
    end if
  end for
end if

Lemma 3 *([15]) If $\overline{P}_i^{(\mu,\nu)}(x)$, $\overline{P}_j^{(\mu,\nu)}(x)$, and $\overline{P}_k^{(\mu,\nu)}(x)$ are severally i, j, and k-th normalized SJPs, then*

$$\begin{aligned} q_{ijk} &= \int_0^1 \overline{P}_i^{(\mu,\nu)}(x)\overline{P}_j^{(\mu,\nu)}(x)\overline{P}_k^{(\mu,\nu)}(x)w(x)dx \\ &= \frac{1}{\sqrt{\theta_i\theta_j\theta_k}} \sum_{r=0}^{i}\sum_{s=0}^{j}\sum_{t=0}^{k} \gamma_r^i\gamma_s^j\gamma_t^k \int_0^1 x^r x^s x^t (1-x)^\mu x^\nu dx \\ &= \frac{1}{\sqrt{\theta_i\theta_j\theta_k}} \sum_{r=0}^{i}\sum_{s=0}^{j}\sum_{t=0}^{k} \gamma_r^i\gamma_s^j\gamma_t^k \frac{\Gamma(r+s+t+\nu+1)\Gamma(\mu+1)}{\Gamma(r+s+t+\mu+\nu+2)}. \end{aligned} \tag{3}$$

In practical calculation on the Jacobi bases, we mostly experienced the product of $\Psi(x)$ and $\Psi^T(x)$, called the operational matrix of product. A common law was expressed to find the $(N+1)\times(N+1)$ operational matrix of product $\widetilde{V}$ as

$$\Psi(x)\Psi^\top(x)V \simeq \widetilde{V}\Psi(x). \tag{4}$$

The next theorem describes the elements of the matrix $\widetilde{V}$.

Theorem 1 *([15]) The elements of matrix $\widetilde{V}$ in equation (4) are calculated as*

$$\widetilde{V}_{jk} = \sum_{i=0}^{N} V_i q_{ijk}, \quad j,k = 0,1,\ldots,N.$$

where q_{ijk} *is obtained in equation* (3) *and* V_i*,* $i = 0, 1, \ldots, N$ *are the elements of vector* V *in equation* (4)*.*

A general form of operational matrix of fractional integration for NSJPs is as follows

$$I_x^{\alpha}\Psi(x) = \mathbf{P}^{\alpha}\Psi(x), \tag{5}$$

where $\mathbf{P}^{\alpha}$ is the operational matrix of fractional integral with dimension $(N+1)\times(N+1)$ on [0, 1]. The elements of matrix $\mathbf{P}^{\alpha}$ are evaluated as follows.

Theorem 2 *([15]) The elements of matrix* $\mathbf{P}^{\alpha}$ *in equation* (5) *are calculated as*

$$\begin{aligned}\mathbf{P}^{\alpha} &= \left\langle I_x^{\alpha}\overline{P}_n(x), \overline{P}_j(x)\right\rangle_w \\ &= \frac{1}{\sqrt{\theta_n\theta_j}}\sum_{k=0}^{n}\gamma_k^n\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\sum_{s=0}^{j}\gamma_s^j\int_0^1 x^{k+\alpha}x^s(1-x)^{\mu}x^{\nu}dx \\ &= \frac{1}{\sqrt{\theta_n\theta_j}}\sum_{k=0}^{n}\sum_{s=0}^{j}\gamma_k^n\gamma_s^j\frac{\Gamma(k+1)\Gamma(k+\alpha+s+\nu+1)\Gamma(\mu+1)}{\Gamma(k+\alpha+1)\Gamma(k+\alpha+s+\mu+\nu+2)},\end{aligned} \tag{6}$$

where

$$I_x^{\alpha}\overline{P}_n(x) = \frac{1}{\sqrt{\theta_n}}\sum_{k=0}^{n}\gamma_k^n I_x^{\alpha}x^k = \frac{1}{\sqrt{\theta_n}}\sum_{k=0}^{n}\gamma_k^n\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}x^{k+\alpha}.$$

2.3 Jacobi wavelets

Wavelets are a family of functions created from dilation and translation of a single function $\Phi(t)$ called the mother wavelet. Whereas the dilation argument a and the translation argument b vary continuously, we have the following family of continuous wavelets [16].

$$\Phi_{a,b}(t) = |a|^{-\frac{1}{2}}\Phi\left(\frac{t-b}{a}\right), \quad a \neq 0.$$

If we confine the arguments a and b to separate deal as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, and $b_0 > 0$, where n and k are positive integers, then we have the following family of separate wavelets

$$\Phi_{k,n}(t) = |a_0|^{\frac{k}{2}}\Phi(a_0^k t - nb_0), \quad \forall\, a, b \in \mathbb{R}, \quad a \neq 0,$$

where $\Phi_{k,n}(t)$ make a wavelet base for $L^2(\mathbb{R})$. In special case, when $a_0 = 2$ and $b_0 = 1$, thus $\Phi_{k,n}(t)$ makes an orthonormal base [17].

Here, we employ NSJPs to create the Jacobi wavelets JWs. The JWs $\varphi_{nm}(t) = \varphi(k, n, m, t)$ have four parameters as $k \in \mathbb{N}$, $n = 1, \ldots, N$, m is the order of Jacobi polynomials, t is the time and $N = 2^{k-1}$. They are defined on $[0, L]$ by

$$\varphi_{nm}(t) = \begin{cases} \sqrt{\frac{N}{L}}\overline{P}_m^{(\mu,\nu)}(\frac{N}{L}t - n + 1), & \frac{(n-1)L}{N} \le t < \frac{nL}{N}, \\ 0, & \text{otherwise}, \end{cases}$$

where $m = 0, \dots, M-1$ and $n = 1, \dots, N$. The dilation argument is $a = \frac{L}{N}$ and the translation argument is $b = \frac{nL}{N}$. The JWs construct an orthonormal base for $L^2_{w_n}[0, L]$, i.e., [18],

$$\langle \varphi_{nm}(t), \varphi_{n'm'}(t) \rangle_{w_n(t)} = \int_0^L \varphi_{nm}(t)\varphi_{n'm'}(t) w_n(t) dt$$

$$= \begin{cases} 1, & (n, m) = (n', m'), \\ 0, & (n, m) \ne (n', m'), \end{cases}$$

where

$$w_n(t) = \begin{cases} w(\frac{N}{L}t - n + 1), & \frac{(n-1)L}{N} \le t < \frac{nL}{N} \\ 0, & \text{otherwise} \end{cases}$$

Any function $y(t) \in L^2_{w_n}[0, L]$ can be expanded by means of JWs as follows

$$y(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_{nm} \varphi_{nm}(t), \tag{7}$$

where $a_{nm} = \langle y(t), \varphi_{nm}(t) \rangle_{w_n}$ and $\langle \cdot, \cdot \rangle$ indicates the inner product in $L^2_{\omega_n}[0, L]$ with respect to the weight function $w_n(x)$. By shorten the infinite series in (7), we gain an estimate phrase of $y(t)$ as follows [18]

$$y(t) \approx y_{NM}(t) = \sum_{n=1}^{N} \sum_{m=0}^{M-1} a_{nm} \varphi_{nm}(t) = A^\top \Phi(t), \tag{8}$$

where A and $\Phi(t)$ are column vectors with NM members as follows

$$A = [a_{10}, \dots, a_{1(M-1)}, a_{20}, \dots, a_{2(M-1)}, \dots, a_{N0}, \dots, a_{N(M-1)}]^\top,$$

$$\Phi(t) = [\varphi_{10}, \dots, \varphi_{1(M-1)}, \varphi_{20}, \dots, \varphi_{2(M-1)}, \dots, \varphi_{N0}, \dots, \varphi_{N(M-1)}]^\top.$$

3 Implementation of the method

The operational matrix of fractional integration of order α for JWs is defined as follows

$$I_t^\alpha \Phi(t) \simeq \mathbf{P}_L^\alpha \Phi(t), \tag{9}$$

where $\mathbf{P}_L^\alpha$ is called the operational matrix of fractional integral of order α for JWs with dimension $MN \times MN$ on the interval $[0, L]$. In the following theorem, we construct the structure and entries of the operational matrix of integration $\mathbf{P}_L^\alpha$.

Theorem 3 *The matrix* $\mathbf{P}_L^\alpha$ *in* (9) *has the following structure*

$$\mathbf{P}_L^\alpha = \left(\frac{L}{N}\right)^\alpha \begin{bmatrix} \mathbf{P}^\alpha & \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{H}_4 & \dots & \mathbf{H}_{N-1} \\ & \mathbf{P}^\alpha & \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \dots & \mathbf{H}_{N-2} \\ & & \mathbf{P}^\alpha & \mathbf{H}_1 & \mathbf{H}_2 & \dots & \mathbf{H}_{N-3} \\ & & & \mathbf{P}^\alpha & \mathbf{H}_1 & \ddots & \vdots \\ & & & & \ddots & \ddots & \mathbf{H}_2 \\ & & & & & \ddots & \mathbf{H}_1 \\ & & & & & & \mathbf{P}^\alpha \end{bmatrix}, \tag{10}$$

where $\mathbf{P}^\alpha$ *is the* $M \times M$ *operational matrix of fractional integration of order* α *for NSJPs on* $[0,1]$ *which was defined in* (5), *and* $\mathbf{H}_r, r = 1, 2, \dots, N-1$ *is an* $M \times M$ *matrix whose elements will be defined in* (19).

Proof First we calculate $I_t^\alpha \varphi_{nm}(t)$ in three different cases.

Case 1: $t < \frac{(n-1)L}{N}$. Since $\varphi_{nm}(t) = 0$ for $t < \frac{(n-1)L}{N}$ then $I_t^\alpha \varphi_{nm}(t) = 0$.

Case 2: $\frac{(n-1)L}{N} < t < \frac{nL}{N}$. In this case

$$\begin{aligned} I_t^\alpha \varphi_{nm}(t) &= \tfrac{1}{\Gamma(\alpha)} \textstyle\int_{\frac{(n-1)L}{N}}^t (t-\tau)^{\alpha-1} \varphi_{nm}(\tau) d\tau \\ &= \sqrt{\tfrac{N}{L}} \tfrac{1}{\Gamma(\alpha)} \textstyle\int_{\frac{(n-1)L}{N}}^t (t-\tau)^{\alpha-1} \overline{P}_m(\tfrac{N}{L}\tau - n + 1) d\tau. \end{aligned}$$

Let $s = \frac{N}{L}\tau - n + 1$ then

$$\begin{aligned} I_t^\alpha \varphi_{nm}(t) &= \sqrt{\tfrac{N}{L}} \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^{\frac{N}{L}t-n+1} (t - \tfrac{L}{N}(s+n-1))^{\alpha-1} \bar{P}_m(s) \tfrac{L}{N} ds \\ &= \sqrt{\tfrac{N}{L}} \left(\tfrac{L}{N}\right)^\alpha \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^X (X-s)^{\alpha-1} \bar{P}_m(s) ds \\ &= \sqrt{\tfrac{N}{L}} \left(\tfrac{L}{N}\right)^\alpha I_X^\alpha \bar{P}_m(X), \end{aligned}$$

where $X = \frac{N}{L}t - n + 1$ and $0 \le X \le 1$.

Case 3: Let $t > \frac{nL}{N}$. For this case we write

$$I_t^\alpha \varphi_{nm}(t) = \sqrt{\frac{N}{L}} \frac{1}{\Gamma(\alpha)} \int_{\frac{(n-1)L}{N}}^{\frac{nL}{N}} (t-\tau)^{\alpha-1} \bar{P}_m(\frac{N}{L}\tau - n + 1) d\tau.$$

Let $s = \frac{N}{L}\tau - n + 1$ then $\tau = \frac{L}{N}(s + n - 1)$, and we have

$$\begin{aligned} I_t^\alpha \varphi_{nm}(t) &= \sqrt{\tfrac{N}{L}} \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^1 (t - \tfrac{L}{N}(s+n-1))^{\alpha-1} \bar{P}_m(s) \tfrac{L}{N} ds \\ &= \sqrt{\tfrac{N}{L}} \tfrac{1}{\Gamma(\alpha)} \left(\tfrac{L}{N}\right)^\alpha \textstyle\int_0^1 (X-s)^{\alpha-1} \bar{P}_m(s) ds \\ &= \sqrt{\tfrac{N}{L}} \left(\tfrac{L}{N}\right)^\alpha \overline{I}_m^\alpha(X), \end{aligned}$$

where $X = \frac{N}{L}t - n + 1$ and $\overline{I}_m^\alpha(X) = \frac{1}{\Gamma(\alpha)} \int_0^1 (X-s)^{\alpha-1} \bar{P}_m(s) ds$.

Thus we summarize the results as follows

$$I_t^\alpha \varphi_{nm}(t) = \begin{cases} 0, & t < \frac{(n-1)L}{N}, \\ \sqrt{\frac{N}{L}}\left(\frac{L}{N}\right)^\alpha I_X^\alpha \bar{P}_m(X), & \frac{(n-1)L}{N} < t < \frac{nL}{N}, \\ \sqrt{\frac{N}{L}}\left(\frac{L}{N}\right)^\alpha \overline{I}_m^\alpha(X), & t > \frac{nL}{N}. \end{cases} \tag{11}$$

Now, we calculate the entries of $(\mathbf{P}_L^\alpha)_{ij}$, $i, j = 1, 2, \dots, NM$. Let i and j have the following integer decomposition

$$i = (n-1)M + m + 1, \quad j = (n'-1)M + m' + 1.$$

Again, there are three possible scenarios

Case 1: Let $n' < n$ then

$$(\mathbf{P}_L^\alpha)_{ij} = \langle I_t^\alpha \varphi_{nm}, \varphi_{n'm'}\rangle_{w_{n'}} = \int_{\frac{(n'-1)L}{N}}^{\frac{n'L}{N}} I_t^\alpha \varphi_{nm}(t)\varphi_{n'm'}(t) w_{n'}(t)dt. \tag{12}$$

By (11) we have $I_t^\alpha \varphi_{nm}(t) = 0$ for $t < \frac{(n-1)L}{N}$, then $(\mathbf{P}_L^\alpha)_{ij} = 0$.

Case 2: Let $n' = n$

$$\begin{aligned}(\mathbf{P}_L^\alpha)_{ij} &= \langle I_t^\alpha \varphi_{nm}, \varphi_{nm'}\rangle_{w_n} = \int_{\frac{(n-1)L}{N}}^{\frac{nL}{N}} I_t^\alpha \varphi_{nm}(t)\varphi_{nm'}(t) w_n(t)dt \\ &= \frac{N}{L}\left(\frac{L}{N}\right)^\alpha \int_{\frac{(n-1)L}{N}}^{\frac{nL}{N}} I_X^\alpha \overline{P}_m(X)\overline{P}_{m'}(\tfrac{N}{L}t - n + 1) w(\tfrac{N}{L}t - n + 1)dt,\end{aligned}$$

we put $X = \frac{N}{L}t - n + 1$ and with (11) we have

$$(\mathbf{P}_L^\alpha)_{ij} = \left(\frac{L}{N}\right)^\alpha \int_0^1 \overline{I}_X^\alpha \overline{P}_m(X)\overline{P}_{m'}(X)w(X)dX = \left(\frac{L}{N}\right)^\alpha \mathbf{P}_{mm'}^\alpha, \tag{13}$$

where $\mathbf{P}^\alpha$ is the operational matrix of fractional integral for normalized SJPs on $[0, 1]$ defined in (5).

Case 3: Let $n' > n$ then

$$\begin{aligned}(\mathbf{P}_L^\alpha)_{ij} &= \langle I_t^\alpha \varphi_{nm}, \varphi_{n'm'}\rangle_{w_{n'}} \\ &= \frac{N}{L}\left(\frac{L}{N}\right)^\alpha \int_{\frac{(n'-1)L}{N}}^{\frac{n'L}{N}} \overline{I}_m^\alpha(X)\overline{P}_{m'}(\tfrac{N}{L}t - n' + 1) w(\tfrac{N}{L}t - n' + 1)dt,\end{aligned}$$

where $X = \frac{N}{L}t - n + 1$. With $s = \frac{N}{L}t - n' + 1$ and (11) we have

$$\langle I_t^\alpha \varphi_{nm}, \varphi_{n'm'}\rangle_{w_{n'}} = \left(\frac{L}{N}\right)^\alpha \int_0^1 I_m^\alpha(s + n' - n)\overline{P}_{m'}(s)w(s)ds. \tag{14}$$

With (12), (13), and (14) we conclude that

$$(\mathbf{P}_L^\alpha)_{ij} = \begin{cases} 0, & n' < n, \\ \left(\frac{L}{N}\right)^\alpha \mathbf{P}_{mm'}^\alpha, & n' = n, \\ \left(\frac{L}{N}\right)^\alpha \int_0^1 \overline{I}_m^\alpha(s+n'-n)\overline{P}_{m'}(s)w(s)ds, & n' > n. \end{cases} \quad (15)$$

With (15) we conclude that the operational matrix of fractional integration for JWs has the structure defined in (16).

To define the matrixes $\mathbf{H}_r$, $r = 1, 2, \ldots, N-1$ we have

$$(\mathbf{H}_r)_{mm'} = \int_0^1 \overline{I}_m^\alpha(s+r)\overline{P}_{m'}(s)w(s)ds, \quad r = 1, \ldots, N-1. \quad (16)$$

Now, we calculate the integral in (16). To this we start with the following integral.

$$\int_0^1 (x-s)^{\alpha-1}s^k ds,$$

taking $x - s = t$ we have

$$\int_0^1 (t-s)^{\alpha-1}s^k ds = \int_{x-1}^x t^{\alpha-1}(x-t)^k dt,$$

after some simplifications we get

$$\int_0^1 (t-s)^{\alpha-1}s^k ds = \sum_{i=0}^k \frac{(-1)^i k!}{i!(k-i)!(\alpha+i)} x^{k-i}\left(x^{\alpha+i} - (x-1)^{\alpha+i}\right). \quad (17)$$

Now, we calculate $\overline{I}_m^\alpha(x)$ as follows

$$\overline{I}_m^\alpha(x) = \frac{1}{\Gamma(\alpha)}\int_0^1 (x-s)^{\alpha-1}\overline{P}_m(s)ds = \frac{1}{\Gamma(\alpha)\sqrt{\theta_m}}\sum_{k=0}^m \gamma_k^m \int_0^1 (x-s)^{\alpha-1}s^k ds,$$

with (17) we get

$$\overline{I}_m^\alpha(x) = \frac{1}{\Gamma(\alpha)\sqrt{\theta_m}}\sum_{k=0}^m \gamma_k^m \sum_{i=0}^k \frac{(-1)^i k!}{i!(k-i)!(\alpha+i)}\left[x^{\alpha+k} - x^{k-i}(x-1)^{\alpha+i}\right]. \quad (18)$$

Substituting (18) in (16) yilds

$$(\mathbf{H}_r)_{mm'} = \frac{1}{\Gamma(\alpha)\sqrt{\theta_m}\sqrt{\theta_{m'}}}\sum_{k=0}^m \gamma_k^m \sum_{i=0}^k \frac{(-1)^i k!}{i!(k-i)!(\alpha+i)}\sum_{j=0}^{m'}\gamma_j^{m'} I_{ijkr}, \quad (19)$$

where

$$I_{ijkr} = \int_0^1 \left[(x+r)^{\alpha+k} - (x+r)^{k-i}(x+r-1)^{\alpha+i}\right] x^j(1-x)^\mu x^\nu dx. \quad (20)$$

I_{ijkr} in (20) can be easily evaluated for given values for parameters i, j, k, r, α, μ, and ν. This completes the proof. □

For instance, with $T = 1$, $M = 4$, $k = 3$, $\alpha = \frac{1}{2}$, and $\mu = \nu = \frac{1}{2}$ the entries of matrices $\mathbf{H}_r, r = 1, 2, 3$ and $\mathbf{P}^\alpha$ are as follows

$$\mathbf{P}^{\frac{1}{2}} = \begin{bmatrix} 0.766 & 0.218 & -0.036 & 0.013 \\ -0.364 & 0.340 & 0.179 & -0.037 \\ 0.119 & -0.192 & 0.249 & 0.153 \\ -0.123 & 0.010 & -0.151 & 0.206 \end{bmatrix}, \quad \mathbf{H}_1 = \begin{bmatrix} 0.610 & -0.102 & 0.031 & -0.012 \\ 0.131 & -0.069 & 0.034 & -0.018 \\ 0.242 & -0.068 & 0.034 & -0.019 \\ 0.067 & -0.045 & 0.028 & -0.018 \end{bmatrix},$$

$$\mathbf{H}_2 = \begin{bmatrix} 0.404 & -0.026 & 0.002 & -0.000 \\ 0.035 & -0.006 & 0.001 & -0.000 \\ 0.138 & -0.010 & 0.001 & -0.000 \\ 0.014 & -0.002 & 0.000 & -0.000 \end{bmatrix}, \quad \mathbf{H}_3 = \begin{bmatrix} 0.327 & -0.013 & 0.000 & -0.000 \\ 0.018 & -0.002 & 0.000 & -0.000 \\ 0.110 & -0.004 & 0.000 & -0.000 \\ 0.007 & -0.000 & 0.000 & -0.000 \end{bmatrix}.$$

The following theorem determines the elements of $\widetilde{\mathbf{A}}$, the operational matrix of product for JWs defined as follows

$$\Phi(x)\Phi^\top(x)A \simeq \widetilde{\mathbf{A}}\Phi(x), \tag{21}$$

where $\widetilde{\mathbf{A}}$ is the $NM \times NM$ matrix, whose elements are a combination of the entries of vector A.

Theorem 4 *The matrix* $\widetilde{\mathbf{A}}$ *in* (21) *has the following format*

$$\widetilde{\boldsymbol{A}} = \sqrt{\frac{N}{L}} \begin{bmatrix} \widetilde{V}_1 & & & \\ & \widetilde{V}_2 & & \\ & & \ddots & \\ & & & \widetilde{V}_N \end{bmatrix} \tag{22}$$

where $\widetilde{V}_k$ ($k = 1, 2, \ldots, N$) is the $M \times M$ product operational matrix for NSJPs defined in (4)*, whose entries are the combination of $a_{k0}, a_{k1}, \ldots, a_{k(M-1)}$.*

Proof To construct the entries $\widetilde{\mathbf{A}}_{ij}$, let

$$i = (n-1)M + m + 1, \qquad j = (n'-1)M + m' + 1,$$

then

$$\begin{aligned} \widetilde{\mathbf{A}}_{ij} &= \langle \varphi_{nm}(t) \sum_{r=0}^{M-1} a_{nr}\varphi_{nr}(t), \varphi_{ns}(t) \rangle_{w_n} \\ &= \sum_{r=0}^{M-1} a_{nr} \int_{\frac{(n-1)L}{N}}^{\frac{nL}{N}} \varphi_{nm}(t)\varphi_{nr}(t)\varphi_{ns}(t)w_n(t)dt \\ &= \sqrt{\tfrac{N}{L}} \sum_{r=0}^{M-1} a_{nr} \int_0^1 \overline{P}_m(X)\overline{P}_r(X)\overline{P}_s(X)w(X)dX \\ &= \sqrt{\tfrac{N}{L}} \sum_{r=0}^{M-1} a_{nr} q_{mrs}, \end{aligned}$$

where $X = \frac{N}{L}t - n + 1$ and q_{mrs} is defined in (3). Then, we conclude that the product operational matrix $\widetilde{\mathbf{A}}$ for JWs can be rewritten to the matrix format (22). □

In this section, we use JWs to solve the fractional Volterra integral equation defined in (1) where $u(t)$ is the unknown function to be found and $0 < \alpha \leq 1$ is a positive real numeral. To solve (1) we commence with

$$u(x) = \Phi^\top(x)A, \quad k(x,t) = \Phi^\top(x)\mathbf{K}\Phi(t), \tag{23}$$

where A is the vector of unknown coefficients, $\mathbf{K}$ is the known matrix and $\Phi(t)$ is JWs vector. Substituting (23) in (1) yields

$$\begin{aligned} wA^\top\Phi(x) &= v(x) + \Phi^\top(x)\mathbf{K}\int_0^x (x-t)^{\alpha-1}\Phi(t)\Phi^\top(t)Adt \\ &= v(x) + \Phi^\top(x)\mathbf{K}\int_0^x (x-t)^{\alpha-1}\widetilde{\mathbf{A}}\Phi(t)dt \\ &= v(x) + \Gamma(\alpha)\Phi^\top(x)\mathbf{K}\widetilde{\mathbf{A}}\mathbf{P}_L^\alpha\Phi(x), \end{aligned} \tag{24}$$

where $\widetilde{\mathbf{A}}$ is the operational matrix of product for vector A defined in (22) and $\mathbf{P}_L^\alpha$ is the operational matrix of fractional integration of order α defined in (10). Collocating (24) at the collocation points defined as follows

$$x_i = \frac{(2i-1)L}{2NM}, \qquad i = 1, 2, \ldots, NM,$$

where $\mathbf{L}$ is the end of domain and defined in (1), we will have

$$wA^\top\Phi(x_i) = v(x_i) + \Gamma(\alpha)\Phi^\top(x_i)\mathbf{K}\widetilde{\mathbf{A}}\mathbf{P}_L^\alpha\Phi(x_i), \quad i = 1, 2, \ldots, NM. \tag{25}$$

Then, by rearranging (25) we get the following linear system of equations

$$\mathbf{G}A = V,$$

where A is the unknown vector, $V = \big[v(x_1), v(x_2), \ldots, v(x_{NM})\big]^\top$ is the right hand side vector and $\mathbf{G}$ is the coefficient matrix.

4 Convergence analysis

In this section, we provide some theorems for the convergence of the approximate solution. First, we prove the following theorem.

Theorem 5 *If a continuous function $y(t) \in L^2[0, L]$ be bounded, then the JWs expansion* (7) *converges to $y(t)$.*

Proof Let $y(t)$ be a bounded real valued function on $[0, L]$. The JWs coefficients of continuous functions $y(t)$ in (7) has the following upper bound.

$$|a_{nm}| = |\langle y(t), \varphi_{nm}(t)\rangle_{w_n}| = \left| \int_{\frac{(n-1)L}{N}}^{\frac{nL}{N}} y(t)\varphi_{nm}(t)w_n(t)dt \right|$$

$$\leq \sqrt{\frac{N}{L}} \int_{\frac{(n-1)L}{N}}^{\frac{nL}{N}} |y(t)| \left| \overline{P}_m^{(\mu,\nu)}(\frac{N}{L}t - n + 1) \right| w(\frac{N}{L}t - n + 1)dt,$$

$$= \sqrt{\frac{L}{N}} \int_0^1 |\overline{P}_m^{(\mu,\nu)}(s)| \left| y(\frac{L}{N}(s + n - 1)) \right| ds,$$

where $s = \frac{N}{L}t - n + 1$. Using generalized mean value theorem for integrals, we have

$$|a_{nm}| \leq \sqrt{\frac{L}{N}} \left| y(\frac{L}{N}(z + n - 1)) \right| \int_0^1 |\bar{P}_m^{(\mu,\nu)}(s)| ds, \quad z \in [0, 1],$$

considering $\int_0^1 |\bar{P}_m^{(\mu,\nu)}(s)ds| \leq \gamma < \infty$, for $m = 0, 1, \ldots, M - 1$, we have

$$|a_{nm}| \leq \sqrt{\frac{L}{N}} \left| y(\frac{L}{N}(z + n - 1)) \right| \gamma \leq \sqrt{\frac{L}{N}} M_y \gamma = \frac{\sqrt{L} M_y \gamma}{2^{\frac{k-1}{2}}},$$

where $N = 2^{k-1}$ and $M_y = \max\limits_{0 \leq t \leq L} |y(t)|$.

Since $y(t)$ is bounded, therefore $\sum\limits_{n,m=0}^{\infty} |a_{nm}|$ is absolutely convergent, then the series $\sum\limits_{n,m=0}^{\infty} a_{nm}$ is convergent. Hence the JWs expansion of $y(t)$ converges uniformly. □

The following theorem is quoted directly from [19].

Theorem 6 *Let the sequence* $\{q_n(x)\}_{n=0}^{\infty}$ *is a system of orthogonal polynomials on* $[a, b]$ *with the weight function* $w(x)$*. Let* $p_n(x)$ *denotes the least square approximation of order* n *for* $y(x)$*. Then if* $y^{(n+1)}$ *is continuous on* $[a, b]$ *there exists a number* ξ *in* (a, b) *such that*

$$\|y(x) - p_n(x)\| = \frac{1}{\mu_{n+1}} \|q_{n+1}(x)\| \frac{|y^{(n+1)}(\xi)|}{(n + 1)!}$$

where μ_{n+1} *is the leading coefficient of* $q_{n+1}(x)$*.*

Let $u_{M-1}(x)$ is the approximation of $u(x)$ by SJPs (2) of order $M - 1$ then we have

$$\|u(x) - u_{M-1}(x)\| = \frac{1}{\gamma_M^M \sqrt{\theta_M}} \frac{|u^{(M)}(\xi)|}{M!} \leq \frac{\overline{U}_M}{\gamma_M^M M! \sqrt{\theta_M}},$$

where $\overline{U}_M = \max_{0\le x\le 1} |u^{(M)}(x)|$.
For the approximation of $u(x)$ by JWs on the interval $[0, T]$, defined in (8) a simple calculation shows that

$$\|u(x) - u_{NM}(x)\| \le \left(\frac{T}{N}\right)^M \frac{\overline{U}_M}{\gamma_M^M M! \sqrt{\theta_M}}.$$

To find an error estimation for the proposed method if we substitute $u_{NM}(x)$ to (1) we get

$$wu_{NM}(x) = v(x) + v_{NM}(x) + \int_0^x (x-t)^{\alpha-1} k(x,t) u_{NM}(t)dt, \qquad (26)$$

where $v_{NM}(x)$ is the perturbation factor. Subtracting (26) and (1) yields

$$we_{NM}(x) = -v_{NM}(x) + \int_0^x (x-t)^{\alpha-1} k(x,t) e_{NM}(t)dt, \qquad (27)$$

where $e_{NM}(x) = u(x) - u_{NM}(x)$ is the error function. Eq. (27) is a new equation of type (1). Now, we can proceed in the way similar to Section 3 to obtain the estimation $\overline{e}_{NM}(x)$ to the error function $e_{NM}(x)$.
Now, $u_{NM}(x) + \overline{e}_{NM}(x)$ is a better estimation for $u(x)$.

5 Numerical experiments

In this section, we present some numerical experiments to illustrate the applicability and accuracy of the proposed method. The following notations are provided to compare the gained results with those of the other methods provided in the literature.

$$\|e_{NM}\|_\infty = \max\{e_{NM}(x_i),\ i = 1, 2, \ldots, NM\},$$

and

$$\|e_{NM}\|_2 = \left(\int_0^T (u(x) - u_{NM}(x))^2 dx\right)^{1/2},$$

where $u(x)$ and $u_{NM}(x)$ are exact and approximate solutions of the test problems respectively and x_i $(i = 1, \ldots, NM)$, are the uniform grids on $[0, T]$.

Example 1 For the first example, we consider the following equation in [11]

$$u(x) = 2\sqrt{\frac{x}{\pi}} - \frac{1}{\sqrt{\pi}} \int_0^x (x-t)^{-1/2} u(t)dt, \quad x \in [0, 1]. \qquad (28)$$

Comparing (28) with (1) we have

$$w = 1, \quad \alpha = 1/2, \quad v(x) = 2\sqrt{\frac{x}{\pi}}, \quad k(x,t) = -\frac{1}{\sqrt{\pi}}.$$

Under these assumptions the exact solution in $u(x) = 1 - e^x \text{erfc}(\sqrt{x})$, where $\text{erfc}(x)$ is the complementary error function defined by

$$\text{erfc}(x) = 1 - \int_0^x e^{-t^2} dt.$$

Eq. (28) was solved by Bernstein polynomials approximation method in [11]. The numerical results of this example are compared with the results of [11] and [20] in Table 1. Table 1 shows that the proposed method is comparable with [11] and better than [20]. The graph of exact and numerical solutions of Example 1 for $M = 3$ and $N = 8$ can be seen in Fig. 1.

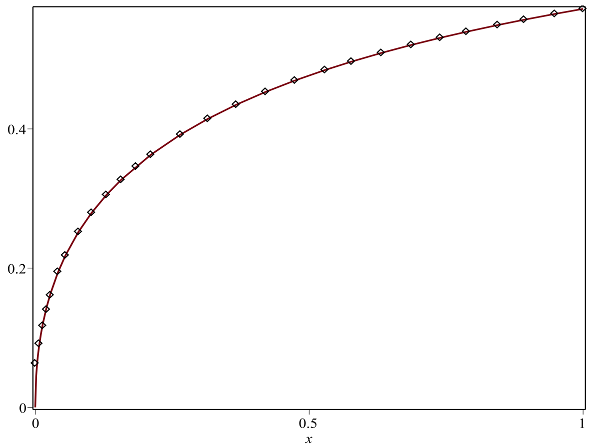

Fig. 1 The graph of exact (solid) and approximate solution (point) for Example 1.

Table 1 Numerical Results for Example 1.

M=3	Proposed Method			[11]	[20]
N	$\|e_{NM}\|_2$	$\|e_{NM}\|_\infty$	n	$\|e_n\|_2$	$\|e_n\|_2$
8	2.9863e-3	5.6771e-3	12	1.5430e-3	1.0074e-2
16	1.5103e-3	3.9431e-3	18	7.7258e-4	6.9999e-3
32	7.6019e-4	2.7430e-3	24	4.57783-4	2.3811e-3
64	3.8161e-4	1.9135e-3			

Example 2 Suppose the following first kind Abel Volterra integral equation [11]

$$e^{-x}(x^5 + x^7 + x^9) = \frac{e^{-x}}{\sqrt{\pi}} \int_0^x (x-t)^{-1/2} u(t) dt \quad x \in [0, 1]. \tag{29}$$

Here,

$$w = 0, \quad \alpha = 1/2, \quad v(x) = -e^{-x}(x^5 + x^7 + x^9), \quad k(x, t) = \frac{e^{-x}}{\sqrt{\pi}}.$$

Under these assumptions the exact solution in

$$u(x) = \frac{5!}{\Gamma(\frac{11}{2})}x^{9/2} + \frac{7!}{\Gamma(\frac{15}{2})}x^{13/2} + \frac{9!}{\Gamma(\frac{19}{2})}x^{17/2}.$$

The numerical results of this example are compared with two other methods in Table 2. The results shows that the new method is better than [20]. In Fig. 2 the graph of exact and numerical solution of Example 2 for $M = 3$ and $N = 8$ is plotted.

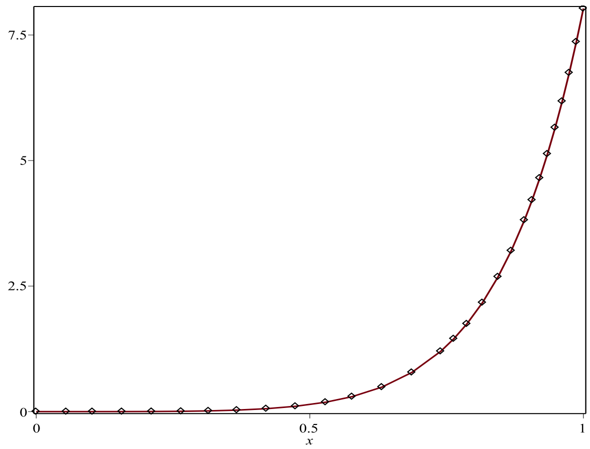

Fig. 2 The graph of exact (solid) and approximate solution (point) for Example 2.

Table 2 Numerical results for Example 2.

M=5	Proposed Method			[11]		[21]
N	$\|e_{NM}\|_2$	$\|e_{NM}\|_\infty$	n	$\|e_n\|_2$	n	$\|e_n\|_2$
8	6.0911e-6	2.6363e-6	12	2.0857e-7	31	2.30e-2
16	1.8734e-7	7.3576e-8	16	1.7367e-8	63	3.00e-4
32	4.5984e-9	6.3884e-9	22	5.3970-10	127	3.89e-5
64	1.4221e-10	2.1018e-10				

Example 3 In this example the following second kind Abel type Volterra integral equation is considered

$$u(x) = 1 + \int_0^x (x-t)^{-1/3} u(t)dt, \quad x \in [0,1]. \tag{30}$$

Here,

$$w = 1, \quad \alpha = 2/3, \quad v(x) = 1, \quad k(x,t) = 1.$$

Under these assumptions the exact solution is

$$u(x) = E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(\alpha k + 1)}.$$

The numerical results of this example are illustrated in Table 3. Similarly, the graph of exact and numerical solution of Example 3 for $M = 5$ and $N = 8$ can be seen in Fig. 3. The results present good agreement between the numerical and exact solution.

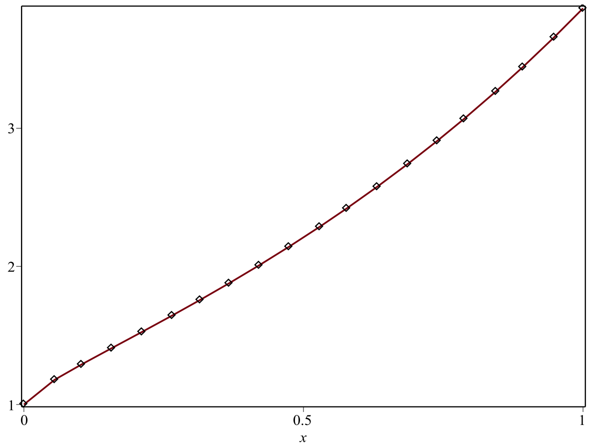

Fig. 3 The graph of exact (solid) and approximate solution (point) for Example 3.

Table 3 Numerical results for Example 3 ($M = 5$).

N	$\|e_{NM}\|_2$	$\|e_{NM}\|_\infty$
8	7.3111e-4	2.3608e-3
16	3.3193e-4	1.5132e-3
32	6.7155e-5	6.1115e-4
64	3.0056e-5	3.8664e-4

Example 4 Suppose the following first kind Abel Volterra integral equation [12]

$$e^{-x}(x^4 + x^6) = \int_0^x (x-t)^{-1/2} e^{-x+t} u(t)dt, \quad x \in [0, 1]. \tag{31}$$

Here,

$$w = 0, \quad \alpha = 1/2, \quad v(x) = -e^{-x}(x^4 + x^6), \quad k(x,t) = e^{-x+t}.$$

Under these assumptions the exact solution in

$$u(x) = e^{-x}\left(\frac{4!}{\Gamma(4.5)}x^{3.5} + \frac{6!}{\Gamma(6.5)}x^{5.5}\right).$$

The numerical results of this example are illustrated and compared with [12] in Table 4. It is obvious that our results are better than the results in [12]. The graph of exact and numerical solution of Example 4 for $M = 5$ and $N = 8$ is plotted in Fig.4.

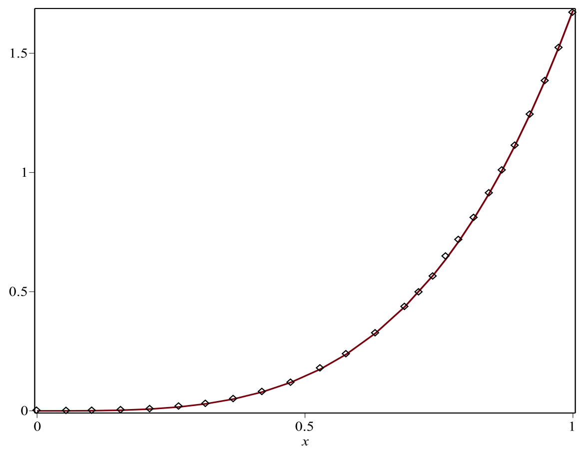

Fig. 4 The graph of exact solution (solid) and approximate solution (point) for Example 4.

Table 4 The Comparison of numerical results for Example 4. Trapezoidal method [21] and hp-collocation method [12].

M=3	Proposed Method			[21]	[12]
N	$\|e_{NM}\|_2$	$\|e_{NM}\|_\infty$	n	$\|e_n\|_\infty$	$\|e_n\|_\infty$
8	1.6288e-4	1.7933e-4	32	2.95e-3	7.29e-4
16	2.0578e-5	2.4123e-5	64	1.04e-3	1.83e-4
32	2.5882e-6	3.1208e-6	128	2.03e-4	4.573-5
64	2.6613e-8	2.2920e-8	256	5.79e-5	1.14e-5

Example 5 The last example is a second kind Abel Volterra integral equation [12]

$$4u(x) = \frac{4}{\sqrt{x+1}} - \arctan\left(\frac{1-x}{1+x}\right) + \frac{\pi}{2} - \int_0^x (x-t)^{-1/2}u(t)dt, \quad x \in [0,1]. \tag{32}$$

Here,

$$w = 1, \quad \alpha = \frac{1}{2}, \quad v(x) = \frac{1}{\sqrt{x+1}} - \frac{1}{4}\arctan\left(\frac{1-x}{1+x}\right) + \frac{\pi}{8}, \quad k(x,t) = -\frac{1}{4}.$$

The exact solution is

$$u(x) = x^2.$$

Table 5 presents a comparison for numerical results of this example and those in [12], and it implies that our method is much better. Also, the graph of exact and numerical solution of Example 5 for $M = 5$ and $N = 8$ can be seen in Fig.5.

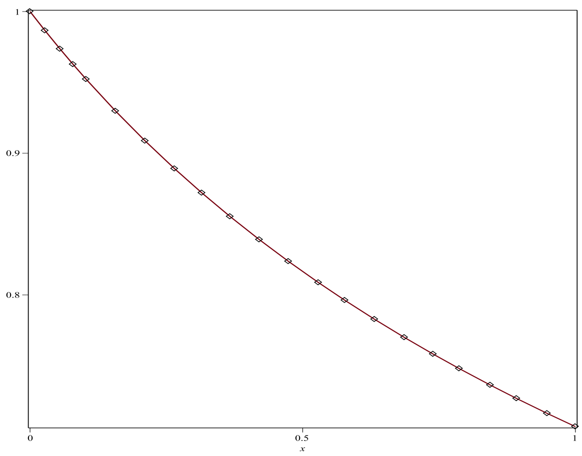

Fig. 5 The graph of exact solution (solid) and approximate solution (point) for Example 5.

Table 5 Numerical results for Example 5.

M=5	Proposed Method			[14]
N	$\|e_{NM}\|_2$	$\|e_{NM}\|_\infty$	n	$\|e_n\|_\infty$
8	8.1451e-8	1.9319e-7	16	4.1538e-3
16	2.7568e-9	8.3635e-9		
32	8.8084e-11	3.1086e-10		

6 Conclusion

In this paper, the operational matrix of Riemann-Liouville fractional integration and the product operational matrix for Jacobi wavelets on an arbitrary interval $[0, L]$ in computed in a new block matrix structure, which reduces computational costs. Then, the new method is applied to the first and second kind weakly singular Volterra integral equation by converting the main problem to the corresponding linear system of algebraic equations. Some numerical experiments are provided and the results are compared with the exact solution or with the results gained by some other recently reported numerical techniques. According to these results, the proposed method has good performance in comparison to the referenced methods.

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